KENNETH J. ARROW'S WORK ON COPING WITH RISK AND UNCERTAINTY

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In World War II, Kenneth Arrow served as weather officer in the US Army Air Corps. Upon his discharge at the war's end he returned, as doctoral student, to Columbia University. Simultaneously, and in parallel, he was appointed Research Associate at the Cowles Commission in the University of Chicago. The Director of the Cowles Commission at the time (1943-1948) was Jacob Marschak, under whose leadership a remarkable group of researchers had assembled. One of the topics which was being explored intensively at the Cowles Commission at the time was decision making under uncertainty. Researchers pursuing this topic in Chicago at the time included, in addition to Kenneth Arrow, Jacob Marschak, Leonard J. Savage, Herman Chernoff and Herman Rubin. Among many landmarks, the work of this group led to two memorable events: first, Jacob Marschak's widely attended Cowles Commission Seminar in July of 1948, in which he unveiled his own version of the Expected Utility hypothesis, and second, Kenneth Arrow's 1949 presentation, at the annual meeting of the American Statistical Association, on "New Developments in the Theory of Choice under Uncertainty". The texts of both presentations appeared subsequently (in 1950 and 1951, respectively) in *Econometrica*. It should be remembered that in the late 1940's behavior under risk and uncertainty had yet to become part of canonical economic theory. Von Neumann's and Morgenstern's axiomatic treatment of the Expected Utility hypothesis had just come out (*Theory of Games and Economic Behavior*, 2nd Edition, 1947) and did not as yet find its way into graduate school economics. And Frank Ramsey's seminal contribution (*Truth and Probability*, 1931) was largely ignored by the Economics profession, as indeed was Daniel Bernoulli's 1738 article on the St. Petersburg Paradox. (Alfred Marshall did mention Bernoulli's contribution in his *Principles*, but only in a remote footnote.) Kenneth Arrow's and Jacob Marschak's contributions/presentations in the late 1940's formed a gate, so to speak, through which the new theory of Decision Making Under Uncertainty was ushered into mainstream Price Theory.

In 1952, Kenneth Arrow was invited to present a paper at a colloquium in Paris on "The Foundations and Applications of Risk-Theory in Economics". Arrow's paper appeared in

1953, in French, in the published proceedings of that colloquium, and an English translation later appeared in the *Review of Economic Studies* (1963-1964) under the title "The Role of Securities in the Optimal Allocation of Risk-Bearing". The Basic idea underlying this paper was to model the uncertainty faced by a decision maker not in terms of the probabilities of various events but in terms of the underlying states-of-nature which make up those events. The set of states-of-nature (or "states-of-the-world") is essentially what statisticians call "the Sample Space" whose (measurable) subsets are the events. Each state-of-nature is one of the conceivable answers to the question "how are things going to turn out?" The "universe of discourse" is thus the *product* of the set of states-of-nature with a conventional commodity space, so that a typical element of this "universe" is basically a *contract* promising to deliver unto its owner, in each state-of-nature, certain quantities of resources, contingent upon that state having occurred. An agent's *preference order* on this product space conveys information not only about how the agent's well-being is served, but also about how likely each state-of-nature is in the agent's eyes. In his 1952 Paris paper, Kenneth Arrow uses this framework to analyze behavior in risk-laden situations. Classical theorems (due also, largely, to Kenneth Arrow), concerning existence and optimality of competitive equilibrium, apply in this expanded setting, provided that the agent's preferences obey the requisite assumptions, including convexity. In other words, the competitive market can be relied upon to allocate both goods *and risks* optimally. To describe the agent's preferences, Arrow invokes a multivariate analogue of von Neumann's and Morgenstern's Expected Utility theorem. Convexity of preferences implies risk aversion, i.e., the utility functions whose expected values are being maximized need to be concave. Arrow points out that state-contingent commodity bundles are, in effect, contingent claims, i.e., they take on a role of generalized *securities*. In the real world, securities are contingent claims, normally denominated in money, not in physical commodities. It is natural, therefore, to ask whether the competitive allocation that emerges in a world with generalized commodity-denominated securities is achievable also when securities are restricted to being payable in money, that is, when the markets for commodity-denominated securities simply do not exist. We have here an early case where a competitive system with *missing markets* is being studied. Arrow's answer is that, yes, Pareto-optimal competitive equilibrium is achievable also when securities are completely monetarized. Moreover, this result remains valid also in a world where securities are restricted not only to being monetarized, but also to being *basic*. (A *basic security* is a contract that promises its holder a reward of one monetary unit—say $1—if some pre-specified state-of-nature occurs and 0 otherwise.) Arrow shows that basic securities have the property of *spanning* the entire space of securities for which competitive optimality holds. In 1959, Gerard Debreu used the same construction in his book *Theory of Value*, after which basic securities have come to be known as "*Arrow-Debreu Securities*."

