Kenneth Arrow's Contributions to General Equilibrium

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It is not easy to separate the significance and influence of the Arrow-Debreu model of general equilibrium from that of mathematical economics itself. In an extraordinary series of papers and books (1951, 1954, 1959, 1971), Ken Arrow and Gerard Debreu settled two of the oldest and most important questions of economics through arguments at least as elegant as any that have ever been given in all of economics, using the techniques of convexity and fixed point theory that are still, after sixty-five years, the most important mathematical devices in mathematical economics. More than any other, their model crystallized the mathematical-axiomatic approach that transformed economics from a field not much more mathematical than its sister social sciences like sociology and psychology into a discipline with the same mathematical rigor as physics and the other hard sciences.[1]

The Arrow Debreu model was simple enough to be understood immediately by mathematicians with no training in economics, yet general enough, given ever subtler interpretations of the notion of commodity, to encompass a large fraction of economics known up until that time, as special cases. Moreover, many subsequent developments in economics could be cast as elementary relaxations of the Arrow-Debreu framework. Today general equilibrium plays an absolutely central role in fields as diverse as international trade, public finance, development, finance, and macroeconomics. When we consider that Arrow not only derived the most fundamental properties of the model (along with Debreu, and McKenzie), but also provided the most significant interpretive extensions, it is no wonder that he remains the youngest Nobel Prize winner in economics. [2]

In 1951 and 1954 Arrow and Debreu sought to give a formal mathematical answer to two questions discussed by Adam Smith in 1776. Can free markets alone coordinate the diverse desires and talents of millions of consumers and producers, all pursuing their own selfish interests without any regard for or knowledge of the others, in a way that promotes the common good? Axioms were provided for the commodity space, for endowments and preferences of each consumer, and for the production possibilities of each firm. Equilibrium was then defined via prices and budget sets, embodying the neoclassical methodological premises of individual rationality, market clearing, perfect competition, and rational expectations.

The first fruit of the more precise formulation of equilibrium was the transparent

[1] Of course economics does not have the same empirical-experimental validity as say physics.

[2] Debreu made one very substantive contribution the others didn't, introducing transversality theory to prove that equilibria are "generically" or almost always locally unique and finite in number. Dierker then showed that generically equilibria are odd in number.

demonstration of the first and second welfare theorems that Arrow and Debreu independently gave in 1951. Particularly noteworthy is the proof that every equilibrium is Pareto optimal, that is, that no other feasible allocation is preferred by every consumer. So simple and illuminating is this demonstration that it is no exaggeration to call it the most important argument in all of neoclassical economic theory. The old proofs, which still linger in many intermediate textbooks, rested on three irrelevant assumptions: differentiable and concave utilities, and strictly positive consumption of every good by every individual. For example, Lange 1942 argued that in equilibrium every consumer's marginal rates of substitution had to equal every producer's marginal rates of transformation, which is tantamount to showing that a welfare function, consisting of the weighted sum of consumer utilities minus the value of their consumptions, has derivative equal to zero at the competitive equilibrium allocation. Concavity shows that welfare is maximized there, and hence no other feasible allocation can Pareto dominate the equilibrium allocation.

While listening to a talk about housing by Franko Modigliani, Arrow realized that most people consume nothing of most goods (for example living in just one particular kind of house), and thus that the prevailing efficiency proofs assumed away all the realistic cases. He soon found a proof that relied on none of the three assumptions, at exactly the same time Debreu was independently finding the same proof. If a competitive allocation could be Pareto dominated by another allocation, then the new allocation must cost each consumer more money at the old equilibrium prices than his old income, otherwise the competitive allocation could not have been optimal for him. But then the total cost over all consumers of the dominating allocation is more than their total income, proving that the dominating allocation is not feasible.

