Econometrica Supplementary Material

SUPPLEMENT TO "THE U.S. PUBLIC DEBT VALUATION PUZZLE"

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APPENDIX A: PROOFS OF PROPOSITIONS

Proposition 1.

PROOF: All objects in this proof are in nominal terms. The government faces the following one-period budget constraint:

$$G_t - T_t + Q^{\$}_{t-1}(1) = \sum_{h=1}^{H} \big(Q^{\$}_t(h) - Q^{\$}_{t-1}(h+1)\big) P^{\$}_t(h), \tag{A.1}$$

where G_t is total nominal government spending, T_t is total nominal government revenue, $Q^{\$}_t(h)$ is the number of nominal zero-coupon bonds of maturity h outstanding in period t each promising to pay back \$1 at time $(t+h)$, and $P^{\$}_t(h)$ is today's price for an h-period zero-coupon bond with \$1 face value. A unit of $(h+1)$-period bond issued at $t-1$ becomes a unit of h-period bond in period t. Moreover, bonds could be issued or redeemed in period t, so that the stock of bonds of each maturity evolves according to $Q^{\$}_t(h) = Q^{\$}_{t-1}(h+1) + \Delta Q^{\$}_t(h)$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons. We assume that H is the longest maturity issued in each period so that $Q^{\$}_{t-1}(H+1) = 0, \forall t$.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G - T$ and one-period debt from last period that is now maturing. The right-hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equaling total income:

$$G_t + Q^{\$}_{t-1}(1) + \sum_{h=1}^{H} Q^{\$}_{t-1}(h+1) P^{\$}_t(h) = T_t + \sum_{h=1}^{H} Q^{\$}_t(h) P^{\$}_t(h),$$

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We can now iterate the budget constraint forward. The period t constraint is given by

$$\begin{aligned}T_t - G_t &= Q^\$_{t-1}(1) - Q^\$_t(1)P^\$_t(1) + Q^\$_{t-1}(2)P^\$_t(1) - Q^\$_t(2)P^\$_t(2)\\ &\quad + Q^\$_{t-1}(3)P^\$_t(2) - Q^\$_t(3)P^\$_t(3) + \cdots - Q^\$_t(H)P^\$_t(H) + Q^\$_{t-1}(H+1)P^\$_t(H).\end{aligned}$$

Consider the period-$t+1$ constraint,

$$\begin{aligned}T_{t+1} - G_{t+1} &= Q^\$_t(1) - Q^\$_{t+1}(1)P^\$_{t+1}(1) + Q^\$_t(2)P^\$_{t+1}(1) - Q^\$_{t+1}(2)P^\$_{t+1}(2) + Q^\$_t(3)P^\$_{t+1}(2)\\ &\quad - Q^\$_{t+1}(3)P^\$_{t+1}(3) + \cdots - Q^\$_{t+1}(H)P^\$_{t+1}(H) + Q^\$_t(H+1)P^\$_{t+1}(H),\end{aligned}$$

multiply both sides by $M^\$_{t+1}$, and take expectations conditional on time t:

$$\begin{aligned}\mathbb{E}_t\big[M^\$_{t+1}(T_{t+1} - G_{t+1})\big] &= Q^\$_t(1)P^\$_t(1) - \mathbb{E}_t\big[Q^\$_{t+1}(1)M^\$_{t+1}P^\$_{t+1}(1)\big] + Q^\$_t(2)P^\$_t(2)\\ &\quad - \mathbb{E}_t\big[Q^\$_{t+1}(2)M^\$_{t+1}P^\$_{t+1}(2)\big] + Q^\$_t(3)P^\$_t(3)\\ &\quad - \mathbb{E}_t\big[Q^\$_{t+1}(3)M^\$_{t+1}P^\$_{t+1}(3)\big] + \cdots + Q^\$_t(H)P^\$_t(H)\\ &\quad - \mathbb{E}_t\big[Q^\$_{t+1}(H)M^\$_{t+1}P^\$_{t+1}(H)\big] + Q^\$_t(H+1)P^\$_t(H+1),\end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t[M^\$_{t+1}] = P^\$_t(1)$, $\mathbb{E}_t[M^\$_{t+1}P^\$_{t+1}(1)] = P^\$_t(2), \cdots,$ $\mathbb{E}_t[M^\$_{t+1}P^\$_{t+1}(H-1)] = P^\$_t(H)$, and $\mathbb{E}_t[M^\$_{t+1}P^\$_{t+1}(H)] = P^\$_t(H+1)$.

Consider the period $t+2$ constraint, multiplied by $M^\$_{t+1}M^\$_{t+2}$ and take time-t expectations:

$$\begin{aligned}&\mathbb{E}_t\big[M^\$_{t+1}M^\$_{t+2}(T_{t+2} - G_{t+2})\big]\\ &\quad = \mathbb{E}_t\big[Q^\$_{t+1}(1)M^\$_{t+1}P^\$_{t+1}(1)\big] - \mathbb{E}_t\big[Q^\$_{t+2}(1)M^\$_{t+1}M^\$_{t+2}P^\$_{t+2}(1)\big] + \mathbb{E}_t\big[Q^\$_{t+1}(2)M^\$_{t+1}P^\$_{t+1}(2)\big]\\ &\qquad - \mathbb{E}_t\big[Q^\$_{t+2}(2)M^\$_{t+1}M^\$_{t+2}P^\$_{t+2}(2)\big] + \mathbb{E}_t\big[Q^\$_{t+1}(3)M^\$_{t+1}P^\$_{t+1}(3)\big] - \cdots\\ &\qquad + \mathbb{E}_t\big[Q^\$_{t+1}(H)M^\$_{t+1}P^\$_{t+1}(H)\big] - \mathbb{E}_t\big[Q^\$_{t+2}(H)M^\$_{t+1}M^\$_{t+2}P^\$_{t+2}(H)\big]\\ &\qquad + \mathbb{E}_t\big[Q^\$_{t+1}(H+1)M^\$_{t+1}P^\$_{t+1}(H+1)\big],\end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M^\$_{t+2}] = P^\$_{t+1}(1)$, $\mathbb{E}_{t+1}[M^\$_{t+2} \times P^\$_{t+2}(1)] = P^\$_{t+1}(2)$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$, we get

$$\begin{aligned}&T_t - G_t + \mathbb{E}_t\big[M^\$_{t+1}(T_{t+1} - G_{t+1})\big] + \mathbb{E}_t\big[M^\$_{t+1}M^\$_{t+2}(T_{t+2} - G_{t+2})\big]\\ &\quad = \sum_{h=0}^{H} Q^\$_{t-1}(h+1)P^\$_t(h) - \mathbb{E}_t\big[Q^\$_{t+2}(1)M^\$_{t+1}M^\$_{t+2}P^\$_{t+2}(1)\big]\\ &\qquad - \mathbb{E}_t\big[Q^\$_{t+2}(2)M^\$_{t+1}M^\$_{t+2}P^\$_{t+2}(2)\big] - \cdots - \mathbb{E}_t\big[Q^\$_{t+2}(H)M^\$_{t+1}M^\$_{t+2}P^\$_{t+2}(H)\big].\end{aligned}$$

Similarly, consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_{t}(h) = \mathbb{E}_t\left[\sum_{j=0}^{J} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right] + \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right],$$

where we used the cumulative SDF notation $M^{\$}_{t,t+j}=\prod_{i=0}^{j} M^{\$}_{t+i}$ and by convention $M^{\$}_{t,t}=M^{\$}_{t}=1$ and $P^{\$}_{t}(0)=1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J\to\infty$:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_{t}(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right] + \lim_{J\to\infty}\mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j}-G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J\to\infty}\mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right]=0,$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise, there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt at the beginning of period t reflects the expected present-discounted value of the current and all future primary surpluses:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_{t}(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right].$$

Finally, using the one-period budget constraint (A.1) again, we obtain the end-of-period market value of government debt, D_t, defined in equation (1) in the main text:

$$D_t=\sum_{h=1}^{H} Q^{\$}_{t}(h)P^{\$}_{t}(h) = \mathbb{E}_t\left[\sum_{j=1}^{\infty} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right].$$

Case With Default.

PROOF: We consider only full default, without loss of generality. Alternatively, we can write the budget constraint that obtains in case of no default at t:

$$G_t + Q^{\$}_{t-1}(1) + \sum_{h=1}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = T_t + \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h),$$

and, in case of default at t, the one-period budget constraint is given by

$$G_t = T_t + \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h).$$

We can now iterate the budget constraint forward. In case of no default, the period t constraint is given by

$$\begin{aligned} T_t - G_t &= Q^{\$}_{t-1}(1) - Q^{\$}_t(1)P^{\$}_t(1) + Q^{\$}_{t-1}(2)P^{\$}_t(1) - Q^{\$}_t(2)P^{\$}_t(2) + Q^{\$}_{t-1}(3)P^{\$}_t(2) \\ &\quad - Q^{\$}_t(3)P^{\$}_t(3) + \cdots - Q^{\$}_t(H)P^{\$}_t(H) + Q^{\$}_{t-1}(H+1)P^{\$}_t(H). \end{aligned}$$

In case of default, the period t constraint is given by

$$T_t - G_t = -Q^{\$}_t(1)P^{\$}_t(1) - Q^{\$}_t(2)P^{\$}_t(2) - Q^{\$}_t(3)P^{\$}_t(3) - Q^{\$}_t(H)P^{\$}_t(H).$$

First, consider the period-$t+1$ constraint in case of no default,

$$\begin{aligned} T_{t+1} - G_{t+1} &= Q^{\$}_t(1) - Q^{\$}_{t+1}(1)P^{\$}_{t+1}(1) + Q^{\$}_t(2)P^{\$}_{t+1}(1) - Q^{\$}_{t+1}(2)P^{\$}_{t+1}(2) + Q^{\$}_t(3)P^{\$}_{t+1}(2) \\ &\quad - Q^{\$}_{t+1}(3)P^{\$}_{t+1}(3) + \cdots - Q^{\$}_{t+1}(H)P^{\$}_{t+1}(H) + Q^{\$}_t(H+1)P^{\$}_{t+1}(H). \end{aligned}$$

Second, consider the period-$t+1$ constraint in case of default,

$$\begin{aligned} T_{t+1} - G_{t+1} &= -Q^{\$}_{t+1}(1)P^{\$}_{t+1}(1) - Q^{\$}_{t+1}(2)P^{\$}_{t+1}(2) - Q^{\$}_{t+1}(3)P^{\$}_{t+1}(3) \\ &\quad - \cdots - Q^{\$}_{t+1}(H)P^{\$}_{t+1}(H). \end{aligned}$$

We use χ_t as an indicator variable for default. To simplify, we consider only full default with zero recovery. This is without loss of generality. Next, multiply both sides of the no default constraint by $(1-\chi_{t+1})M^{\$}_{t+1}$, and take expectations conditional on time t:

$$\begin{aligned} &\mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi_{t+1})(T_{t+1}-G_{t+1})\big] \\ &\quad = Q^{\$}_t(1)\mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi_{t+1})\big] - \mathbb{E}_t\big[Q^{\$}_{t+1}(1)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)\big] \\ &\qquad + \mathbb{E}_t\big[(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)\big]Q^{\$}_t(2) \\ &\qquad - \mathbb{E}_t\big[Q^{\$}_{t+1}(2)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(2)\big] + \mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(2)\big]Q^{\$}_t(3) \\ &\qquad - \mathbb{E}_t\big[Q^{\$}_{t+1}(3)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(3)\big] + \cdots + Q^{\$}_t(H)\mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi)P^{\$}_{t+1}(H-1)\big] \\ &\qquad - \mathbb{E}_t\big[Q^{\$}_{t+1}(H)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(H)\big] + Q^{\$}_t(H+1)\mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)\big], \end{aligned}$$

and multiply both sides of the default constraint by $M^{\$}_{t+1}\chi_{t+1}$,

$$\begin{aligned}
&\mathbb{E}_t\big[M^{\$}_{t+1}\chi_{t+1}(T_{t+1}-G_{t+1})\big] \\
&\quad = -\mathbb{E}_t\big[Q^{\$}_{t+1}(1)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(1)\big] - \mathbb{E}_t\big[Q^{\$}_{t+1}(2)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(2)\big] \\
&\qquad - \mathbb{E}_t\big[Q^{\$}_{t+1}(3)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(3)\big] - \cdots - \mathbb{E}_t\big[Q^{\$}_{t+1}(H)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(H)\big].
\end{aligned}$$

By adding these two constraints, we obtain the following expression:

$$\begin{aligned}
&\mathbb{E}_t\big[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\big] \\
&\quad = Q^{\$}_t(1)\mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi_{t+1})\big] - \mathbb{E}_t\big[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)\big] \\
&\qquad + \mathbb{E}_t\big[(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)\big]Q^{\$}_t(2) \\
&\qquad - \mathbb{E}_t\big[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)\big] + \mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(2)\big]Q^{\$}_t(3) \\
&\qquad - \mathbb{E}_t\big[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)\big] + \cdots + Q^{\$}_t(H)\mathbb{E}_t\big[M_{t+1}(1-\chi)P^{\$}_{t+1}(H-1)\big] \\
&\qquad - \mathbb{E}_t\big[Q^{\$}_{t+1}(H)M_{t+1}P^{\$}_{t+1}(H)\big] + Q^{\$}_t(H+1)\mathbb{E}_t\big[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)\big].
\end{aligned}$$

This can be restated as

$$\begin{aligned}
\mathbb{E}_t\big[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\big] = {}& Q^{\$}_t(1)P^{\$}_t(1) - \mathbb{E}_t\big[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)\big] + Q^{\$}_t(2)P^{\$}_t(2) \\
&- \mathbb{E}_t\big[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)\big] + Q^{\$}_t(3)P^{\$}_t(3) \\
&- \mathbb{E}_t\big[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)\big] + \cdots + Q^{\$}_t(H)P^{\$}_t(H) \\
&- \mathbb{E}_t\big[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)\big] + Q^{\$}_t(H+1)P^{\$}_t(H+1),
\end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})] = P^{\$}_t(1)$, $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(1)] = P^{\$}_t(2)$, $\cdots$, $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H-1)] = P^{\$}_t(H)$, and $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)] = P^{\$}_t(H+1)$.

The rest of the proof is essentially unchanged. Consider the period $t+2$ constraint, multiplied by $M^{\$}_{t+1}M^{\$}_{t+2}(1-\chi_{t+2})$ in the no-default case, and $M^{\$}_{t+1}M^{\$}_{t+2}(\chi_{t+2})$ for the default case, and take time-t expectations (after adding default and no-default states):

$$\begin{aligned}
&\mathbb{E}_t\big[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2}-G_{t+2})\big] \\
&\quad = \mathbb{E}_t\big[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)\big] - \mathbb{E}_t\big[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)\big] \\
&\qquad + \mathbb{E}_t\big[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)\big] - \mathbb{E}_t\big[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)\big] \\
&\qquad + \mathbb{E}_t\big[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)\big] - \cdots + \mathbb{E}_t\big[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)\big] \\
&\qquad - \mathbb{E}_t\big[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)\big] + \mathbb{E}_t\big[Q^{\$}_{t+1}(H+1)M^{\$}_{t+1}P^{\$}_{t+1}(H+1)\big],
\end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M^{\$}_{t+2}(1-\chi_{t+2})] = P^{\$}_{t+1}(1)$, $\mathbb{E}_{t+1}[M^{\$}_{t+2}(1-\chi_{t+2})P^{\$}_{t+2}(1)] = P^{\$}_{t+1}(2)$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$, we get

$$
\begin{aligned}
&T_t - G_t + \mathbb{E}_t\big[M^{\$}_{t+1}(T_{t+1} - G_{t+1})\big] + \mathbb{E}_t\big[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2} - G_{t+2})\big] \\
&\quad = \sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) - \mathbb{E}_t\big[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)\big] \\
&\qquad - \mathbb{E}_t\big[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)\big] - \cdots - \mathbb{E}_t\big[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)\big].
\end{aligned}
$$

Similarly, consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get

$$
\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = \mathbb{E}_t\left[\sum_{j=0}^{J} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right] + \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right],
$$

where we used the cumulative SDF notation $M^{\$}_{t,t+j} = \prod_{i=0}^{j} M^{\$}_{t+i}$ and by convention $M^{\$}_{t,t} = M^{\$}_t = 1$ and $P^{\$}_t(0) = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \to \infty$:

$$
\begin{aligned}
\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) &= \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right] \\
&\quad + \lim_{J\to\infty} \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right].
\end{aligned}
$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j} - G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$
\lim_{J\to\infty} \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right] = 0,
$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise, there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today,

$$
\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right].
$$

Using the one-period budget constraint (A.1) again, we obtain the expression for the end-of-period market value of government debt, D_t, in equation (1):

$$D_t = \sum_{h=1}^{H} Q_t^{\$}(h) P_t^{\$}(h) = \mathbb{E}_t\left[\sum_{j=1}^{\infty} M_{t,t+j}^{\$}(T_{t+j} - G_{t+j})\right].$$

Q.E.D.