In the 1950s, Kenneth Arrow repeatedly taught a graduate micro-theory course at Stanford University, in which decision making under uncertainty was a central chapter. The Expected Utility hypothesis and its history, dating back to Bernoulli's 1738 article, were highlighted. Arrow reminded his students that, historically, discussions of decision making in risk-infected situations were held almost invariably in the context of parlor games and social gambling, where the entity being affected by risk was *money*. (the most notable exception was Pascal's Wager.) It was natural therefore that attitudes towards risk should rest upon attitudes towards money (or wealth). Risk aversion, in particular, was identified from the very start with decreasing marginal utility, i.e., with the concavity of the utility that the agent derives from money. The degree to which an agent was risk-averse was thus naturally related to the "degree of concavity" of the agent's utility function. It was by no means the case that agents were always risk-averse. People are known not only for purchasing actuarially unfair insurance (a clear sign of risk aversion) but often also for purchasing lottery tickets and engaging in other types of actuarially biased gambling schemes (a sign of risk-seeking). This phenomenon comes up in a paper by Arrow, titled "The Theory of Risk Aversion", which he delivered in Helsinki, Finland, as part of a lecture series, in 1963. Arrow is visibly unhappy with the well-known proposal, made in 1948 by M. Friedman and L. J. Savage, where the *behavioral* coexistence of opposites (gambling and insurance) is explained by *theoretical* coexistence of opposites (a convex dent in an otherwise concave utility function). Arrow thinks that, unlike insurance, gambling may be due not to the agent's attitude towards *wealth* but rather to the agent's attitude towards *probabilities* (a cognitive tendency to exaggerate very small probabilities). It is here that he quotes the Southern preacher who speaks to his flock thus: "Brethern, here there is a great difficulty; let us face it firmly and pass on". Be that as it may, Arrow at this point sets out to characterize an agent's degree of risk aversion in terms of the *extent of concavity* of that agent's utility function. (It turned out that in parallel, and independently, John W. Pratt [1964] was working along the same lines.) Not surprisingly, the degree of an agent's risk aversion has to do with the rate of decline of that agent's marginal utility of wealth. Given the agent's utility function, u, Arrow defines two indices of concavity, namely,

$-dlogu'(x)/dx$ and $-dlogu'(x)/dlogx$. The first is the index of **absolute** risk aversion (measuring the agent's dislike of risk in gambles which are specified in absolute amounts) and the second is the index of **relative** risk aversion (measuring the agent's dislike of risk when gambles are specified in *percentages* of the agent's current wealth). Both indices are nonnegative (if u is concave) and both are invariant under replacement of u by a linear transformation of itself. It seemed reasonable to hypothesize that absolute risk aversion *decreases* with wealth whereas relative risk aversion *increases* with wealth. Both of these hypotheses were shown to yield reasonable comparative statics predictions. Alas, they also imply that the quadratic utility function must be abandoned and, with it, so must mean-variance analysis.

The 1950's was also a decade in which intense research effort was devoted to the study of dynamic planning, where risk was unfolding gradually over time. (The influential text on this at the time was Richard Bellman's *Dynamic Programming* of 1957.) The central idea was that, with risk being resolved successively, one's task was not to look for a *plan* telling you what's to be done in every future period, but for a *policy* (or a *strategy*) telling you how best to respond, in each future period, to the information that will have become available until that period. Much research at the time went into the study of optimal policies in *inventory management*, where demand in future periods is random. Here Kenneth Arrow, in collaboration with T.E. Harris and J. Marschak (*Econometrica*, 1951) was once again at the forefront. Arrow, Harris and Marschak were among the first to explore the so called "two-bin" inventory policies (also known as (s, S)-policies) having the form: (1) Replenish your inventory if, and only if, current inventory has fallen below the level s; (2) when replenishing, bring inventory level up to the level S. (obviously, $s < S$.) A great deal of effort went, in those years, into exploring optimal inventory policies and the conditions under which they were in fact of the simple two-bin variety. Much of that research took place at Stanford University, where a center for research in mathematical economics and operations research had been created, with funding from the U.S. office of Naval Research and the Ford Foundation. In 1958, Stanford University Press published a volume titled *Studies in the Mathematical Theory of Inventory and Production* under the editorship of Kenneth Arrow, Samuel Karlin and Herbert Scarf. This volume consisted of 17 separate essays, with no fewer than seven authored or co-authored by Kenneth Arrow.