Arrow and Debreu together, and McKenzie separately, gave the touchstone proof in economics, of the existence of competitive equilibrium. Leon Walras, the inventor of general equilibrium, had informally suggested that equilibrium would always exist, because for every market there is a corresponding price, and by increasing the price in markets for which there is too much demand and decreasing the other prices, the economy would grope its way to equilibrium in which demand equals supply for all markets. Arrow and Hurwicz realized early on that proving the convergence of Walras' tatonnement dynamics was problematic in general, though they did prove convergence for several important kinds of economies beginning in 1958. The great mathematicians von Neumann and Wald recognized that Brouwer's fixed point theorem might be useful to prove the existence of equilibrium, but each was only able to prove the existence of equilibrium in very special cases.[3]

Arrow and Debreu independently realized, on reading the general proof of existence of Nash

[3] Von Neumann used Brouwer's theorem to prove the existence of a stationary growth path that ignored consumer preferences in 1937. In the existence proof Wald published in English in 1952 he implicitly assumed that every consumer had the same preferences. There is some testimony that Wald had written another unpublished existence proof based on Brouwer's Fixed Point Theorem that was more general.

equilibrium for games by John Nash in 1950, that a general existence proof for competitive economies must be attainable, even though it had eluded von Neumann and Wald. Their first hurdle in adapting Nash's logic was that, unlike the strategy sets of players in games, consumer choice sets (i.e. their budget sets) were not fixed, but depended on changing endogenous prices. Arrow introduced an anti-consumer for each consumer, whose goal was to punish his doppelganger if he spent too much money. Debreu introduced a price player and the idea of a generalized game in which the strategy sets could depend on the moves of other players. Once they realized they were essentially proving the same theorem, the two combined their efforts in improving Debreu's proof.[4]

By humbly paying careful attention to the precise formulation of the details of the model, especially zero consumption, they were able bring to bear powerful mathematical tools in a sustained argument to prove a theorem vastly more general than anything that came before. Brouwer's fixed point theorem cannot naturally be used to show that demand equals supply, but only that demand is less than or equal to supply. Arrow had observed that when a price is zero, and the good is free like air, it is natural for there to be excess supply. So they modified the definition of equilibrium to allow supply to exceed demand, but only for goods with zero price. The two of them also noticed that when income becomes zero while a price simultaneously hits zero, then demand can jump discontinuously. They recognized the need to make assumptions ruling this eventuality out, but still preserving the possibility that each individual might consume zero of many goods.

Part of the astonishing generality of the Arrow Debreu model is due to the simultaneous and equal treatment it gives to producers and consumers.[5] The classical economists, including Smith, Ricardo, Marx, and Cassel, had more or less assumed that relative prices are determined by fixed coefficients of production. The marginalists, like Jevons and Menger, claimed that price is determined by marginal utility. The Arrow-Debreu model recognized that each price was determined by its own supply and demand (as in the traditional picture of the cross from elementary economics) and also by prices of other commodities in a general equilibrium. This gave full scope to the possibility that indirect consequences can reverse the intentions of economic agents, as when producers striving to increase their profits compete so hard that they drive profits to zero. Arrow called this phenomenon of unintended consequences the most important idea in the social sciences.

The main engine for generalizing the Arrow Debreu model lies in the various interpretations of commodity that Arrow was able to conjure. Hicks, perhaps anticipated by Fisher, was the first to suggest an elaborate notion of commodity, in which two identical apples, that were deliverable at different time periods, were regarded as different commodities. Thus saving, or the lending of money, can be thought of as the trade today of an apple in exchange for a future dated

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[4] It is fascinating that Arrow and Debreu twice independently proved nearly identical theorems. Arrow suggested that ideas like convexity and Brouwer were in the air at Cowles, where they both visited, and at Princeton.

[5] One interesting difference, often remarked upon by Arrow, is that their assumptions about consumers apply to each individually, whereas it is necessary to make an assumption relating the productivity of producers, as Koopmans had shown in his activity analysis.

apple, exactly like the trade of an apple for an orange; the ratio of the apple prices can then be interpreted as the gross real rate of interest. The mundane general equilibrium model of trading apples for oranges can be used without substantive change to explain the real rate of interest, which is the starting point of macroeconomics and finance. Similarly the same commodity or the same labor at different geographical locations can be treated like different commodities, and again the general equilibrium model can encompass international trade.

Arrow took a more imaginative tack in extending commodities. In 1969 he added a firm technology that transformed each (public) good into many copies, each indexed by a different consumer name. This enabled the Arrow-Debreu model to include the theory of public goods and externalities, for example making clear why the efficient production of private goods equates marginal cost with each consumer's marginal utility, whereas the efficient production of public goods equates marginal cost with the sum of consumers' marginal utilities, as Samuelson had pointed out. It also made clear that the efficiency losses from externalities could be ascribed to missing markets.[6]

Arrow's boldest stroke by far was in imagining in 1953 that a New York apple is a different commodity depending on how much it is snowing in Paris. By distinguishing physical objects depending on the state of nature (which includes a complete description of all uncertainty, even those apparently unrelated to the object), Arrow was able to analyze the optimal allocation of risk with exactly the same general equilibrium apparatus used to analyze the exchange of apples and oranges, and thereby to usher in the field of modern finance.