Proposition 2. Since

$$D_t = \sum_{h=1}^{H} Q_t^{\$}(h) P_t^{\$}(h),$$

we have

$$R_{t+1}^D D_t = \sum_{h=0}^{\infty} Q_t^{\$}(h+1) P_{t+1}^{\$}(h) = \left(P_{t+1}^T + T_{t+1}\right) - \left(P_{t+1}^G + G_{t+1}\right)$$
$$= P_t^T R_{t+1}^T - P_t^G R_{t+1}^G.$$

Taking expectations, we obtain equation (3) in the main text. *Q.E.D.*

Proposition 3.

PROOF: We follow the proof in the working paper version of Backus, Boyarchenko, and Chernov (2018) (Appendix A: Long Horizons). For a given pricing kernel $M_{t,t+1}$, Hansen and Scheinkman (2009) consider the problem of finding a positive dominant eigenvalue ν and associated positive eigenfunction v_t satisfying

$$\mathbb{E}_t[M_{t,t+1} v_{t+1}] = \nu v_t. \tag{A.2}$$

The dominant eigenvalue of a matrix is real (not complex) and is strictly greater in absolute values than all other eigenvalues. This gives rise to the following decomposition of the pricing kernel: $M_{t,t+1} = M_{t,t+1}^1 M_{t,t+1}^2$:

$$M_{t,t+1}^1 = M_{t,t+1} v_{t+1}/\nu v_t,$$
$$M_{t,t+1}^2 = \nu v_t / v_{t+1}.$$

By construction, $M_{t,t+1}^1$ is the martingale component with $\mathbb{E}_t[M_{t,t+1}^1] = 1$. Backus, Boyarchenko, and Chernov (2018) show that the long bond yields converge to $-\log\nu$, the long bond one-period bond return converges to $\lim_{n\to\infty} R_{t,t+1}^n = \frac{1}{M_{t,t+1}^2} = v_{t+1}/\nu v_t$, and its expected value to $\mathbb{E}[\log R_{t,t+1}^{\infty}] = -\log\nu$.

Next, we value claims to uncertain cash flows. We first consider the GDP claim. Let $G_{t,t+1}$ denote its one-period growth rate, with $G_0 = 1$. Let ξ and u_t be the dominant eigenvalue and eigenfunction of $G_{t,t+1}$, respectively, satisfying

$$\mathbb{E}_t[G_{t,t+1} u_{t+1}] = \xi u_t.$$

Following Backus, Boyarchenko, and Chernov (2018), we define the transformed pricing kernel $\widehat{M}_{t,t+1} = M_{t,t+1} G_{t,t+1}$, and consider the problem of finding its dominant eigenvalue:

$$\mathbb{E}_t[\widehat{M}_{t,t+1} \widehat{v}_{t+1}] = \mathbb{E}_t[M_{t,t+1} G_{t,t+1} \widehat{v}_{t+1}] = \widehat{\nu}\widehat{v}_t. \tag{A.3}$$

Define the entropy of a positive random variable x as $L_t(x) = \log \mathbb{E}_t(x) - \mathbb{E}_t(\log x)$. The entropy of the SDF is a generalized version of its volatility. Define the coentropy of two positive random variables x_1 and x_2 as the difference between the entropy of their product and the sum of their entropies (Backus, Boyarchenko, and Chernov (2018)): $C_t(x_1, x_2) = L_t(x_1 x_2) - L_t(x_1) - L_t(x_2)$. This measure of dependence is the analog of a covariance. By Hansen (2012), the n-period risk premium is implied from the coentropy. Formally, the risk premium on a claim to this cash flow over holding period n in excess of the return on a risk-free bond with maturity n is given by

$$\begin{aligned}
&\frac{1}{n}\log \mathbb{E}_t[G_{t,t+n}] - \frac{1}{n}\log \mathbb{E}_t[M_{t,t+n}G_{t,t+n}] + \frac{1}{n}\log \mathbb{E}_t[M_{t,t+n}] \\
&\quad = \frac{1}{n}L_t(G_{t,t+n}) - \frac{1}{n}L_t(M_{t,t+n}G_{t,t+n}) + \frac{1}{n}L_t(M_{t,t+n}) \\
&\quad = -\frac{1}{n}C_t(G_{t,t+n}, M_{t,t+n}). \qquad \text{(A.4)}
\end{aligned}$$

Using the definition of coentropy, we obtain the following expression for the long-run risk premium:

$$\lim_{n\to\infty} -\frac{1}{n}C_t(M_{t,t+n}, G_{t,t+n}) = \log \nu + \log \xi - \log \widehat{\nu}.$$

In the case of a stationary cash flow, the martingale component of the cash flow $G^1_{t,t+1} = 1$. Therefore, $\widehat{\nu} = \nu\xi$, $M^1_{t,t+1} = \widehat{M}^1_{t,t+1}$, and the risk premium over long horizons is zero.

Now, consider the tax claim with a cash flow process G^T_t, which satisfies $G^T_0 = c^T$ and whose one-period growth rate solves the eigenvalue problem:

$$\mathbb{E}_t\big[G^T_{t,t+1}u^T_{t+1}\big] = \xi^T u^T_t,$$

and

$$\mathbb{E}_t\big[M_{t,t+1}G^T_{t,t+1}\widehat{v}^T_{t+1}\big] = \widehat{\nu}^T\widehat{v}^T_t.$$

Then the risk premium on the long-run tax strip is implied from the coentropy:

$$\lim_{n\to\infty} -\frac{1}{n}C_t\big(M_{t,t+n}, G^T_{t,t+n}\big) = \log \nu + \log \xi^T - \log \widehat{\nu}^T.$$

As the tax cash flow is cointegrated with the GDP cash flow, the two processes have identical martingale components. Then $\widehat{\nu}^T = \widehat{\nu}$ and $\xi^T = \xi$, implying that the two claims must have the same risk premium in the long run. The same argument applies to the government spending cash flow.

The long-term real bond is the highest-return asset in a model where the SDF only contains a transitory component (Alvarez and Jermann (2005)). Conversely, Alvarez and Jermann (2005) show that when the SDF contains a permanent component, assets whose cash-flows are subject to permanent shocks, such as the GDP, tax, or spending claims, have long-run expected returns that exceed the yield on the long-term real bond: $\log \xi - \log \widehat{\nu} > -\log \nu$. *Q.E.D.*

Proposition 4.

PROOF: Start from the government budget constraint,

$$T_t - G_t = Q^{\$}_{t-1}(1) + \sum_{h=1}^{H-1} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) - \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h),$$

which is the same as (A.1) because $Q^{\$}_{t-1}(H+1) = 0$. We present the proof for the general case with default. Consider the period-$(t+1)$ constraint, multiplied by $M^{\$}_{t+1}(1-\chi_{t+1})$ and by $M^{\$}_{t+1}(\chi_{t+1})$, respectively, and take expectations conditional at time t. The first component is

$$\begin{aligned}
&\mathbb{E}_t\big[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})(1-\chi_{t+1})\big] \\
&\quad = \mathbb{E}_t\Bigg[M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_t(1) + \sum_{h=1}^{H-1} M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_t(h+1)P^{\$}_{t+1}(h) \\
&\qquad - \sum_{h=1}^{H} M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\Bigg] \\
&\quad = Q^{\$}_t(1)P^{\$}_t(1)e^{-\lambda_t(1)} + \sum_{h=1}^{H-1} Q^{\$}_t(h+1)P^{\$}_t(h+1)e^{-\lambda_t(h+1)} \\
&\qquad - \mathbb{E}_t\Bigg[\sum_{h=1}^{H} M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\Bigg].
\end{aligned}$$

The second component is

$$\mathbb{E}_t\big[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})\chi_{t+1}\big] = \mathbb{E}_t\Bigg[-\sum_{h=1}^{H} \chi_{t+1}M^{\$}_{t,t+1}Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\Bigg],$$

assuming no recovery in case of default. Combining the two components, we obtain

$$\begin{aligned}
\mathbb{E}_t\big[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})\big] &= Q^{\$}_t(1)P^1_t e^{-\lambda_t(1)} + \sum_{h=1}^{H-1} Q^{\$}_t(h+1)P^{\$}_t(h+1)e^{-\lambda_t(h+1)} \\
&\quad - \mathbb{E}_t\Bigg[\sum_{h=1}^{H} M^{\$}_{t,t+1}Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\Bigg].
\end{aligned}$$

Combining the period-t and period-$t+1$ constraints, we get

$$\begin{aligned}
&(T_t - G_t) + \mathbb{E}_t\big[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})\big] \\
&\quad = Q^{\$}_{t-1}(1) + \sum_{h=1}^{H-1} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) - \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h)\big(1 - e^{-\lambda^{(h)}_t}\big)
\end{aligned}$$

$$- \mathbb{E}_t\left[\sum_{h=1}^{H} M^{\$}_{t,t+1} Q^{\$}_{t+1}(h) P^{\$}_{t+1}(h)\right].$$

This expression can be written more compactly as

$$\sum_{h=0}^{H-1} Q^{\$}_{t-1}(h+1) P^{\$}_{t}(h) - \mathbb{E}_t\left[M^{\$}_{t,t+1} \sum_{h=1}^{H} Q^{\$}_{t+1}(h) P^{\$}_{t+1}(h)\right]$$
$$= \mathbb{E}_t\left[\sum_{j=0}^{1} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right] + \sum_{h=1}^{H} Q^{\$}_{t}(h) P^{\$}_{t}(h)\left(1 - e^{-\lambda_t^{(h)}}\right).$$

We can iterate this expression to the infinite horizon. If the TVC condition

$$\lim_{k\to\infty} \mathbb{E}_t\left[M^{\$}_{t,t+k} \sum_{h=1}^{H} Q^{\$}_{t+k}(h) P^{\$}_{t+k}(h)\right] = 0,$$

holds, then the debt value is the present value of current and future surpluses and seigniorage revenues from issuing bonds that earn convenience yields

$$\sum_{h=0}^{H-1} Q^{\$}_{t-1}(h+1) P^{\$}_{t}(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right]$$
$$+ \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} \sum_{h=1}^{H} Q^{\$}_{t+j}(h) P^{\$}_{t+j}(h)\left(1 - e^{-\lambda_{t+j}(h)}\right)\right].$$

Using the one-period budget constraint (A.1), we obtain the end-of-period market value of debt D_t:

$$\begin{aligned} D_t &= \sum_{h=1}^{H} Q^{\$}_{t}(h) P^{\$}_{t}(h) \\ &= \mathbb{E}_t\left[\sum_{j=1}^{\infty} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right] \\ &\quad + \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} \sum_{h=1}^{H} Q^{\$}_{t+j}(h) P^{\$}_{t+j}(h)\left(1 - e^{-\lambda_{t+j}(h)}\right)\right]. \end{aligned} \tag{A.5}$$

Q.E.D.

Proof of Convenience Yield Differentiation.

PROOF:

$$\frac{\partial \frac{\overline{PV}^{S}_{t}}{Y_t}(z=0)}{\partial \lambda_0} = \frac{\partial \exp\left(pd^{Y}_{0}\right)}{\partial \lambda_0}(\tau_0 + k_0 - g_0) + \exp\left(pd^{Y}_{0}\right)\frac{\partial k_0}{\partial \lambda_0} + \frac{\partial k_0}{\partial \lambda_0} \tag{A.6}$$

Note

$$pd_0^Y = -\frac{\left(y_0^\$(1) + yspr_0^\$ + rp_0^Y\right) - (x_0 + \pi_0)}{\left(1 - \kappa_1^Y\right)} + \frac{\kappa_0^Y}{\left(1 - \kappa_1^Y\right)}, \tag{A.7}$$

$$\kappa_1^Y = \frac{e^{pd_0^Y}}{e^{pd_0^Y} + 1}, \qquad \kappa_0^Y = \log\left(1 + \exp\left(pd_0^Y\right)\right) - \kappa_1^Y pd_0^Y. \tag{A.8}$$

Substitute in κ_1^Y and κ_0^Y,

$$pd_0^Y = (x_0 + \pi_0) - \left(y_0^\$(1) + yspr_0^\$ + rp_0^Y\right) + \log\left(1 + \exp\left(pd_0^Y\right)\right). \tag{A.9}$$

The true risk-free rate $y_0^\$(1)$ moves one-for-one with λ_0: $\partial y_0^\$(1)/\partial\lambda_0 = 1$. The GDP risk premium is allowed to change with the convenience yield: $\partial rp_0^Y/\partial\lambda_0 \leq 0$. In the stark form of broad convenience yields $\partial rp_0^Y/\partial\lambda_0 = 0$. In the other extreme of narrow convenience yields $\partial rp_0^Y/\partial\lambda_0 = -1$. In the intermediate cases, $0 \leq 1 + \partial rp_0^Y/\partial\lambda_0 \leq 1$.

Then

$$\begin{aligned}\frac{\partial pd_0^Y}{\partial\lambda_0} &= -\left(1 + \frac{\partial rp_0^Y}{\partial\lambda_0}\right) + \frac{\exp\left(pd_0^Y\right)}{1 + \exp\left(pd_0^Y\right)}\frac{\partial pd_0^Y}{\partial\lambda_0} \\ &= -\left(1 + \frac{\partial rp_0^Y}{\partial\lambda_0}\right) + \kappa_1^Y\frac{\partial pd_0^Y}{\partial\lambda_0} = -\frac{\left(1 + \frac{\partial rp_0^Y}{\partial\lambda_0}\right)}{1 - \kappa_1^Y}.\end{aligned} \tag{A.10}$$

Since $k_0 \approx d_0\lambda_0$,

$$\frac{\partial\dfrac{\overline{PV}_t^S}{Y_t}(z=0)}{\partial\lambda_0} = -\exp\left(pd_0^Y\right)\frac{(\tau_0 + k_0 - g_0)}{\left(1 - \kappa_1^Y\right)}\left(1 + \frac{\partial rp_0^Y}{\partial\lambda_0}\right) + \exp\left(pd_0^Y\right)d_0 + d_0 \tag{A.11}$$

so that

$$\frac{\partial\dfrac{\overline{PV}_t^S}{Y_t}(z=0)}{\partial\lambda_0} \geq 0 \text{ iff } \frac{(\tau_0 + k_0 - g_0)}{\left(1 - \kappa_1^Y\right)}\left(1 + \frac{\partial rp_0^Y}{\partial\lambda_0}\right) \leq d_0\left(1 + \exp\left(-pd_0^Y\right)\right). \tag{A.12}$$

Q.E.D.

APPENDIX B: VAR ESTIMATION

B.1. *Cointegration Tests*

We perform a Johansen cointegration test by first estimating the vector error correction model:

$$\Delta w_t = A\left(B' w_{t-1} + c\right) + D\Delta w_{t-1} + \boldsymbol{\varepsilon}_t, \quad \text{where } w_t = \begin{pmatrix} \log T_t \\ \log G_t \\ \log \mathrm{GDP}_t \end{pmatrix}.$$

Both the trace test and the max eigenvalue test do not reject the null of cointegration rank 2 (with p-values of 0.96) or rank 1 (with p-values of 0.11), but reject the null of cointegration rank 0 (with p-values of 0.01). These results are in favor of cointegration relationships between variables in w_t. We also conduct the Phillips–Ouliaris cointegration test on $\{w_t\}$ with a truncation lag parameter of 2, and reject the null hypothesis that w is not cointegrated with a p-value of 0.012.

B.2. *The* $\mathbf{\Psi}$ *Matrix*

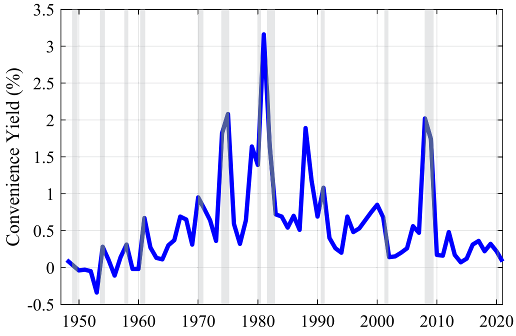
Table B.1 reports the estimated $\mathbf{\Psi}$ matrix.

TABLE B.1

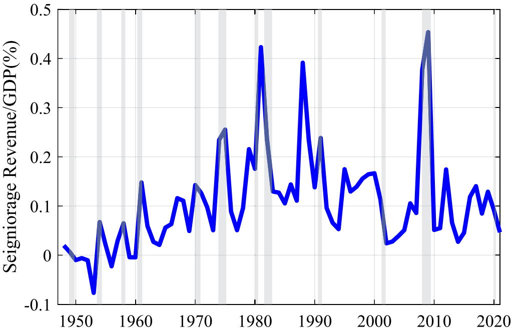
VAR ESTIMATES.