In the early 1960s, Kenneth Arrow began to devote considerable attention to *medical care* and to what distinguishes, from an economic point of view, medical care from other goods and services. Clearly, risk and uncertainty are a major part of the story. In 1963, Arrow's study of "Uncertainty and the Welfare Economics of Medical Care" came out in the *American Economic Review*. Later on, in 1974, came "Optimal Insurance and Generalized Deductibles", published in the *Scandinavian Actuarial Journal*, which could be thought of as a sequel to the 1963 *AER* paper. These studies build on the general "states-of-nature" approach, first analyzed by Arrow in his presentation in 1952 at the Paris colloquium, discussed above, on "the Foundation and Applications of Risk-Theory in Economics". In the medical care context, a "state-of-nature" included a complete specification of every individual's state of health, as well as these individuals' holdings of other "commodities". For simplicity, these other commodities were restricted to include the individuals' state-specific cash holdings and nothing else. The individual's preferences among "commodity bundles" would now include their probability assessments for such things as present and future health or ill-health, whether their own or that of others. In this vast state space, a Walrasian economy can be unleashed and, provided that classical assumptions (convexity, etc.) are met, a Pareto-efficient competitive equilibrium could be shown to exist. Health insurance in this system can be described as a *trade* where, in return for relinquishing certain amounts of money in certain states (i.e., making certain premium payments), the insured receives certain benefits in certain other states. The equilibrium allocation would thus include equilibrium health insurance for all concerned. By definition, the insurance contract held by an individual in equilibrium must be preferred by that individual to all other affordable insurance contracts. What, then, does the best affordable insurance contract look like? To answer this question, Arrow turns once again to Expected Utility Theory. Letting a_s be post-benefit income in state-of-nature, s, the individual is taken to maximize the expression $\sum p_s u(a_s)$ where p_s stands for the probability of state s and u is the utility function. (Later, utility is taken to be state-dependent, so the individual is seen as maximizing $\sum p_s u_s(a_s)$, where u_s is utility in state s.) The best insurance coverage, from the point of view of the insured, is as follows: The set of states-of-nature is partitioned into two subsets, say S and S', such that for states in S no insurance coverage is provided at all, and for states is S' complete insurance is provided, covering the health care cost actually incurred in full, *save for* some pre-specified deductible, to be borne by the insured. The levels of the deductible and the premium (i.e., the price of the insurance policy) are taken by the insured as given.

Will a competitive system be capable, through the interplay of market forces, of delivering equilibrium, with all agents—insurers, insured individuals, health-care providers, etc.—

optimizing, and markets being cleared? Kenneth Arrow appears dubious. Not only are markets in the medical care industry unlikely to deliver efficient equilibria, these markets in fact are unlikely to arise in the first place. This phenomenon of missing markets is referred to by Arrow as **non-marketability**, which appears to pervade the medical care and medical insurance worlds and to impede the rise of appropriate markets through which resources might be optimally allocated. Among these impediments are features like increasing returns on the providers' side and health-externalities and moral hazard on the receivers' side. "Society", says Arrow in his 1963 *AER* paper, "will seek to achieve optimality by non-market means".

On April 13, 1978, Kenneth Arrow delivered a talk at Columbia University, titled "A Cautious Case for Socialism." Later that year, he published that talk as an essay, in the Fall 1978 issue of *Dissent*. On p. 478 he writes:

> In the ideal theory of the competitive economy, market-clearing prices serve as the communication links that brings into coherence the widely dispersed knowledge about the needs and production possibilities of the members of the economy. In the absence of suitable markets, other coordinating and communicating mechanisms are needed for efficiency.

He surely had medical care and medical insurance in mind.

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