Before Arrow, uncertainty was represented in financial theory by joint normal distributions, which he replaced with arbitrary random variables. As Arrow explained, this was a natural step for somebody steeped in statistics. But it immediately implied that the output of a firm, or the payoff of a bond, could be thought of exactly the same way as a basket of commodities or fruit, where the bond payoff in each state corresponds to the quantity of a different kind of fruit. It meant that the payoff of an insurance contract could be modeled by the cash flows it provides in each state, without worrying that it is not normally distributed. This immediately led Arrow to describe what are now called Arrow securities, namely contingent securities that pay one dollar (or one apple) in exactly one state. The price of a bond or an insurance contract is then just the sum of the prices of its constituent Arrow securities, just as the price of a basket of fruit is the sum of its individual fruit prices. Following Arrow, modern financial theorists soon introduced the fundamental vocabulary of the field, talking of *state prices* for evaluating securities, and the payoffs of some benchmark securities *spanning* the payoffs of other target securities, and the *no-arbitrage* relationship that must hold between the target security price and the spanning benchmark securities prices. The celebrated Black-Scholes model of option

[6] Arrow seemed unaware in 1969 of the related work of Liindahl in 1919. Foley (1970) formalized Arrow's somewhat informal remarks.

pricing, especially in its binomial tree formulation, fits perfectly into this Arrow framework.

By laying out an axiomatic framework for general equilibrium, the Arrow-Debreu model opened the door for further work simply by relaxing some of the assumptions, or adjusting the model.[7] The most fundamental change, which Arrow introduced casually in 1953, was to allow for multiple budget constraints (instead of just one budget constraint for each individual) in tandem with replacing the Arrow securities with a smaller number of more realistic composite securities, as Radner formally did in 1968. This has come to be called general equilibrium with incomplete markets, or GEI, and is the setting for most of modern finance, starting with the modern version of the capital asset pricing model. Rationality in GEI means that agents foresee correctly what future prices will be, conditional on the state. Of course because of the missing Arrow securities, equilibrium is no longer Pareto efficient. In fact it turns out that equilibria are typically not even constrained efficient in the sense that the existing securities could have been traded differently to make everybody better off.[8] The model invites the possibility of government intervention, as in macroeconomics.

Much of the clamor in the 1970s for 'microeconomic foundations to macroeconomics' was a desire to see an axiomatic clarity similar to that of the Arrow-Debreu model applied to other areas of economics. Lucas, Kydland, Prescott, Sargent, and Townsend and others made general equilibrium the central paradigm in macroeconomics by adding a government sector and asymmetric information to the previous model (as well as infinite time).[9] They banished the ad hoc aggregate consumption functions and behavioral forecasts of classical macroeconomics in favor of agents a la Arrow and Debreu. Macroeconomics has evolved to rely less and less on asymmetric information, but more and more on the other parts of general equilibrium that Arrow created or fostered. Since the crisis more emphasis has been placed on credit and credit constraints, which Arrow foresaw as early as 1978.

[7] One natural extension was to allow for an infinite number of consumers, or even a continuum of consumers, as Aumann did in 1964. In that setting one can investigate the implications of perfect competition for cooperative games. One celebrated theorem is the equivalence of the Core and the set of competitive equilibria, as proved in a variety of settings by Arrow's friends Scarf, Debreu, and Aumann. A second extension is to allow for an infinity of commodities, as Samuelson did in 1958, and as Bewley did in 1972, and as many others have since, to represent an infinite time horizon.

[8] This was proved by Arrow's students Geanakoplos and Polemarchakis in 1986, using a framework from Stiglitz 1982.

[9] Arrow consistently encouraged young people to introduce asymmetric information into perfect competition. In the 1970s and 80s asymmetric information became a major component of game theory, partly spurred by Arrow's longtime friend and summer companion Bob Aumann. At the same time or a little later asymmetric information became central in macroeconomics and finance.

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