		1	2	3	4	5	6	7	8	9	10	11
						Panel (A) $\mathbf{\Psi}$						
		π_{t-1}	$y^{\$}_{t-1}(1)$	$yspr^{\$}_{t-1}$	x_{t-1}	Δd_{t-1}	d_t	pd_{t-1}	$\Delta\log\tau_{t-1}$	$\log\tau_{t-1}$	$\Delta\log g_{t-1}$	$\log g_{t-1}$
1	π_t	**0.48**	**0.20**	*−0.52*	−0.03	0.02	−0.01	0.00	**0.11**	**−0.07**	−0.01	*0.05*
2	$y^{\$}_t(1)$	0.04	**0.86**	−0.06	0.16	*0.07*	−0.01	0.01	−0.03	−0.01	0.01	**0.07**
3	$yspr^{\$}_t$	−0.06	−0.04	**0.40**	**−0.12**	**−0.03**	−0.01	**−0.01**	0.02	0.02	0.00	−0.02
4	x_t	−0.19	**0.38**	*0.97*	0.21	0.09	0.03	*0.02*	−0.07	−0.09	−0.02	0.07
5	$\Delta\log d_t$	−0.13	**−0.80**	*−2.35*	0.35	**0.28**	**−0.12**	−0.00	−0.31	−0.21	−0.18	0.19
6	$\log d_t$	−0.13	**−0.80**	*−2.35*	0.35	**0.28**	**0.88**	−0.00	−0.31	−0.21	−0.18	0.19
7	pd^M_t	*−2.66*	−0.43	−0.31	−1.35	−0.35	−0.11	**0.68**	0.06	0.40	0.34	−0.54
8	$\Delta\log\tau_t$	**−0.71**	**0.76**	−0.97	0.02	0.12	−0.03	*0.04*	**0.35**	**−0.61**	0.08	0.11
9	$\log\tau_t$	**−0.71**	**0.76**	−0.97	0.02	0.12	−0.03	*0.04*	**0.35**	**0.39**	0.08	0.11
10	$\Delta\log g_t$	**1.05**	−0.25	0.15	−0.18	**−0.31**	0.05	−0.04	*0.36*	−0.19	**0.38**	**−0.60**
11	$\log g_t$	**1.05**	−0.25	0.15	−0.18	**−0.31**	0.05	−0.04	*0.36*	−0.19	**0.38**	**0.40**
						Panel (B) $100\times\mathbf{\Sigma}^{\frac{1}{2}}$						
		$\boldsymbol{\varepsilon}^{\pi}_t$	$\boldsymbol{\varepsilon}^{Y}_t(1)$	$\boldsymbol{\varepsilon}^{Yspr}_t$	$\boldsymbol{\varepsilon}^{x}_t$	$\boldsymbol{\varepsilon}^{\Delta d}_t$		$\boldsymbol{\varepsilon}^{pd}_t$	$\boldsymbol{\varepsilon}^{\Delta\log\tau}_t$		$\boldsymbol{\varepsilon}^{\Delta\log g_t}$	
1	π_t	1.06	0	0	0	0	0	0	0	0	0	0
2	$y^{\$}_t(1)$	0.34	1.21	0	0	0	0	0	0	0	0	0
3	$yspr^{\$}_t$	−0.06	−0.32	0.41	0	0	0	0	0	0	0	0
4	x_t	0.18	0.80	−0.14	1.83	0	0	0	0	0	0	0
5	$\Delta\log d_t$	−1.54	0.36	−0.74	−0.65	4.54	0	0	0	0	0	0
6	$\log d_t$	−1.54	0.36	−0.74	−0.65	4.54	0	0	0	0	0	0
7	pd^M_t	−2.49	0.29	0.23	−2.70	−3.94	0	14.14	0	0	0	0
8	$\Delta\log\tau_t$	0.34	0.77	−0.19	1.55	0.65	0	0.35	2.92	0	0	0
9	$\log\tau_t$	0.34	0.77	−0.19	1.55	0.65	0	0.35	2.92	0	0	0
10	$\Delta\log g_t$	0.30	−1.35	0.17	−2.95	−1.22	0	0.18	0.56	0	4.13	0
11	$\log g_t$	0.30	−1.35	0.17	−2.95	−1.22	0	0.18	0.56	0	4.13	0

Note: Panel (a) reports our estimate of the VAR transition matrix $\mathbf{\Psi}$. Numbers in bold have t-statistics in excess of 1.96 in absolute value. Numbers in italics have t-statistics in excess of 1.645 but below 1.96. Panel (b) reports our estimate of the VAR innovation matrix $\mathbf{\Sigma}^{\frac{1}{2}}$, multiplied by 100 for readability.



(a) Convenience Yield



(b) Seigniorage Revenue/GDP

FIGURE B.1.—Time Series of Convenience Yield and Seigniorage Revenue. The figure plots the convenience yield cy_t and the seigniorage revenue/GDP ratio. The sample is annual, 1947–2020.

B.3. *Convenience Yield and Seigniorage Revenue*

Figure B.1 plots the time series of the convenience yield cy_t in the left-hand side panel and the time series of the seigniorage revenue-to-GDP k_t in the right-hand side panel. Both objects are defined in the main text.

APPENDIX C: DERIVATION OF THE UPPER BOUND

This section derives the Campbell and Shiller (1988) decomposition for the spending and the tax claim, which is used in the derivation of the upper bound.

C.1. *Campbell–Shiller Decomposition of Tax and Spending Claims*

Consider the return on a claim to the government's tax revenue:

$$r_{t+1}^T = \log \frac{P_{t+1}^T + T_{t+1}}{P_t^T} = \log \frac{T_{t+1}}{T_t} \frac{\left(1 + \exp\left(pd_{t+1}^T\right)\right)}{\exp\left(pd_t^T\right)}.$$

We use pd_t^T to denote the log price-dividend ratio on the tax revenue claim: $pd_t^T = \log P_t^T - \log T_t$, where price is measured at the end of the period and the dividend flow is over the same period. Campbell and Shiller (1988) log-linearize the return equation around the mean log price/dividend ratio to derive the following expression for log returns on the tax claim:

$$r_{t+1}^T = \Delta \log T_{t+1} + \kappa_1^T pd_{t+1}^T + \kappa_0^T - pd_t^T,$$

with linearization coefficients as functions of the mean of the log price/dividend ratio pd_0^T:

$$\kappa_1^T = \frac{e^{pd_0^T}}{e^{pd_0^T} + 1} < 1, \qquad \kappa_0^T = \log\left(1 + \exp\left(pd_0^T\right)\right) - \kappa_1^T pd_0^T.$$

By iterating forward on the linearized return equation, imposing a no-bubble condition: $\lim_{j\to\infty} (\kappa_1^T)^j pd_{t+j}^T = 0$, and taking expectations, we derive the following expression for the log price/dividend ratio of the tax claim:

$$pd_t^T = \frac{\kappa_0^T}{1-\kappa_1^T} + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^T)^{j-1}\Delta \log T_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^T)^{j-1} r_{t+j}^T\right].$$

We use rp_t^T to denote the risk premium on tax claims relative to the long bond:

$$\mathbb{E}_t[r_{t+1}^T] = yspr_t^\$ + y_t^\$(1) + rp_t^T.$$

We assume constant risk premia on the tax and spending claims, which we denote as rp_0^T and rp_0^G. We use $\boldsymbol{e}_{y1}$ and $\boldsymbol{e}_{yspr}$ to denote the column vectors that select the short rate and the yield spread. Because the state vector follows VAR(1) dynamics, we can compute the expected return as follows:

$$\mathbb{E}_t[r_{t+j}^T] = y_0^\$(1) + yspr_0^\$ + rp_0^T + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}^{j-1} z_t \tag{C.1}$$

The DR (discount rate) term is given by the following expression:

$$DR_t^T \overset{\text{def}}{=} \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^T)^{j-1} r_{t+j}^T\right] = \frac{y_0^\$(1) + yspr_0^\$ + rp_0^T}{1-\kappa_1^T} + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'(I - \kappa_1^T\boldsymbol{\Psi})^{-1} z_t.$$

The CF (cash flow) term is given by the following expression:

$$CF_t^T \overset{\text{def}}{=} \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^T)^{j-1}\Delta \log T_{t+j}\right] = \frac{x_0 + \pi_0}{1-\kappa_1^T} + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta\tau})'\boldsymbol{\Psi}(I - \kappa_1^T\boldsymbol{\Psi})^{-1} z_t,$$

We end up with the following expressions for the price/dividend ratio on the tax and spending claims:

$$pd_t^T = pd_0^T + \left[(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta\tau})'\boldsymbol{\Psi} - (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\right](I - \kappa_1^T\boldsymbol{\Psi})^{-1} z_t, \tag{C.2}$$

where $\boldsymbol{e}_{\Delta\tau}$ selects the tax-to-GDP growth rate in the state vector, and $(pd_0^T, \kappa_0^T, \kappa_1^T)$ solve

$$\begin{aligned} pd_0^T &= \frac{x_0 + \pi_0 - y_0^\$(1) - yspr_0^\$ - rp_0^T}{(1-\kappa_1^T)} + \frac{\kappa_0^T}{(1-\kappa_1^T)}, \\ \kappa_1^T &= \frac{e^{pd_0^T}}{e^{pd_0^T}+1}, \qquad \kappa_0^T = \log(1 + \exp(pd_0^T)) - \kappa_1^T pd_0^T. \end{aligned} \tag{C.3}$$

Similarly, the log price/dividend ratio of the spending claim:

$$pd_t^G = \frac{\kappa_0^G}{1-\kappa_1^G} + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^G)^{j-1}\Delta \log G_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^G)^{j-1} r_{t+j}^G\right].$$

We can derive a similar expression for the spending claim.

We use $\widetilde{CF}_t^i$ and $\widetilde{DR}_t^i$ to denote the mean-zero time-varying components of the cash flow and discount rate terms. The implied present value of surpluses/GDP ratio is given by

$$\frac{PV_t^S}{Y_t} = \tau_t \exp\left(pd_0^T + \widetilde{CF}_t^T - \widetilde{DR}_t^T\right) - g_t \exp\left(pd_0^G + \widetilde{CF}_t^G - \widetilde{DR}_t^G\right).$$

To derive some intuition, we can evaluate the expression at $z = 0$, that is, when all variables are at their unconditional mean. In this case, the present value of surpluses/GDP ratio is given by

$$\frac{PV_t^S}{Y_t}(z = 0) = \tau_0 \exp\left(pd_0^T\right) - g_0 \exp\left(pd_0^G\right).$$

Upper Bound on Debt Valuation. To derive an upper bound, we equate the expected returns on taxes and spending to the expected return on GDP: $rp_0^Y = rp_0^G = rp_0^T$. This delivers an upper bound on the valuation of future surpluses, because it maximizes the value of the tax claim, and minimizes the value of the spending claim. Given these two assumptions, we derive the following expression for the implied log price/dividend ratio on the tax claim and the spending claim:

$$\begin{aligned} pd_t^T &= pd_0^Y + \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\kappa_1^Y\right)^{j-1}\left(\Delta \log T_{t+j} - (x_0 + \pi_0)\right)\right] \\ &\quad - \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\kappa_1^Y\right)^{j-1}\left(r_{t+j}^Y - \left(yspr_0^\$ + y_0^\$(1) + rp_0^Y\right)\right)\right], \\ pd_t^G &= pd_0^Y + \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\kappa_1^Y\right)^{j-1}\left(\Delta \log G_{t+j} - (x_0 + \pi_0)\right)\right] \\ &\quad - \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\kappa_1^Y\right)^{j-1}\left(r_{t+j}^Y - \left(yspr_0^\$ + y_0^\$(1) + rp_0^Y\right)\right)\right]. \end{aligned} \tag{C.4}$$

The long-run growth rate of tax and spending equals the long-run growth rate of output: $x_0 + \pi_0$. That follows directly from co-integration. We use a constant GDP risk premium rp_0^Y. We can back this number out of the unconditional equity risk premium by unlevering the equity premium. We use $\boldsymbol{e}_\pi$ to denote a column vector of zero with a 1 as the first element. The DR (discount rate) term is defined by

$$DR_t^T = DR_t^G = DR_t^Y = \frac{y_0^\$(1) + yspr_0^\$ + rp_0^Y}{1 - \kappa_1^Y} + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\left(I - \kappa_1^Y \boldsymbol{\Psi}\right)^{-1} \boldsymbol{z}_t.$$

The CF (cash flow) term for the tax claim is defined by

$$CF_t^T = \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\kappa_1^Y\right)^{j-1} \Delta \log T_{t+j}\right] = \frac{x_0 + \pi_0}{1 - \kappa_1^Y} + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta\tau})' \boldsymbol{\Psi}\left(I - \kappa_1^Y \boldsymbol{\Psi}\right)^{-1} \boldsymbol{z}_t.$$

The CF (cash flow) term for the spending claim is defined by

$$CF_t^G = \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^Y)^{j-1}\Delta\log G_{t+j}\right] = \frac{x_0+\pi_0}{1-\kappa_1^Y} + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta g})'\boldsymbol{\Psi}(I-\kappa_1^Y\boldsymbol{\Psi})^{-1}z_t.$$

We use $\widetilde{CF}_t^i$ and $\widetilde{DR}_t^i$ to denote the time-varying components. Hence, we end up with the following expressions for the price/dividend ratio on the tax and spending claims:

$$\begin{aligned} pd_t^T &= pd_0^Y + \big[(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta\tau})'\boldsymbol{\Psi} - (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\big](I-\kappa_1^Y\boldsymbol{\Psi})^{-1}z_t, \\ pd_t^G &= pd_0^Y + \big[(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta g})'\boldsymbol{\Psi} - (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\big](I-\kappa_1^Y\boldsymbol{\Psi})^{-1}z_t. \end{aligned} \tag{C.5}$$

A first-order Taylor expansion yields the following expression:

$$\frac{\overline{PV}_t^S}{Y_t} \approx (\tau_t - g_t)\exp(pd_0^Y) + \tau_t(\widetilde{CF}_t^T - \widetilde{DR}_t^T)\exp(pd_0^Y) - g_t\exp(pd_0^Y)(\widetilde{CF}_t^G - \widetilde{DR}_t^G).$$

This expression can be simplified. We obtain the following intuitive expression for an upper bound on the PDV of surpluses:

$$\frac{\overline{PV}_t^S}{Y_t} \approx \exp(pd_0^Y)\big((\tau_t - g_t)(1-\widetilde{DR}_t^Y) + \tau_t\widetilde{CF}_t^T - g_t\widetilde{CF}_t^G\big).$$

Suppose the country currently runs a primary surplus of zero. The discount rate effects cancel out, again to a first-order approximation. When the country runs a zero primary surplus, the upper bound on the value of debt/GDP is positive only if the expected tax revenue growth exceeds expected spending growth:

$$\frac{\overline{PV}_t^S}{Y_t} \approx \exp(pd_0^Y)\tau_t(\widetilde{CF}_t^T - \widetilde{CF}_t^G) > 0 \quad \textit{iff}\ \widetilde{CF}_t^T > \widetilde{CF}_t^G.$$

This can be further simplified to yield the following expression:

$$\frac{\overline{PV}_t^S}{Y_t} \approx \exp(pd_0^Y)\tau_t(\boldsymbol{e}_{\Delta\tau} - \boldsymbol{e}_{\Delta g})'\boldsymbol{\Psi}(I-\kappa_1^Y\boldsymbol{\Psi})^{-1}z_t.$$

The discount rate dynamics and the dynamics of GDP growth are irrelevant (to a first-order approximation) for the upper bound. What matters is the dynamics in tax/GDP and spending/GDP. In other words, the expected cumulative effect of mean reversion in taxes has to outweigh the expected cumulative effect of mean-reversion in spending.

Upper Bound With Convenience Yields. We can write the intertemporal budget constraint with convenience yield seigniorage revenue in (A.5) as

$$D_t = \mathbb{E}_t\left[\sum_{j=1}^{\infty}M_{t,t+j}^{\$}(T_{t+j}+K_{t+j})\right] + K_t - \mathbb{E}_t\left[\sum_{j=1}^{\infty}M_{t,t+j}^{\$}G_{t+j}\right].$$

where

$$K_{t+j} \stackrel{\text{def}}{=} \sum_{h=1}^{H} Q_{t+j}^{\$}(h) P_{t+j}^{\$}(h)\left(1 - e^{-\lambda_{t+j}(h)}\right), \quad \forall j \geq 0.$$

We use the variable TK to represent the combined tax and seigniorage revenues as a fraction of the current tax revenue.

Next, we include the seigniorage revenue from the convenience yields in the government revenue. The observed nominal Treasury yield $y_t^{\$}(1)$ is given by the following expression:

$$y_t^{\$}(1) = \rho_t^{\$}(1) - \lambda_t,$$

where $\rho_t^{\$}(1)$ is the one-period nominal risk-free rate. We include the one-period nominal risk-free rate in the VAR:

$$\boldsymbol{e}_{y1}' \boldsymbol{z}_t \stackrel{\text{def}}{=} \rho_t^{\$}(1) = y_t^{\$}(1) + \lambda_t.$$

The DR (discount rate) term is defined by

$$DR_t^i = \frac{y_0^{\$}(1) + \lambda_0 + yspr_0^{\$} + rp_0^i}{1 - \kappa_1^i} + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\left(I - \kappa_1^i \boldsymbol{\Psi}\right)^{-1} \boldsymbol{z}_t,$$

where $i \in \{TK, G\}$. The implied upper bound is given by

$$\frac{\overline{PV}_t^S}{Y_t} = k_t + (\tau_t + k_t)\exp\left(pd_0^T + \widetilde{CF}_t^{TK} - \widetilde{DR}_t^{TK}\right) - g_t \exp\left(pd_0^G + \widetilde{CF}_t^G - \widetilde{DR}_t^G\right).$$

Alternatively, we can evaluate the expression at $z = 0$, that is, when all variables are at their unconditional mean:

$$\frac{\overline{PV}_t^S}{Y_t}(z = 0) = k_0 + (\tau_0 + k_0)\exp\left(pd_0^T\right) - g_0 \exp\left(pd_0^G\right),$$

where pd_0^i, κ_0^i, κ_1^i solve

$$pd_0^i = -\frac{\left(y_0^{\$}(1) + \lambda_0 + yspr_0^{\$} + rp_0^i\right) - (x_0 + \pi_0)}{\left(1 - \kappa_1^i\right)} + \frac{\kappa_0^i}{\left(1 - \kappa_1^i\right)},$$

$$\kappa_1^i = \frac{e^{pd_0^i}}{e^{pd_0^i} + 1}, \qquad \kappa_0^i = \log\left(1 + \exp\left(pd_0^i\right)\right) - \kappa_1^i pd_0^i.$$

Reverse-Engineering. Finally, we could also reverse-engineer the discount rate for the tax claim that would imply the value of the debt/output ratio equal to the debt/output ratio. Suppose we fix the risk premium on the spending claim. Then we can determine rp_t^T such that it solves the following equation:

$$\frac{D_t}{Y_t} = \tau_t \exp\left(pd_0^T - \frac{rp_t^T}{1 - \kappa_1^T} + CF_t^T - DR_t^T\right) - g_t \exp\left(pd_0^Y - \frac{rp_t^Y}{1 - \kappa_1^Y} + CF_t^G - DR_t^G\right).$$

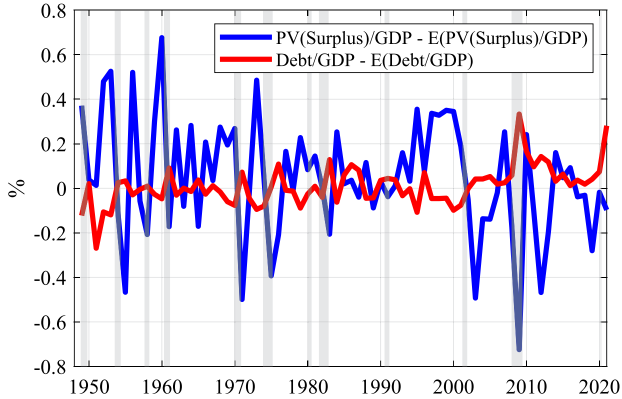


FIGURE C.1.—Cyclicality of the Innovation in the PDV of Surpluses and Debt. The figure plots the innovation in $\overline{PV}^S$ to GDP ratio, and the innovation in debt to GDP ratio. The sample period is from 1947 to 2020.

C.2. *Additional Upper Bound Results*

This section reports additional results for the upper bound calculation.

C.2.1. *Cyclicality of the Upper Bound*

To study the cyclical behavior of the PDV of surpluses, we calculate the innovation in the expected present discounted value of the surplus, scaled by GDP:

$$\frac{\overline{PV}^S_{t+1}}{Y_{t+1}} - \mathbb{E}_t\left[\frac{\overline{PV}^S_{t+1}}{Y_{t+1}}\right] = \left(\frac{P^T_{t+1}}{Y_{t+1}} - \mathbb{E}_t\left[\frac{P^T_{t+1}}{Y_{t+1}}\right]\right) - \left(\frac{P^G_{t+1}}{Y_{t+1}} - \mathbb{E}_t\left[\frac{P^G_{t+1}}{Y_{t+1}}\right]\right).$$

The above innovation can be constructed from our estimated VAR model (equation (13) in the paper). We compare it to that of the market value of debt:

$$\frac{D_{t+1}}{Y_{t+1}} - \mathbb{E}_t\left[\frac{D_{t+1}}{Y_{t+1}}\right].$$

The latter can be constructed by estimating an equation for debt/GDP, which depends on its own lag and the lagged state variables in the VAR.

Figure C.1 shows the results. NBER recessions periods are shaded in gray. The blue line shows that, in recessions, fiscal fundamentals deteriorate and result in a downward shock to our measure of fiscal backing, even relative to a falling GDP. At the same time, the market value of debt increases (red line). The wedge between the two increases in recessions.

C.2.2. *Varying the Level of the GDP Risk Premium*

Figure C.2 plots the upper bound when the GDP risk premium is set to 2.5%. Lowering the risk premium mainly increases the confidence intervals, because occasionally the valuation ratios become much larger. As a result, the upper bound is mostly just outside the 95% confidence interval, until the GFC. Starting with the GFC, the debt/output ratio is comfortably outside of the 95% confidence interval.

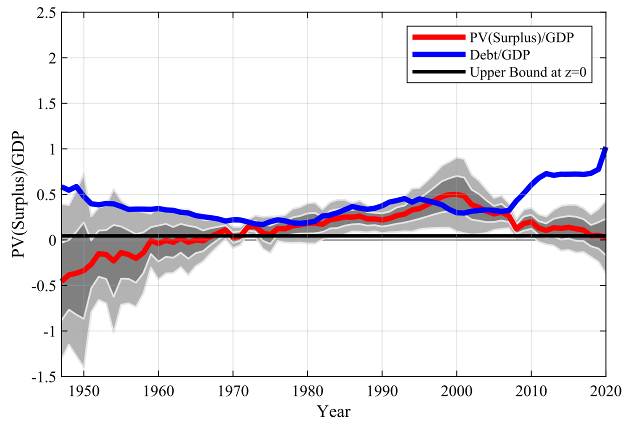


FIGURE C.2.—Upper Bound on the Value of Surpluses/GDP. The figure plots the upper bound on the present value of government surpluses in equation (14), the steady-state upper bound evaluated at $z = 0$ in equation (15), and the actual debt/output ratio. We report the benchmark case with a GDP risk premium of 2.5%. The sample period is from 1947 to 2020.

Figure C.3 plots the upper bound on the PDV of surpluses relative to GDP for values of the GDP risk premium ranging from 1.5% to 6% per year. As in the benchmark upper bound exercise, we set the risk premia on tax and spending claims equal to that on the GDP claim for this calculation. As we lower the GDP risk premium, the present value of government surpluses increases. Government surpluses are a risky cash flow stream whose present value increases as we lower the risk premium. When the risk premium is very low at 1.5% per year, the average upper bound is about 70% of GDP, which is higher than the average debt/GDP ratio in the post-war period. While this result may lead one to think that a low risk premium resolves the puzzle, the right panel of Figure C.3 shows that this low risk premium setting generates a very high price/dividend ratio for the GDP claim. Since the price of the GDP claim is the value of total wealth in the economy, this

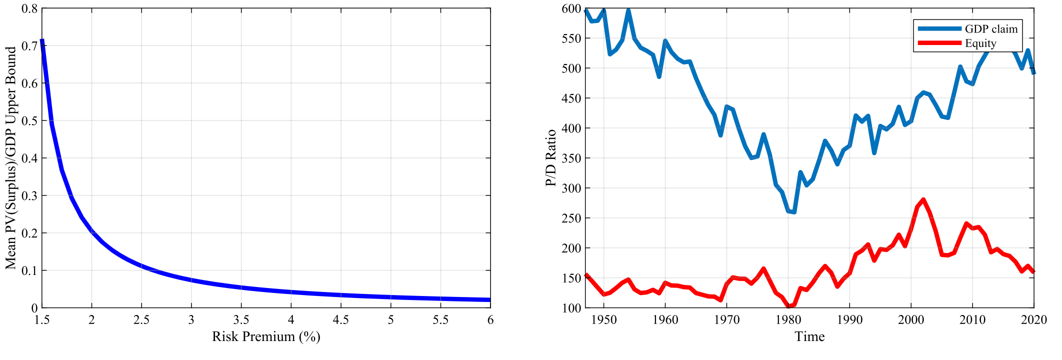


FIGURE C.3.—Upper Bound Fiscal Backing as we Vary GDP Risk Premium. The left panel plots the average upper-bound estimate of the present value of surpluses as a function of the GDP risk premium rp_0^Y. The right panel plots the price-dividend ratio of the GDP claim when the risk premium is 1.5% (blue line). It also plots the price-dividend ratio on equity (red line), using an equity risk premium equal to the assumed GDP risk premium scaled by the inverse of the leverage ratio.

low risk premium implies an implausible amount of wealth of about 450 times GDP.[22] It also generates a very high stock price/dividend ratio, plotted in red. The price/dividend ratio of 150 is more than 4 times higher than the observed price/dividend ratio in the data, which averages 35 for the post-war period.

C.2.3. *Time-Varying GDP Risk Premium*

We use $\widetilde{\widetilde{DR}}_t^Y$ to denote the discount rate component on the GDP claim with time-varying GDP risk premia. We use $\widetilde{DR}_t^Y$ to denote the discount rate component obtained with constant GDP risk premia, the measure computed in the main text. We seek to characterize the gap between the time-varying measure of the upper bound and the measure computed using the constant risk premium. Below is an ex-expression for the upper bound with time-varying GDP risk premia minus the upper bound with constant GDP risk premia:

$$\begin{aligned}
&\mathbb{E}\big[\big(\tau_t \exp\big(pd_0^Y + \widetilde{CF}_t^T\big) - g_t \exp\big(pd_0^Y + \widetilde{CF}_t^G\big)\big)\big(\exp\big(-\widetilde{\widetilde{DR}}_t^Y\big) - \exp\big(-\widetilde{DR}_t^Y\big)\big)\big] \\
&\quad = \mathrm{Cov}\big(\tau_t \exp\big(pd_0^Y + \widetilde{CF}_t^T\big) - g_t \exp\big(pd_0^Y + \widetilde{CF}_t^G\big), \exp\big(-\widetilde{\widetilde{DR}}_t^Y\big) - \exp\big(-\widetilde{DR}_t^Y\big)\big),
\end{aligned}$$

where the second equality follows if $\mathbb{E}(\exp(\widetilde{\widetilde{DR}}_t^Y)) = \mathbb{E}(\exp(\widetilde{DR}_t^Y))$. The upper-bound measure with constant GDP risk premium equals the upper-bound measure with time-varying GDP risk premium only if the GDP risk premium and the primary surplus are orthogonal. Given the countercyclical nature of the output risk premium and the procyclical nature of the surplus process, we expect this covariance term to be positive, because higher than average valuation ratios would coincide with higher than average surpluses. This implies that our baseline measure understates the true upper bound. We show now that this understatement is quantitatively small.

Using the log-linearized expression for the returns, the log of the valuation ratio for the tax claim is given by the following expression:

$$pd_t^T = \frac{\kappa_0^T}{1-\kappa_1^T} + \mathbb{E}_t\left[\sum_{j=1}^{\infty} \big(\kappa_1^T\big)^{j-1} \Delta \log T_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty} \big(\kappa_1^T\big)^{j-1} r_{t+j}^T\right].$$

When the (GDP) risk premia are constant, we obtain the expression using the VAR in equations (C.2) and (C.3). The expression for the spending claim is analogous. To compute the upper bound, we set $rp_0^T = rp_0^G = rp_0^Y$.

Next, we allow for time-varying GDP risk premia. In each period t, we infer the risk premium rp_t^Y from the equity P/D ratio and the leverage ratio, assuming the term structure of GDP risk premium is flat. Recall that the log of the equity P/D ratio can be stated

[22]Since the price of the GDP claim is the value of total wealth in the economy, this low risk premium implies an implausible amount of wealth of about 450 times GDP. Most of this is human wealth. In a similar no-arbitrage model, Lustig, Van Nieuwerburgh, and Verdelhan (2013) estimate that only 8% of total wealth is financial wealth. Using that conservative number produces per capita financial wealth of around \$916 trillion or around \$2.7 million per household. According to the U.S. Flow of Funds, U.S. household wealth excluding human wealth is only \$167 trillion (including real estate) in 2022 Q.4. (Table B.101 in Flow of Funds.) This low risk premium and high multiple produces 5.5 times too much nonhuman wealth. We have now effectively replaced the bond valuation puzzle with another puzzle: the missing wealth puzzle. The GDP multiple of 450 is much higher than the multiple of 85 estimated by Lustig, Van Nieuwerburgh, and Verdelhan (2013) for the total wealth/consumption ratio, a closely related concept.

as

$$
\begin{aligned}
& pd_t^M - pd_0^M \\
& \quad = \left[(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta d})' \boldsymbol{\Psi} - (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\right]\left(I - \kappa_1^M \boldsymbol{\Psi}\right)^{-1} z_t - \sum_{j=1}^{\infty} \left(\kappa_1^M\right)^{j-1} \left(rp_t^M - rp_0^M\right) \\
& \quad = \left[(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta d})' \boldsymbol{\Psi} - (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\right]\left(I - \kappa_1^M \boldsymbol{\Psi}\right)^{-1} z_t - \frac{1}{1 - \kappa_1^M}\left(rp_t^M - rp_0^M\right),
\end{aligned}
$$

where rp_0^M is given by the following expression, obtained by rearranging the expression for pd_0^M:

$$
rp_0^M = x_0 + \pi_0 - y_0^\$(1) - yspr_0^\$ - \left(pd_0^M\left(1 - \kappa_1^M\right) - \kappa_0^M\right).
$$

We can back out the following expression for the risk premium on the stock market:

$$
\left(rp_t^M - rp_0^M\right) = \left(1 - \kappa_1^M\right)\left\{\left[(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta d})' \boldsymbol{\Psi} - (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\right]\left(I - \kappa_1^M \boldsymbol{\Psi}\right)^{-1} z_t - \boldsymbol{e}_{pd}' z_t\right\}.
$$

With leverage ratio ℓ_t =firm debt/firm asset, we obtain the GDP risk premium as $rp_t^Y = rp_t^M(1 - \ell_t)$. We obtain a new formula for the log valuation ratios:

$$
\begin{aligned}
pd_t^T = {} & pd_0^T + \left[(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta\tau})' \boldsymbol{\Psi} - (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\right]\left(I - \kappa_1^T \boldsymbol{\Psi}\right)^{-1} z_t \\
& - \frac{1}{1 - \kappa_1^Y}\left(rp_t^Y - rp_0^Y\right). \qquad \text{(C.6)}
\end{aligned}
$$

We assume that the average unlevered equity premium is the GDP risk premium: $rp_0^Y = rp_0^M(1 - \ell_0)$.

The true upper bound on the present value of government surplus is given by

$$
\frac{\overline{PV}_t^S}{Y_t} = \tau_t \exp\left(pd_0^Y + \widetilde{CF}_t^T - \widetilde{\widetilde{DR}}_t^Y\right) - g_t \exp\left(pd_0^Y + \widetilde{CF}_t^G - \widetilde{\widetilde{DR}}_t^Y\right), \qquad \text{(C.7)}
$$

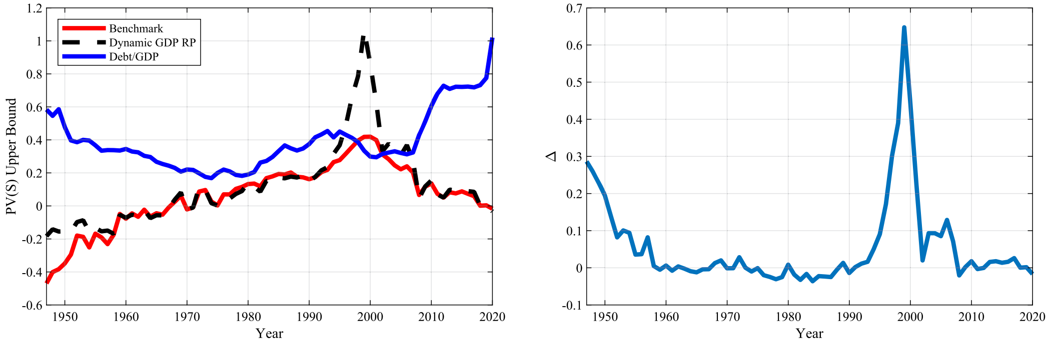
where the unconditional valuation ratios and the linearization constants $pd_0^Y = pd_0^T = pd_0^G$, $\kappa_0^Y = \kappa_0^T = \kappa_0^G$, and $\kappa_1^Y = \kappa_1^T = \kappa_1^G$ solve the system of equations in (C.3), where the discount rate component with time-varying GDP risk premia is given by

$$
\widetilde{\widetilde{DR}}_t^Y = (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\left(I - \kappa_1^Y \boldsymbol{\Psi}\right)^{-1} z_t + \frac{1}{1 - \kappa_1^Y}\left(rp_t^Y - rp_0^Y\right).
$$

The measure in the paper uses the constant GDP risk premium when to evaluate this discount rate component:

$$
\widetilde{DR}_t^Y = (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\left(I - \kappa_1^Y \boldsymbol{\Psi}\right)^{-1} z_t.
$$

The left panel in Figure C.4 plots the upper bound with time-varying risk premia against the upper bound with the constant GDP risk premium. With the exception of the late 90s, the two measures are quite close. However, in the late 90s, the upper bound with the time-varying GDP risk premium spikes.



(a) Upper Bound/GDP (b) Gap/GDP

FIGURE C.4.—Upper Bound. The left panel of the figure plots the upper bound with the time-varying risk premium in equation (C.7) and the constant GDP risk premium. The right panel of the figure plots the Δ in equation (C.8).

We can also derive the following expression for the gap between the two upper-bound measures:

$$\Delta_t = \left(\tau_t \exp\left(pd_0^Y + \widetilde{CF}_t^T\right) - g_t \exp\left(pd_0^Y + \widetilde{CF}_t^G\right)\right)\left(\exp\left(-\widetilde{\widetilde{DR}}_t^Y\right) - \exp\left(-\widetilde{DR}_t^Y\right)\right). \quad \text{(C.8)}$$

As a result, we obtain the following expression for the gap between the two measures:

$$\Delta_t = \left(\tau_t \exp\left(pd_0^Y + \widetilde{CF}_t^T\right) - g_t \exp\left(pd_0^Y + \widetilde{CF}_t^G\right)\right) \exp\left(-\widetilde{DR}_t^Y\right)$$
$$\times \left[\exp\left(-\frac{1}{1-\kappa_1^Y}\left(rp_t^Y - rp_0^Y\right)\right) - 1\right],$$

where $rp_t^Y = rp_t^M(1-\ell_t)$, and $(rp_t^M - rp_0^M) = (1-\kappa_1^M)\{[\mathbf{\Psi}'(\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{\Delta d}) - \boldsymbol{e}_{y1} - \boldsymbol{e}_{yspr}]'(I - \kappa_1^M \mathbf{\Psi})^{-1} z_t - \boldsymbol{e}_{pd}' z_t\}$.

The right panel in Figure C.4 plots the gap Δ_t. The average gap is 5.67% of GDP. While introducing time-varying risk premia does increase the upper bound, it does not materially change our quantitative conclusions.

C.2.4. *Including the Debt/Output Ratio in the State Vector*

Table C.1 lists all the state variables in the VAR for the case with the debt/output ratio included in the state vector. Figure C.5 plots upper bound for the case with the debt/output ratio in the VAR. In the left (right) panel, we plot the case with $rp_0^Y = 3\%$ ($rp_0^Y = 2.5\%$).

C.2.5. *Longer Sample*

As the average growth rates of tax and spending were very different before and after 1947, we demean the tax and spending growth rates by the subsample means in 1930–1946 and in 1947–2020 before we estimate the VAR.

Figure C.6 plots the upper bound constructed in the longer sample for the case of $rp_0^Y = 3\%$ and the case of $rp_0^Y = 2.5\%$. Lowering the risk premium does not have a first-order effect on the upper bound itself, but it widens the size of the confidence interval.

TABLE C.1

STATE VARIABLES.

Position	Variable	Mean	Description
1	π_t	π_0	Log Inflation
2	$y_t^\$(1)$	$y_0^\$(1)$	Log 1-Year Nominal Yield
3	$yspr_t^\$$	$yspr_0^\$$	Log 5-Year Minus Log 1-Year Nominal Yield Spread
4	x_t	x_0	Log Real GDP Growth
5	$\Delta \log d_t$	μ_d	Log Stock Dividend-to-GDP Growth
6	$\log d_t$	$\log d_0$	Log Stock Dividend-to-GDP Level
7	pd_t^M	pd_0^M	Log Stock Price-to-Dividend Ratio
8	$\Delta \log \tau_t$	μ_τ	Log Tax Revenue-to-GDP Growth
9	$\log \tau_t$	$\log \tau_0$	Log Tax Revenue-to-GDP Level
10	$\Delta \log g_t$	μ_g	Log Spending-to-GDP Growth
11	$\log g_t$	$\log g_0$	Log Spending-to-GDP Level
12	$\Delta \log debt_t$	μ_{debt}	Log Debt-to-GDP Growth
13	$\log debt_t$	$\log debt_0$	Log Debt-to-GDP Level

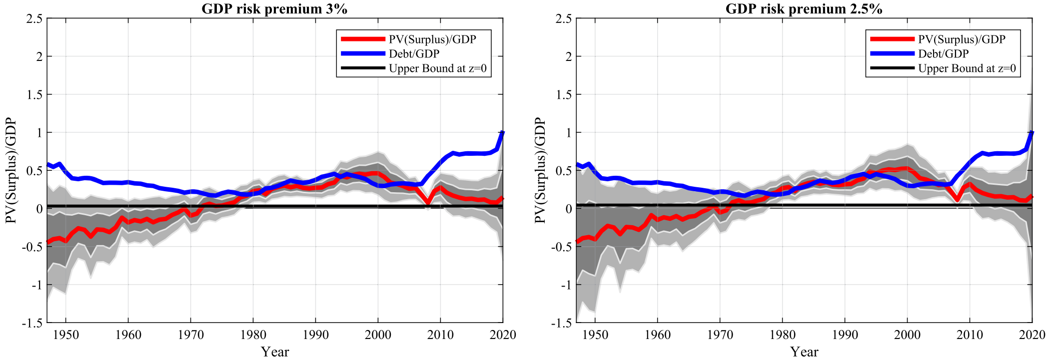


FIGURE C.5.—Upper Bound on the Value of Surpluses/GDP with Debt in the VAR. The figure plots the upper bound on the present value of government surpluses in equation (14), the steady-state upper bound evaluated at $z = 0$ in equation (15), and the actual debt/output ratio. We report the benchmark case with a GDP risk premium of 3% and 2.5%. The sample period is from 1947 to 2020.

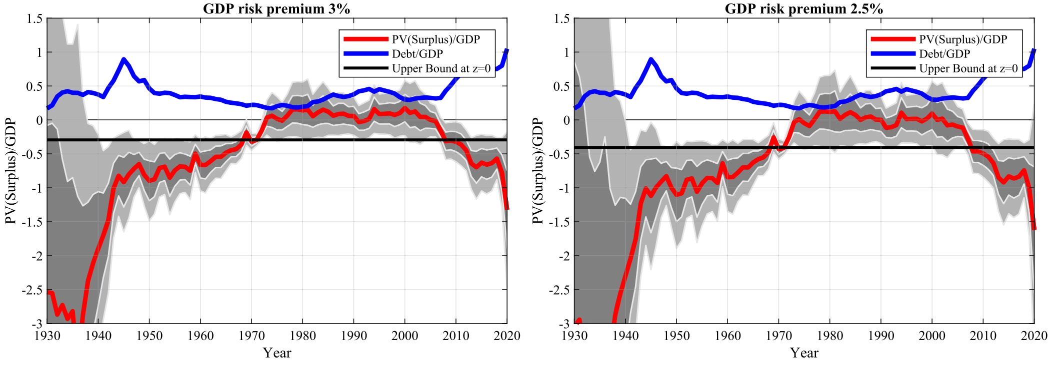


FIGURE C.6.—Upper Bound on the Value of Surpluses/GDP: Longer Sample. The figure plots the upper bound on PDV of surpluses/GDP in equation (14), the steady-state upper bound evaluated at $z = 0$ in equation (15) and the actual debt/output ratio. The GDP risk premium is set to 3% or 2.5%. The sample period is from 1947 to 2020.

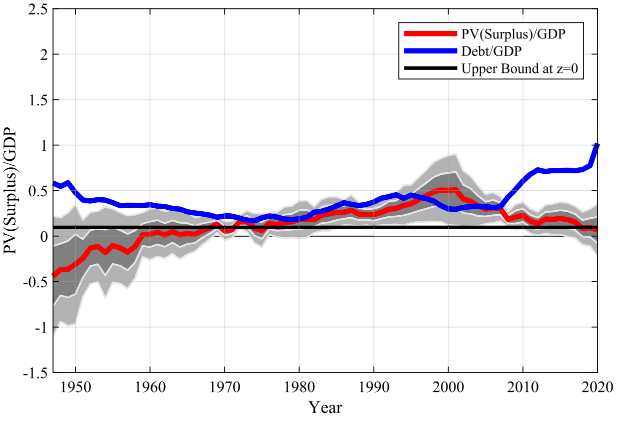


FIGURE C.7.—Upper Bound on the Value of Surpluses/GDP with Convenience Yields. The figure plots the upper bound on PDV of surpluses/GDP in equation (14), the steady-state upper bound evaluated at $z = 0$ in equation (15) and the actual debt/output ratio. The GDP risk premium is set to 2.7% to maintain the same discount rate on the GDP claim despite a higher risk-free rate without convenience yield. The sample period is from 1947 to 2020.

C.2.6. *Convenience Yields*

Figure C.7 plots upper bound for the case of convenience yields with $rp_0^Y = 3\%$. Over the sample period from 1947 to 2020, the average convenience yield λ_0 is 0.56% per year.

APPENDIX D: DYNAMIC ASSET PRICING MODEL

This section derives the expression for equilibrium asset prices in the dynamic asset pricing model.

D.1. *Risk-Free Rate*

The real short yield $y_t(1)$, or risk-free rate, satisfies $\mathbb{E}_t[\exp\{m_{t+1} + y_t(1)\}] = 1$. Solving out this Euler equation, we get

$$\begin{aligned} y_t(1) &= y_t^\$(1) - \mathbb{E}_t[\pi_{t+1}] - \frac{1}{2}(\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}\boldsymbol{e}_\pi + (\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_t \\ &= y_0(1) + \big[(\boldsymbol{e}_{yn})' - (\boldsymbol{e}_\pi)'\boldsymbol{\Psi} + (\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1\big]z_t, \end{aligned} \tag{D.1}$$

$$y_0(1) \overset{\text{def}}{=} y_0^\$(1) - \pi_0 - \frac{1}{2}(\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}\boldsymbol{e}_\pi + (\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0. \tag{D.2}$$

where we used the expression for the real SDF,

$$\begin{aligned} m_{t+1} = m_{t+1}^\$ + \pi_{t+1} &= -y_t^\$(1) - \frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1} + \pi_0 + (\boldsymbol{e}_\pi)'\boldsymbol{\Psi}z_t + (\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1} \\ &= -y_t(1) - \frac{1}{2}(\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}\boldsymbol{e}_\pi + (\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_t - \frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \big(\boldsymbol{\Lambda}_t' - (\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{\frac{1}{2}}\big)\boldsymbol{\varepsilon}_{t+1}. \end{aligned}$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

D.2. *Nominal and Real Term Structure*

PROPOSITION 1: *Nominal bond yields are affine in the state vector*:

$$y_t^{\$}(h) = -\frac{A^{\$}(h)}{h} - \frac{\boldsymbol{B}^{\$}(h)'}{h} z_t,$$

where the coefficients $A^{\$}(h)$ *and* $\boldsymbol{B}^{\$}(h)$ *satisfy the following recursions*:

$$A^{\$}(h+1) = -y_0^{\$}(1) + A^{\$}(h) + \frac{1}{2}\boldsymbol{B}^{\$}(h)'\boldsymbol{\Sigma}\boldsymbol{B}^{\$}(h) - \boldsymbol{B}^{\$}(h)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0, \tag{D.3}$$

$$\boldsymbol{B}^{\$}(h+1)' = \boldsymbol{B}^{\$}(h)'\boldsymbol{\Psi} - (\boldsymbol{e}_{yn})' - \boldsymbol{B}^{\$}(h)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1, \tag{D.4}$$

initialized at $A^{\$}(0) = 0$ *and* $\boldsymbol{B}^{\$}(0) = \boldsymbol{0}$.

PROOF: We conjecture that the $t+1$-price of a τ-period bond is exponentially affine in the state:

$$\log\bigl(P_{t+1}^{\$}(h)\bigr) = A^{\$}(h) + \boldsymbol{B}^{\$}(h)'z_{t+1},$$

and solve for the coefficients $A^{\$}(h+1)$ and $\boldsymbol{B}^{\$}(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P_t^{\$}(h+1) &= \mathbb{E}_t\bigl[\exp\bigl\{m_{t+1}^{\$} + \log\bigl(P_{t+1}^{\$}(h)\bigr)\bigr\}\bigr] \\
&= \mathbb{E}_t\left[\exp\left\{-y_t^{\$}(1) - \frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1} + A^{\$}(h) + \boldsymbol{B}^{\$}(h)'z_{t+1}\right\}\right] \\
&= \exp\left\{-y_0^{\$}(1) - (\boldsymbol{e}_{yn})'z_t - \frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t + A^{\$}(h) + \boldsymbol{B}^{\$}(h)'\boldsymbol{\Psi}z_t\right\} \\
&\quad \times \mathbb{E}_t\bigl[\exp\bigl\{-\boldsymbol{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1} + \boldsymbol{B}^{\$}(h)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1}\bigr\}\bigr].
\end{aligned}$$

We use the log-normality of $\boldsymbol{\varepsilon}_{t+1}$ and substitute for the affine expression for $\boldsymbol{\Lambda}_t$ to get

$$\begin{aligned}
P_t^{\$}(h+1) = \exp\Bigl\{&-y_0^{\$}(1) - (\boldsymbol{e}_{yn})'z_t + A^{\$}(h) + \boldsymbol{B}^{\$}(h)'\boldsymbol{\Psi}z_t \\
&+ \frac{1}{2}\boldsymbol{B}^{\$}(h)'\boldsymbol{\Sigma}\boldsymbol{B}^{\$}(h) - \boldsymbol{B}^{\$}(h)'\boldsymbol{\Sigma}^{\frac{1}{2}}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1 z_t)\Bigr\}.
\end{aligned}$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(h+1))$:

$$\log\bigl(P_t^{\$}(h+1)\bigr) = A^{\$}(h+1) + \boldsymbol{B}^{\$}(h+1)'z_t,$$

where $A^{\$}(h+1)$ satisfies (D.3) and $\boldsymbol{B}^{\$}(h+1)$ satisfies (D.4). The relationship between log bond prices and bond yields is given by $-\log(P_t^{\$}(h))/h = y_t^{\$}(h)$. *Q.E.D.*

Define the one-period return on a nominal zero-coupon bond as

$$r_{t+1}^{\$}(h) = \log\bigl(P_{t+1}^{\$}(h)\bigr) - \log\bigl(P_t^{\$}(h+1)\bigr).$$

The nominal bond risk premium on a bond of maturity h is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

$$\begin{aligned}\mathbb{E}_t\big[r^\$_{t+1}(h)\big] - y^\$_t(1) + \frac{1}{2}\operatorname{Var}_t\big[r^\$_{t+1}(h)\big] &= -\operatorname{Cov}_t\big[m^\$_{t+1}, r^\$_{t+1}(h)\big] \\ &= \boldsymbol{B}^\$(h)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_t.\end{aligned}$$

Real bond yields, $y_t(h)$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A(h+1) = -y_0(1) + A(h) + \frac{1}{2}\big(\boldsymbol{B}(h)\big)'\boldsymbol{\Sigma}\big(\boldsymbol{B}(h)\big) - \big(\boldsymbol{B}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\big(\boldsymbol{\Lambda}_0 - \boldsymbol{\Sigma}^{\frac{1}{2}\prime}\boldsymbol{e}_\pi\big), \quad \text{(D.5)}$$

$$\boldsymbol{B}(h+1)' = -(\boldsymbol{e}_{yn})' + \big(\boldsymbol{e}_\pi + \boldsymbol{B}(h)\big)'\big(\boldsymbol{\Psi} - \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1\big). \quad \text{(D.6)}$$

For $h = 1$, we recover the expression for the risk-free rate in (D.1)–(D.2).

D.3. *Stocks*

D.3.1. *Aggregate Stock Market*

We define the real return on the aggregate stock market as $R^M_{t+1} = \frac{P^M_{t+1}+D^M_{t+1}}{P^M_t}$, where P^M_t is the ex-dividend price on the equity market. A log-linearization delivers

$$r^M_{t+1} = \kappa^M_0 + x_{t+1} + \Delta d_{t+1} + \kappa^M_1 pd^M_{t+1} - pd^M_t. \quad \text{(D.7)}$$

The unconditional mean log real stock return is $r^M_0 = \mathbb{E}[r^M_t]$, the unconditional mean real dividend growth rate is $x_0 + \mu_d$, and $pd^M_0 = \mathbb{E}[pd^M_t]$ is the unconditional average log price-dividend ratio on equity. The linearization constants κ^M_0 and κ^M_1 are defined as

$$\kappa^M_1 = \frac{e^{pd^M_0}}{e^{pd^M_0}+1} < 1 \quad \text{and} \quad \kappa^M_0 = \log\big(e^{pd^M_0}+1\big) - \frac{e^{pd^M_0}}{e^{pd^M_0}+1}pd^M_0. \quad \text{(D.8)}$$

Our state vector z contains the (demeaned) log real dividend-to-GDP ratio growth rate on the stock market, $\Delta d_{t+1} - \mu_d$, and the (demeaned) log price-dividend ratio $pd^M - pd^M_0$.

$$\begin{aligned}pd^M_t(h) &= pd^M_0 + (\boldsymbol{e}_{pd})'z_t, \\ x_t + \Delta d_t &= x_0 + \mu_d + (\boldsymbol{e}_{\text{divm}})'z_t,\end{aligned}$$

where $\boldsymbol{e}_{\text{divm}} = \boldsymbol{e}_{\Delta d} + \boldsymbol{e}_x$, and $\boldsymbol{e}_{pd}$, $\boldsymbol{e}_{\Delta d}$, and $\boldsymbol{e}_x$ select the log pd ratio, the log dividend/GDP ratio growth rate, and the real GDP growth rate in the state vector, respectively.

We *define* the log return on the stock market so that the log return equation holds exactly, given the time series for $\{\Delta d^M_t, pd^M_t\}$. Rewriting (D.7),

$$\begin{aligned}r^M_{t+1} - r^M_0 &= \big[\big(\boldsymbol{e}_{\text{divm}} + \kappa^M_1\boldsymbol{e}_{pd}\big)'\boldsymbol{\Psi} - (\boldsymbol{e}_{pd})'\big]z_t + \big(\boldsymbol{e}_{\text{divm}} + \kappa^M_1\boldsymbol{e}_{pd}\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1}, \\ r^M_0 &= x_0 + \mu_d + \kappa^M_0 - pd^M_0\big(1-\kappa^M_1\big).\end{aligned}$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$
\begin{aligned}
1 &= \mathbb{E}_t\left[M_{t+1}\frac{P^M_{t+1}+D^M_{t+1}}{P^M_t}\right] \\
&= \mathbb{E}_t\left[\exp\left\{m^\$_{t+1}+\pi_{t+1}+r^M_{t+1}\right\}\right] \\
&= \mathbb{E}_t\Bigg[\exp\Bigg\{-y^\$_{t,1}-\frac{1}{2}\boldsymbol{\Lambda}'_t\boldsymbol{\Lambda}_t-\boldsymbol{\Lambda}'_t\boldsymbol{\varepsilon}_{t+1}+\pi_0+(\boldsymbol{e}_\pi)'z_{t+1}+r^M_0 \\
&\quad +\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}\right)'z_{t+1}-(\boldsymbol{e}_{pd})'z_t\Bigg\}\Bigg] \\
&= \exp\left\{-y^\$_0(1)-\frac{1}{2}\boldsymbol{\Lambda}'_t\boldsymbol{\Lambda}_t+\pi_0+r^M_0+\left[\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Psi}-(\boldsymbol{e}_{pd})'-(\boldsymbol{e}_{yn})'\right]z_t\right\} \\
&\quad \times\mathbb{E}_t\left[\exp\left\{-\boldsymbol{\Lambda}'_t\boldsymbol{\varepsilon}_{t+1}+\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1}\right\}\right] \\
&= \exp\left\{r^M_0+\pi_0-y^\$_0(1)+\left[\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Psi}-(\boldsymbol{e}_{pd})'-(\boldsymbol{e}_{yn})'\right]z_t\right\} \\
&\quad \times\exp\left\{\frac{1}{2}\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Sigma}\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)-\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_t\right\}.
\end{aligned}
$$

Taking logs on both sides delivers

$$
\begin{aligned}
& r^M_0+\pi_0-y^\$_0(1)+\left[\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Psi}-(\boldsymbol{e}_{pd})'-(\boldsymbol{e}_{yn})'\right]z_t \\
&\quad +\frac{1}{2}\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Sigma}\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)=\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_t, \quad \text{(D.9)} \\
& \mathbb{E}_t\left[r^{m,\$}_{t+1}\right]-y^\$_{t,1}+\frac{1}{2}\operatorname{Var}_t\left[r^{m,\$}_{t+1}\right]=-\operatorname{Cov}_t\left[m^\$_{t+1},r^{m,\$}_{t+1}\right].
\end{aligned}
$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$
\begin{aligned}
& \mathbb{E}_t\left[r^M_{t+1}\right]-y_{t,1}+\frac{1}{2}\operatorname{Var}_t\left[r^M_{t+1}\right]=-\operatorname{Cov}_t\left[m_{t+1},r^M_{t+1}\right], \\
& r^M_0-y_0(1)+\left[\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}+\boldsymbol{e}_\pi\right)'\boldsymbol{\Psi}-(\boldsymbol{e}_{pd})'-(\boldsymbol{e}_{yn})'-(\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Lambda}_1\right]z_t \\
&\quad +\frac{1}{2}\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}\right)'\boldsymbol{\Sigma}\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}\right)=\left(\boldsymbol{e}_{\text{divm}}+\kappa^M_1\boldsymbol{e}_{pd}\right)'\boldsymbol{\Sigma}^{1/2}\left(\boldsymbol{\Lambda}_t-\left(\boldsymbol{\Sigma}^{1/2}\right)'\boldsymbol{e}_\pi\right).
\end{aligned}
$$

We combine the terms in $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_1$ on the right-hand side and plug in for $y_0(1)$ from (D.2) to get

$$r_0^M + \pi_0 - y_{0,1}^{\$} + \frac{1}{2}(\boldsymbol{e}_\pi)'\mathbf{\Sigma}\boldsymbol{e}_\pi + \frac{1}{2}\big(\boldsymbol{e}_{\text{divm}} + \kappa_1^M \boldsymbol{e}_{pd}\big)'\mathbf{\Sigma}\big(\boldsymbol{e}_{\text{divm}} + \kappa_1^M \boldsymbol{e}_{pd}\big) + (\boldsymbol{e}_\pi)'\mathbf{\Sigma}\big(\boldsymbol{e}_{\text{divm}} + \kappa_1^M \boldsymbol{e}_{pd}\big)$$
$$+ \big[\big(\boldsymbol{e}_{\text{divm}} + \kappa_1^M \boldsymbol{e}_{pd} + \boldsymbol{e}_\pi\big)'\mathbf{\Psi} - (\boldsymbol{e}_{pd})' - (\boldsymbol{e}_{yn})'\big]z_t$$
$$= \big(\boldsymbol{e}_{\text{divm}} + \kappa_1^M \boldsymbol{e}_{pd}\big)'\mathbf{\Sigma}^{1/2}\mathbf{\Lambda}_t + (\boldsymbol{e}_\pi)'\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{\Lambda}_0 + (\boldsymbol{e}_\pi)'\mathbf{\Sigma}^{1/2}\mathbf{\Lambda}_1 z_t.$$

This recovers equation (D.9).

D.3.2. *Dividend Strips*

Price-Dividend Ratios of Dividend Strips.

PROPOSITION 2: *Log price-dividend ratios on dividend strips are affine in the state vector*:

$$pd_t^M(h) = A^M(h) + \boldsymbol{B}^M(h)'z_t,$$

where the coefficients $A^M(h)$ *and* $\boldsymbol{B}^M(h)$ *follow recursions*:

$$A^M(h+1) = A^M(h) + x_0 + \mu_d - y_0^{\$}(1) + \pi_0$$
$$+ \frac{1}{2}\big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\mathbf{\Sigma}\big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)$$
$$- \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{\Lambda}_0, \tag{D.10}$$

$$\boldsymbol{B}^M(h+1)' = \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\mathbf{\Psi} - (\boldsymbol{e}_{yn})' - \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{\Lambda}_1, \tag{D.11}$$

initialized at $A_0^M = 0$ *and* $\boldsymbol{B}_0^M = 0$.

PROOF: We conjecture the affine structure and solve for the coefficients $A^M(h+1)$ and $\boldsymbol{B}^M(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\exp\big(pd_t^M(h+1)\big) = \mathbb{E}_t\bigg[M_{t+1}\exp\big(pd_{t+1}^M(h)\big)\frac{D_{t+1}^M}{D_t^M}\bigg]$$
$$= \mathbb{E}_t\big[\exp\big\{m_{t+1}^{\$} + \pi_{t+1} + x_{t+1} + \Delta d_{t+1} + pd_{t+1}^M(h)\big\}\big]$$
$$= \mathbb{E}_t\bigg[\exp\bigg\{-y_{t,1}^{\$} - \frac{1}{2}\mathbf{\Lambda}_t'\mathbf{\Lambda}_t - \mathbf{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1} + \pi_0 + (\boldsymbol{e}_\pi)'z_{t+1} + x_0 + \mu_d$$
$$+ (\boldsymbol{e}_{\text{divm}})'z_{t+1} + A^M(h) + \boldsymbol{B}(h)^{m\prime}z_{t+1}\bigg\}\bigg]$$
$$= \exp\bigg\{-y_0^{\$}(1) - (\boldsymbol{e}_{yn})'z_t - \frac{1}{2}\mathbf{\Lambda}_t'\mathbf{\Lambda}_t + \pi_0 + (\boldsymbol{e}_\pi)'\mathbf{\Psi}z_t + x_0 + \mu_d$$
$$+ (\boldsymbol{e}_{\text{divm}})'\mathbf{\Psi}z_t + A^M(h) + \boldsymbol{B}(h)^{m\prime}\mathbf{\Psi}z_t\bigg\}$$
$$\times \mathbb{E}_t\big[\exp\big\{-\mathbf{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1} + \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\mathbf{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1}\big\}\big].$$

We use the log-normality of $\boldsymbol{\varepsilon}_{t+1}$ and substitute for the affine expression for $\boldsymbol{\Lambda}_t$ to get

$$\begin{aligned} pd_t^M(h+1) = {} & -y_0^\$(1) + \pi_0 + x_0 + \mu_d + A^M(h) + \big[\big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Psi} - (\boldsymbol{e}_{yn})'\big]\boldsymbol{z}_t \\ & + \frac{1}{2}\big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Sigma}\big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big) \\ & - \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1\boldsymbol{z}_t). \end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^D(h+1)$:

$$pd_t^M(h+1) = A^M(h+1) + \boldsymbol{B}^M(h+1)'\boldsymbol{z}_t,$$

where

$$\begin{aligned} A^M(h+1) = {} & A^M(h) + x_0 + \mu_d - y_0^\$(1) + \pi_0 \\ & + \frac{1}{2}\big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Sigma}\big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big) \\ & - \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0, \\ \boldsymbol{B}^M(h+1)' = {} & \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Psi} - (\boldsymbol{e}_{yn})' - \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1. \end{aligned}$$

We recover the recursions in (D.10) and (D.11) after using equation (D.2). *Q.E.D.*

We define the dividend strip risk premium as

$$\begin{aligned} \mathbb{E}_t\big[r_{t+1}^M(h)\big] - y_{t,1}^\$ + \frac{1}{2}\operatorname{Var}_t\big[r_{t+1}^M(h)\big] & = -\operatorname{Cov}_t\big(m_{t+1}^\$, r_{t+1}^M(h)\big) \\ & = \big(\boldsymbol{e}_{\text{divm}} + \boldsymbol{e}_\pi + \boldsymbol{B}^M(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_t. \end{aligned}$$

Dividend Futures: Price and Return. The price of a dividend futures contract that delivers one year's worth of nominal dividends at time $t+h$, divided by the current dividend, is equal to

$$\frac{F_t^M(h)}{D_t^M} = \exp\big(pd_t^M(h) + hy_t^\$(h)\big),$$

where $pd_t^M(h)$ is the log spot price-dividend ratio on the dividend strip of maturity h. Using the affine expressions for the strip price-dividend ratio and the nominal bond price, it can be written as

$$\frac{F_t^M(h)}{D_t^M} = \exp\big(A^M(h) - A^\$(h) + \big(\boldsymbol{B}^M(h) - \boldsymbol{B}^\$(h)\big)'\boldsymbol{z}_t\big).$$

The one-period holding period return on the dividend future of maturity h is

$$R_{t+1}^{\text{Fut},M}(h) = \frac{F_{t+1}^M(h-1)}{F_t^M(h)} - 1 = \frac{F_{t+1}^M(h-1)/D_{t+1}^M}{F_t^M(h)/D_t^M}\frac{D_{t+1}^M}{D_t^M} - 1.$$

It can be written as

$$\begin{aligned}\log\bigl(1+R_{t+1}^{\mathrm{Fut},M}(h)\bigr) &= A^M(h-1)-A^\$(h-1)-A^M(h)+A^\$(h)+x_0+\mu_d+\pi_0\\ &\quad+\bigl(\boldsymbol{B}^M(h-1)-\boldsymbol{B}^\$(h-1)+\boldsymbol{e}_{\mathrm{divm}}+\boldsymbol{e}_\pi\bigr)'\boldsymbol{z}_{t+1}-\bigl(\boldsymbol{B}^M(h)-\boldsymbol{B}^\$(h)\bigr)'\boldsymbol{z}_t.\end{aligned}$$

The expected log return, which is already a risk premium on account of the fact that the dividend future takes out the return on an equal-maturity nominal Treasury bond, equals

$$\begin{aligned}&\mathbb{E}_t\bigl[\log\bigl(1+R_{t+1}^{\mathrm{Fut},M}(h)\bigr)\bigr]\\ &\quad= A^M(h-1)-A^\$(h-1)-A^M(h)+A^\$(h)+x_0+\mu_d+\pi_0\\ &\qquad+\bigl[\bigl(\boldsymbol{B}^M(h-1)-\boldsymbol{B}^\$(h-1)+\boldsymbol{e}_{\mathrm{divm}}+\boldsymbol{e}_\pi\bigr)'\boldsymbol{\Psi}-\bigl(\boldsymbol{B}^M(h)-\boldsymbol{B}^\$(h)\bigr)'\bigr]\boldsymbol{z}_t.\end{aligned}$$

Given that the state variable z_t is mean-zero, the first row denotes the unconditional dividend futures risk premium.

D.4. *Government Spending and Tax Revenue Claims*

This Supplemental Appendix computes P_t^T, the value of a claim to future tax revenues, and P_t^G, the value of a claim to future government spending. It contains the proof for Proposition 5.

D.4.1. *Spending Claim*

Nominal government spending growth equals

$$\Delta\log G_{t+1}=\Delta\log g_{t+1}+x_{t+1}+\pi_{t+1}=x_0+\pi_0+\mu_0^G+(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi)'\boldsymbol{z}_{t+1}. \qquad \text{(D.12)}$$

We conjecture the log price-dividend ratios on spending strips are affine in the state vector:

$$\log\bigl(pd_t^G(h)\bigr)=A^G(h)+\boldsymbol{B}^G(h)'\boldsymbol{z}_t.$$

We solve for the coefficients $A^G(h+1)$ and $\boldsymbol{B}^G(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}\exp\bigl(pd_t^G(h+1)\bigr) &= \mathbb{E}_t\left[M_{t+1}\exp\bigl(pd_{t+1}^G(h)\bigr)\frac{G_{t+1}}{G_t}\right]\\ &= \mathbb{E}_t\bigl[\exp\bigl\{m_{t+1}^\$+\Delta\log g_{t+1}+x_{t+1}+\pi_{t+1}+pd_{t+1}^G(h)\bigr\}\bigr]\\ &= \exp\left\{-y_0^\$(1)-(\boldsymbol{e}_{yn})'\boldsymbol{z}_t-\frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t+\mu^G+x_0+\pi_0\right.\\ &\qquad\left.+\bigl(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^G(h)\bigr)'\boldsymbol{\Psi}\boldsymbol{z}_t+A^G(h)\right\}\\ &\qquad\times\mathbb{E}_t\bigl[\exp\bigl\{-\boldsymbol{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1}+\bigl(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^G(h)\bigr)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1}\bigr\}\bigr].\end{aligned}$$

We use the log-normality of $\boldsymbol{\varepsilon}_{t+1}$ and substitute for the affine expression for $\boldsymbol{\Lambda}_t$ to get

$$\begin{aligned}
&\exp\bigl(pd_t^G(h+1)\bigr) \\
&\quad = \exp\Bigl\{-y_0^\$(1) + \mu^G + x_0 + \pi_0 + \bigl((\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Psi} - (\boldsymbol{e}_{yn})'\bigr)\boldsymbol{z}_t \\
&\qquad + A^G(h) + \frac{1}{2}(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Sigma}(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h)) \\
&\qquad - (\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Sigma}^{\frac{1}{2}}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1\boldsymbol{z}_t)\Bigr\}.
\end{aligned}$$

Taking logs and collecting terms, we obtain

$$\begin{aligned}
A^G(h+1) &= -y_0^\$(1) + \mu^G + x_0 + \pi_0 + A^G(h) \\
&\quad + \frac{1}{2}(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Sigma}(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h)) \\
&\quad - (\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0, \\
\boldsymbol{B}^G(h+1)' &= (\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Psi} - (\boldsymbol{e}_{yn})' - (\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1,
\end{aligned}$$

and the price-dividend ratio of the ex-dividend spending claim is

$$\sum_{h=1}^{\infty}\exp\bigl(A^G(h+1) + \boldsymbol{B}^G(h+1)'\boldsymbol{z}_t\bigr).$$

Derivation of Risk Premium. We note that the 1-period holding return on a spending strip is

$$\exp\bigl(r_{t+1}^G(h)\bigr) = \exp\bigl\{\Delta\log g_{t+1} + x_{t+1} + \pi_{t+1} + pd_{t+1}^G(h) - pd_t^G(h+1)\bigr\}$$

so that the Euler equation is $\mathbb{E}_t[\exp(m_{t+1}^\$ + r_{t+1}^G(h))] = 1$.

We can express the expected return as

$$\begin{aligned}
\mathbb{E}_t\bigl[r_{t+1}^G(h)\bigr] &= -\mathbb{E}_t\bigl[m_{t+1}^\$\bigr] - \frac{1}{2}\operatorname{Var}_t\bigl(m_{t+1}^\$\bigr) - \frac{1}{2}\operatorname{Var}_t\bigl(r_{t+1}^G(h)\bigr) - \operatorname{Cov}_t\bigl(m_{t+1}^\$, r_{t+1}^G(h)\bigr) \\
&= y_t^\$(1) - \frac{1}{2}\operatorname{Var}_t\bigl(r_{t+1}^G(h)\bigr) - \operatorname{Cov}_t\bigl(m_{t+1}^\$, r_{t+1}^G(h)\bigr)
\end{aligned}$$

and the risk premium is

$$\begin{aligned}
\mathbb{E}_t\bigl[r_{t+1}^G(h)\bigr] - y_t^\$(1) &= -\frac{1}{2}\operatorname{Var}_t\bigl(r_{t+1}^G(h)\bigr) + \operatorname{Cov}_t\bigl(\boldsymbol{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1}, r_{t+1}^G(h)\bigr) \\
&= (\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Sigma}^{\frac{1}{2}}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1\boldsymbol{z}_t) \\
&\quad - \frac{1}{2}(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h))'\boldsymbol{\Sigma}(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h)).
\end{aligned}$$

To evaluate the risk premium for the entire duration of the strip, we define the holding-period risk premium as

$$\frac{1}{h}\sum_{k=0}^{h-1}\mathbb{E}_t\big[r^G_{t+k+1}(h-k)-y^\$_{t+k}(1)\big]$$

when the state variable is at $z_t=0$, the expected holding-period risk premium simplifies to

$$\begin{aligned}&\frac{1}{h}\sum_{k=0}^{h-1}(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0\\&\quad-\frac{1}{2}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^G(h-k)\big)'\boldsymbol{\Sigma}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^G(h-k)\big).\end{aligned}\tag{D.13}$$

Entire Spending Claim. Next, we define the nominal holding period return on the claim as $R^G_{t+1}=\frac{G_{t+1}+P^G_{t+1}}{P^G_t}$. We log-linearize the return around $z_t=0$:

$$r^G_{t+1}=\kappa^G_0+\Delta\log G_{t+1}+\kappa^G_1 pd^G_{t+1}-pd^G_t.\tag{D.14}$$

where $pd^G_t\stackrel{\text{def}}{=}\log P^G_t-\log G_t$. The unconditional mean log return of the spending claim is $r^G_0=\mathbb{E}[r^G_t]$.

We obtain pd^G_0 from the precise valuation formula at $z_t=0$. We define linearization constants κ^G_0 and κ^G_1 as

$$\kappa^G_1=\frac{e^{pd^G_0}}{e^{pd^G_0}+1}<1\quad\text{and}\quad\kappa^G_0=\log\big(e^{pd^G_0}+1\big)-\frac{e^{pd^G_0}}{e^{pd^G_0}+1}pd^G_0.\tag{D.15}$$

Then, under a log-linear approximation, the unconditional expected return is

$$r^G_0=x_0+\pi_0+\kappa^G_0-pd^G_0\big(1-\kappa^G_1\big).\tag{D.16}$$

The log ex-dividend price-dividend ratio on the entire spending claim is affine in the state vector and verify the conjecture by solving the Euler equation for the claim:

$$pd^G_t=pd^G_0+\big(\bar{\boldsymbol{B}}^G\big)'z_t\tag{D.17}$$

This allows us to write the return as

$$r^G_{t+1}=r^G_0+\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\kappa^G_1\bar{\boldsymbol{B}}^G\big)'z_{t+1}-\big(\bar{\boldsymbol{B}}^G\big)'z_t.\tag{D.18}$$

PROOF: Starting from the Euler equation,

$$\begin{aligned}1\quad&=\mathbb{E}_t\big[\exp\{m^\$_{t+1}+r^G_{t+1}\}\big]\\&=\exp\bigg\{-y^\$_0(1)-(\boldsymbol{e}_{yn})'z_t-\frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t+r^G_0+\big[\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\kappa^G_1\bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Psi}-\big(\bar{\boldsymbol{B}}^G\big)'\big]z_t\bigg\}\\&\quad\times\mathbb{E}_t\big[\exp\{-\boldsymbol{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1}+\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\kappa^G_1\bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1}\}\big].\end{aligned}$$

We use the log-normality of $\boldsymbol{\varepsilon}_{t+1}$ and substitute for the affine expression for $\boldsymbol{\Lambda}_t$ to get

$$1 = \exp\Bigg\{ r_0^G - y_0^\$(1) + \big[\big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Psi} - \big(\bar{\boldsymbol{B}}^G\big)' - (\boldsymbol{e}_{yn})'\big]\boldsymbol{z}_t + \underbrace{\frac{1}{2}\big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Sigma}\big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \bar{\boldsymbol{B}}^G\big)}_{\text{Jensen}} - \big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1 \boldsymbol{z}_t)\Bigg\}.$$

Taking logs and collecting terms, we obtain the following system of equations:

$$r_0^G - y_0^\$(1) + \text{Jensen} = \big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0 \tag{D.19}$$

and

$$\big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Psi} - \big(\bar{\boldsymbol{B}}^G\big)' - (\boldsymbol{e}_{yn})' = \big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \bar{\boldsymbol{B}}^G\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1. \tag{D.20}$$

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The right-hand side is the unconditional covariance of the log SDF with the log return. This equation describes the unconditional risk premium on the claim to government spending. Equation (D.20) describes the time-varying component of the government spending risk premium. Given $\boldsymbol{\Lambda}_1$, the system of N equations (D.20) can be solved for the vector $\bar{\boldsymbol{B}}^G$:

$$\bar{\boldsymbol{B}}^G = \big(I - \kappa_1^G\big(\boldsymbol{\Psi} - \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1\big)'\big)^{-1}\big[\big(\boldsymbol{\Psi} - \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1\big)'(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi) - \boldsymbol{e}_{yn}\big]. \tag{D.21}$$

Q.E.D.

D.4.2. *Revenue Claim*

Nominal government revenue growth equals

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^T + (\boldsymbol{e}_{\Delta\tau} + \boldsymbol{e}_x + \boldsymbol{e}_\pi)'\boldsymbol{z}_{t+1}. \tag{D.22}$$

where $\tau_t = T_t/\text{GDP}_t$ is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero. The remaining proof exactly mirrors the proof for government spending with

$$pd_t^T \stackrel{\text{def}}{=} \log\left(\frac{P_t^T}{T_t}\right) = pd_0^T + \big(\boldsymbol{B}^T\big)'\boldsymbol{z}_t \tag{D.23}$$

$$r_{t+1}^T = r_0^T + \big(\boldsymbol{e}_{\Delta\tau} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^G \boldsymbol{B}^T\big)'\boldsymbol{z}_{t+1} - \big(\boldsymbol{B}^T\big)'\boldsymbol{z}_t, \tag{D.24}$$

and

$$\begin{aligned} r_0^T &= x_0 + \pi_0 + \kappa_0^T - pd_0^T\big(1 - \kappa_1^T\big), \\ r_0^T - y_0^\$(1) + \text{Jensen} &= \big(\boldsymbol{e}_{\Delta\tau} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^T \boldsymbol{B}^T\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0. \end{aligned} \tag{D.25}$$

D.5. *Peso Exercise*

Baseline Case. To compute the implied probability of the peso event that justifies the valuation gap, we need to price a claim to the government spending in the presence of such peso probability. We use $P_t^{G,\text{Peso}}$ to denote its valuation. Then

$$P_t^{G,\text{Peso}} \stackrel{\text{def}}{=} (1-\phi_t)\mathbb{E}_t\big[M_{t+1}^{\$}\big(G_{t+1}+P_{t+1}^{G,\text{Peso}}\big)\big]+\phi_t(1-\ell)P_t^G,$$

and our objective is to find the probability of the peso event, ϕ_t, such that

$$D_t = P_t^T - P_t^{G,\text{Peso}}.$$

We assume that in period t, agents expect the same probability ϕ_t for peso events in all future periods: $\mathbb{E}_t[\phi_{t+j}] = \phi_t$. To solve $P_t^{G,\text{Peso}}$, we define an auxiliary variable $P_t^{\text{Cut},\text{Peso}}$, which denotes the present value of the spending cut:

$$P_t^{\text{Cut},\text{Peso}} \stackrel{\text{def}}{=} P_t^G - P_t^{G,\text{Peso}} = \mathbb{E}_t\big[M_{t+1}^{\$}\big(\phi_t\big(\ell\cdot\big(G_{t+1}+P_{t+1}^G\big)\big)+(1-\phi_t)P_{t+1}^{\text{Cut},\text{Peso}}\big)\big].$$

Since $P_t^{\text{Cut},\text{Peso}}$ does not have a simple affine form, it is easier to express it explicitly as an infinite sum of discounted future cash flows:

$$\begin{aligned}
P_t^{\text{Cut},\text{Peso}} &= \phi_t\mathbb{E}_t\big[M_{t,t+1}^{\$}\ell\big(G_{t+1}+P_{t+1}^G\big)\big]+(1-\phi_t)\phi_t\mathbb{E}_t\big[M_{t,t+2}^{\$}\ell\big(G_{t+2}+P_{t+2}^G\big)\big] \\
&\quad + (1-\phi_t)^2\phi_t\mathbb{E}_t\big[M_{t,t+3}^{\$}\ell\big(G_{t+3}+P_{t+3}^G\big)\big]+\cdots \\
&= \ell\phi_t\mathbb{E}_t\left[\sum_{j=1}^{\infty}M_{t,t+j}^{\$}G_{t+j}\right]+\ell(1-\phi_t)\phi_t\mathbb{E}_t\left[\sum_{j=2}^{\infty}M_{t,t+j}^{\$}G_{t+j}\right] \\
&\quad + \ell(1-\phi_t)^2\phi_t\mathbb{E}_t\left[\sum_{j=3}^{\infty}M_{t,t+j}^{\$}G_{t+j}\right]+\cdots \\
&= \ell G_t\left(\phi_t\sum_{j=1}^{\infty}e^{A^G(j)+\boldsymbol{B}^G(j)'z_t}+(1-\phi_t)\phi_t\sum_{j=2}^{\infty}e^{A^G(j)+\boldsymbol{B}^G(j)'z_t}\right. \\
&\quad \left. + (1-\phi_t)^2\phi_t\sum_{j=3}^{\infty}e^{A^G(j)+\boldsymbol{B}^G(j)'z_t}+\cdots\right).
\end{aligned}$$

Countercyclical Peso Probability. We also consider the peso exercise in which the probability of spending cut is decreasing in the GDP growth shock:

$$\mathbb{P}(\text{Peso}_{t+k}|\boldsymbol{\varepsilon}_{t+k}) \stackrel{\text{def}}{=} \tilde{\phi}_{t+k} = 1-(1-\phi_t)\exp\left(-\frac{1}{2}\nu^2+\nu\boldsymbol{\varepsilon}_{t+k}^x\right), \quad \forall k\geq 1.$$

In this case, the present value of the spending cut is

$$P_t^{\text{Cut},\text{Peso}} = \mathbb{E}_t\big[M_{t,t+1}^{\$}\big(\tilde{\phi}_{t+1}\big(\ell\cdot\big(G_{t+1}+P_{t+1}^G\big)\big)+(1-\tilde{\phi}_{t+1})P_{t+1}^{\text{Cut},\text{Peso}}\big)\big].$$

Then

$$
\begin{aligned}
P_t^{\mathrm{Cut,Peso}} &= \ell \mathbb{E}_t\Bigg[\big(1-(1-\tilde{\phi}_{t+1})\big)M_{t,t+1}^{\$}G_{t+1} + \left(1-\prod_{k=1}^{2}(1-\tilde{\phi}_{t+k})\right)M_{t,t+2}^{\$}G_{t+2} \\
&\quad + \left(1-\prod_{k=1}^{3}(1-\tilde{\phi}_{t+k})\right)M_{t,t+3}^{\$}G_{t+3} + \cdots\Bigg] \\
&= \ell P_t^G - \ell \mathbb{E}_t\Bigg[(1-\tilde{\phi}_{t+1})M_{t,t+1}^{\$}G_{t+1} + \prod_{k=1}^{2}(1-\tilde{\phi}_{t+k})M_{t,t+2}^{\$}G_{t+2} \\
&\quad + \prod_{k=1}^{3}(1-\tilde{\phi}_{t+k})M_{t,t+3}^{\$}G_{t+3} + \cdots\Bigg] \\
&= \ell P_t^G - \ell \mathbb{E}_t\big[V_{t,t+1}^G + V_{t,t+2}^G + V_{t,t+3}^G + \cdots\big].
\end{aligned}
$$

Conjecture and verify

$$
\exp\big(A^{\mathrm{Peso}}(h) + \boldsymbol{B}^{\mathrm{Peso}}(h)'\boldsymbol{z}_t\big)G_t = \mathbb{E}_t\left[M_{t,t+h}^{\$}G_{t+h}\prod_{k=1}^{h}(1-\tilde{\phi}_{t+k})\right] = V_{t,t+h}^G,
$$

which can be solved recursively by

$$
\begin{aligned}
&\exp\big(A^{\mathrm{Peso}}(h+1) + \boldsymbol{B}^{\mathrm{Peso}}(h+1)'\boldsymbol{z}_t\big)G_t \\
&\quad = \mathbb{E}_t\big[(1-\tilde{\phi}_{t+1})M_{t,t+1}^{\$}\exp\big(A^{\mathrm{Peso}}(h) + \boldsymbol{B}^{\mathrm{Peso}}(h)'\boldsymbol{z}_{t+1}\big)G_{t+1}\big], \\
&\exp\big(A^{\mathrm{Peso}}(h+1) + \boldsymbol{B}^{\mathrm{Peso}}(h+1)'\boldsymbol{z}_t\big) \\
&\quad = (1-\phi_t)\mathbb{E}_t\left[\exp\left(-\frac{1}{2}\nu^2 + \nu\boldsymbol{\varepsilon}_{t+1}^x\right)M_{t,t+1}^{\$}\exp\big(A^{\mathrm{Peso}}(h) + \boldsymbol{B}^{\mathrm{Peso}}(h)'\boldsymbol{z}_{t+1}\big)\frac{G_{t+1}}{G_t}\right]
\end{aligned}
$$

with terminal condition

$$
\begin{aligned}
G_t\exp\big(A^{\mathrm{Peso}}(1) + \boldsymbol{B}^{\mathrm{Peso}}(1)'\boldsymbol{z}_t\big) &= \mathbb{E}_t\big[(1-\tilde{\phi}_{t+1})M_{t,t+1}^{\$}G_{t+1}\big] \\
&= (1-\phi_t)\mathbb{E}_t\left[\exp\left(-\frac{1}{2}\nu^2 + \nu\boldsymbol{\varepsilon}_{t+1}^x\right)M_{t,t+1}^{\$}G_{t+1}\right].
\end{aligned}
$$

The recursion can be simplified as

$$
\begin{aligned}
&\exp\big(A^{\mathrm{Peso}}(h+1) + \boldsymbol{B}^{\mathrm{Peso}}(h+1)'\boldsymbol{z}_t\big) \\
&\quad = (1-\phi_t)\exp\bigg\{-\frac{1}{2}\nu^2 - y_0^{\$}(1) - (\boldsymbol{e}_{yn})'\boldsymbol{z}_t - \frac{1}{2}\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t + \mu^G + x_0 + \pi_0 \\
&\qquad + \big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^G(h)\big)'\boldsymbol{\Psi}\boldsymbol{z}_t + A^{\mathrm{Peso}}(h)\bigg\} \\
&\qquad \times \mathbb{E}_t\big[\exp\big\{-\boldsymbol{\Lambda}_t'\boldsymbol{\varepsilon}_{t+1} + \big(\boldsymbol{e}_{\Delta g} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \boldsymbol{B}^{\mathrm{Peso}}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\varepsilon}_{t+1} + \nu\boldsymbol{e}_x'\boldsymbol{\varepsilon}_{t+1}\big\}\big]
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\phi_t)\exp\Big\{-y_0^{\$}(1)+\mu^G+x_0+\pi_0+\big((\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^G(h))'\boldsymbol{\Psi}-(\boldsymbol{e}_{yn})'\big)\boldsymbol{z}_t \\
&\quad + A^{\text{Peso}}(h)+\frac{1}{2}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^G(h)\big) \\
&\quad - \big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^G(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}(\boldsymbol{\Lambda}_0+\boldsymbol{\Lambda}_1\boldsymbol{z}_t) \\
&\quad + \big(-(\boldsymbol{\Lambda}_0+\boldsymbol{\Lambda}_1\boldsymbol{z}_t)'+\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\big)\nu\boldsymbol{e}_x\Big\}.
\end{aligned}
$$

Matching terms, we obtain

$$
\begin{aligned}
A^{\text{Peso}}(h+1) &= \log(1-\phi_t)-y_0^{\$}(1)+\mu^G+x_0+\pi_0+A^{\text{Peso}}(h) \\
&\quad + \frac{1}{2}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big) \\
&\quad - \big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0+\big(-(\boldsymbol{\Lambda}_0)' \\
&\quad + \big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\big)\nu\boldsymbol{e}_x, \\
\boldsymbol{B}^{\text{Peso}}(h+1)' &= \big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Psi}-(\boldsymbol{e}_{yn})' \\
&\quad - \big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_1-\nu\boldsymbol{e}_x'\boldsymbol{\Lambda}_1.
\end{aligned}
$$

Finally, the risk premia of the $V_{t,t+h}^G$ claims can be expressed as

$$
\begin{aligned}
rp^{\text{Peso}}(h+1) &= -\frac{1}{2}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}\big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big) \\
&\quad + \big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Lambda}_0+\big((\boldsymbol{\Lambda}_0)' \\
&\quad - \big(\boldsymbol{e}_{\Delta g}+\boldsymbol{e}_x+\boldsymbol{e}_\pi+\boldsymbol{B}^{\text{Peso}}(h)\big)'\boldsymbol{\Sigma}^{\frac{1}{2}}\big)\nu\boldsymbol{e}_x.
\end{aligned}
$$

To understand this risk premium, note that the present value of the spending claim in the presence of peso probability can be expressed as

$$
\begin{aligned}
P_t^{G,\text{Peso}} &\stackrel{\text{def}}{=} P_t^G - P_t^{\text{Cut,Peso}} \\
&= P_t^G - \big(\ell P_t^G - \ell\mathbb{E}_t\big[V_{t,t+1}^G+V_{t,t+2}^G+V_{t,t+3}^G+\cdots\big]\big) \\
&= (1-\ell)P_t^G + \ell\mathbb{E}_t\big[V_{t,t+1}^G+V_{t,t+2}^G+V_{t,t+3}^G+\cdots\big].
\end{aligned}
$$

So, effectively the peso exercise partially replaces P_t^G, the spending claim without possible cuts, with $\mathbb{E}_t[V_{t,t+1}^G+V_{t,t+2}^G+V_{t,t+3}^G+\cdots]$ to generate the spending claim in the presence of possible cuts.

We report the term structure of the risk premium $rp^{\text{Peso}}(h+1)$ for $V_{t,t+h}^G$ in Figure D.1. We can see that the presence of the countercyclical spending cut significantly raises the risk premium on the spending claim, and hence lowers its valuation.

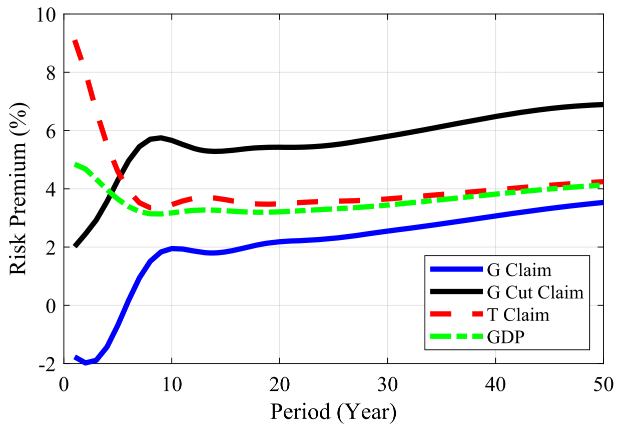


FIGURE D.1.—Term Structure of Risk Premia on the T-Claim and the G-Claim. This figure plots the cumulative risk premia on the spending strips, the spending strips with possible cuts, the tax strips, and the GDP strips in our benchmark model against the holding period.

APPENDIX E: ESTIMATION OF MARKET PRICES OF RISK

This Supplemental Appendix contains the details on the moments we use to estimate the market price of risk parameters and the resulting parameter estimates. Our estimation procedure proceeds in two steps. In the first step, we shut down time variation in the risk prices by setting all entries in the $\mathbf{\Lambda}_1$ matrix to zero. We estimate the $\mathbf{\Lambda}_0$ vector that generates the best fit for the asset pricing moments that we describe below. Given the small dimension of the $\mathbf{\Lambda}_0$ vector, this first step is fast. In the second step, we introduce time-varying risk prices by allowing nonzero values in the $\mathbf{\Lambda}_1$ matrix. In this step, we adjust the moments we use slightly. As the $\mathbf{\Lambda}_1$ matrix is two-dimensional, this step is slower. We run the estimation code until the value function converges. We describe the two steps in detail below.

E.1. *The Constant Risk Price Model*

First, we consider a model with constant risk prices. Then SDF can be written as

$$m^{\$}_{t+1} = -y^{\$}_t(1) - \frac{1}{2}\mathbf{\Lambda}_0'\mathbf{\Lambda}_0 - \mathbf{\Lambda}_0'\boldsymbol{\varepsilon}_{t+1}.$$

We estimate this model using the following moments:

1. No arbitrage restriction on the 5-year bond: Given that the observed 5-year nominal bond yield is one of the elements of the state vector, we insist that the 5-year term nominal yield implied by the asset pricing model matches the 5-year bond yield in the data (in the state space) closely. In this first step with constant risk prices, we only consider the unconditional moment:

$$-A^{\$}(5)/5 = y^{\$}_0(5). \tag{E.1}$$

2. Nominal and real yield curve: We match nominal yields for tenors $\tau = 1, 2, 5, 10, 20, 30$ years and real yields for tenors $\tau = 5, 7, 10, 20, 30$ years. We only consider the unconditional moments, which are the average differences between the observed and the model-implied yields for each tenor.

3. No arbitrage restriction on stock market: Given that the price-dividend ratio and the dividend growth rate of the aggregate stock market portfolio are in the state space, the state implies a time series for the stock return. Therefore, we insist that the model-implied stock return matches the VAR-implied stock return. In this first step, we only consider the unconditional moment, matching the model-implied unconditional equity risk premium (left-hand side) to its empirical counterpart (right-hand side):

$$\left(\boldsymbol{e}_{\Delta d} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^M \boldsymbol{e}_{pd}\right)' \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\Lambda}_0 = r_0^M + \pi_0 - y_0^\$(1) + \frac{1}{2}\text{Jensen}. \tag{E.2}$$

4. Equity price-dividend ratio: We match the average equity price/dividend ratio in model and data.
5. Equity term structure: We match the 1-year and 2-year dividend strips' price/dividend ratios, the ratio of the sum of the one and 2-year dividend strips values to the aggregate equity claim's valuation, as well as the unconditional risk premium on the dividend futures averaged across the first 7 annual strips. The model-implied expressions for the dividend strip price-dividend ratios and the risk premium on dividend futures are derived in Supplemental Appendix D.3.2. The empirical counterparts for the strip price-dividend ratios are available for the period 1996–2009 from van Binsbergen, Brandt, and Koijen (2012) while the dividend futures returns are available for the period 2002–2014 from van Binsbergen and Koijen (2017).
6. GDP risk premium: The price of risk for the GDP shocks is the key variable in our model, which drives the risk premia of tax and spending claims. We assume that the GDP claim is equivalent to an unlevered equity claim. We compute the unlevered corporate asset excess return by multiplying the equity excess return with one minus the aggregate corporate leverage ratio. In our sample, the average leverage ratio (i.e., debt/total asset) from the flow of funds is 21%. We match the GDP strip's risk premium to the equity strip's risk premium times one minus the leverage ratio for strip tenors 1–10, 20, 30, $\dots$, 90 years.
7. Good deal bound: We compute the realized nominal pricing kernel in our sample. In the spirit of Cochrane and Saa-Requejo (2000), we impose a penalty for its standard deviation in excess of 2.
8. Regularity conditions: For tenors 100, 200, 400, 700, 1000, 2000, 3000, 4000 years, we add penalties to (i) force the GDP strip's nominal discount rate to stay above the average nominal GDP growth, (ii) force the nominal-real spread to stay above average inflation, (iii) force the bond risk premium to stay below the GDP risk premium, (iv) force bond return volatility to be bounded below 30%, and (v) force bond risk premium to stay below 2%.

The risk price estimates $\boldsymbol{\Lambda}_0$ are given by

$$\boldsymbol{\Lambda}_0 = \begin{bmatrix} \mathbf{0.52} & 0 & \mathbf{-1.25} & \mathbf{2.07} & \mathbf{0.64} & 0 & \mathbf{0.48} & 0 & 0 & 0 & 0 \end{bmatrix}'$$

and $\boldsymbol{\Lambda}_1$ is all zero.

Figure E.1 plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. Figure E.2 plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real Treasury bond yields (Treasury Inflation Indexed securities).

The top panels of Figure E.3 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. The bottom left panel

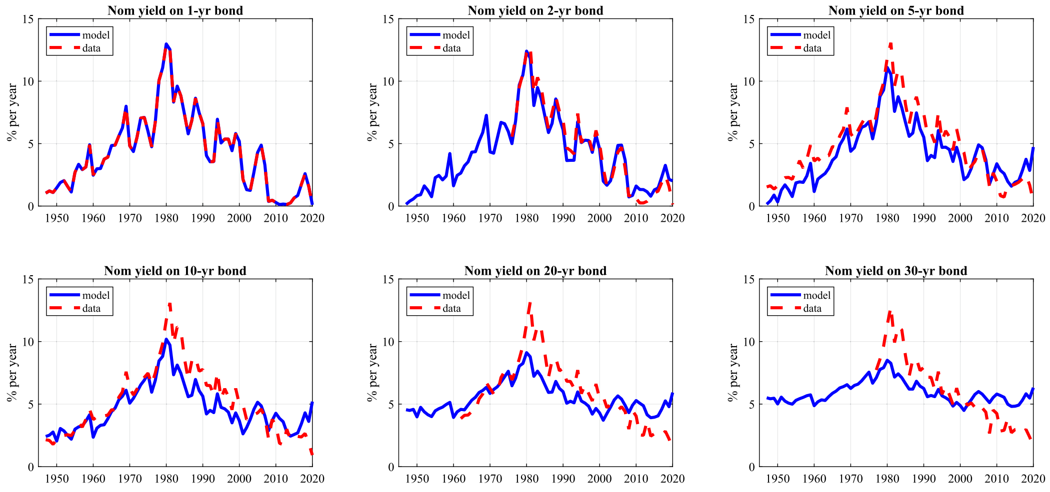


FIGURE E.1.—Dynamics of the Nominal Term Structure of Interest Rates. The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. Yields are measured at the end of the year. Data are from FRED and FRASER. The sample is annual, 1947–2020.

of Figure E.3 plots the dynamics of the nominal bond risk premium, defined as the expected excess return on the 5-year nominal bond. Since there is no time variation in risk premia in this model, the model's nominal bond risk premium is zero. The bottom right panel plots a decomposition of the nominal bond yield on a 5-year bond into the 5-year real bond yield, annual expected inflation over the next 5 years, and the annualized 5-year inflation risk premium. Figure E.4 shows the equity risk premium, the expected excess return on the aggregate stock market portfolio, in the left panel and the corresponding

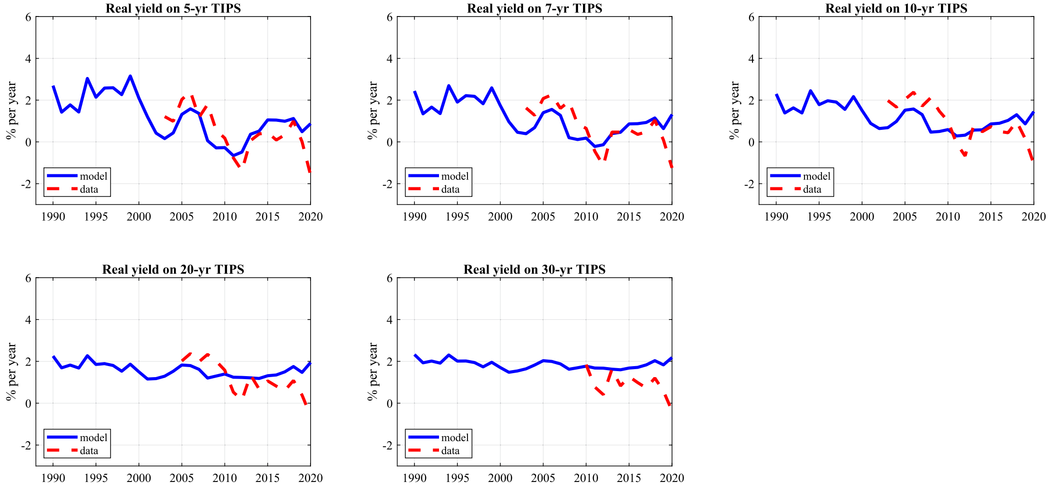


FIGURE E.2.—Dynamics of the Real Term Structure of Interest Rates. The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real bond yields. Data are from FRED and start in 2003. For ease of readability, we start the graph in 1990 but the model of course implies a real yield curve for the entire 1947–2020 period.

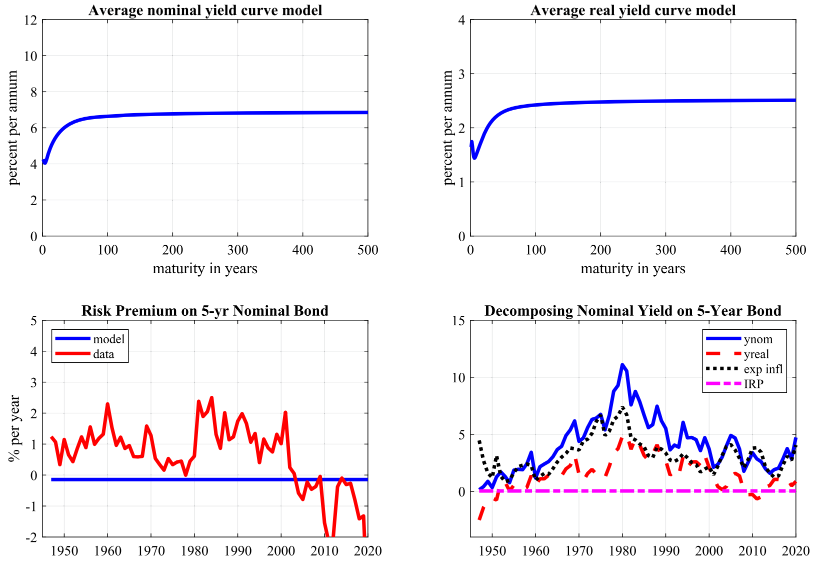


FIGURE E.3.—Long-term Yields and Bond Risk Premia. The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 to 500 years. The bottom left panel plots the nominal bond risk premium on the 5-year bond in model and data. The nominal bond risk premium is measured as the 5-year bond yield minus the expected 1-year bond yield over the next 5 years. The bottom right panel decomposes the model's 5-year nominal bond yield into the 5-year real bond yield, the 5-year expected inflation, and the 5-year inflation risk premium.

price-dividend ratio in the right panel. Since this model has a homoscedastic SDF, the equity risk premium is constant.

Figure E.5 plots the model implied present value of government surpluses. Since this figure is similar to Figure 9, which reports the benchmark case with time-varying market prices of risk, time variation in the market price of risk is not crucial for the main conclusions regarding the bond valuation puzzle. Figure E.6 plots the model-implied term structure of risk premia. It too is similar to Figure 10, which reports the benchmark case with time-varying risk premia.

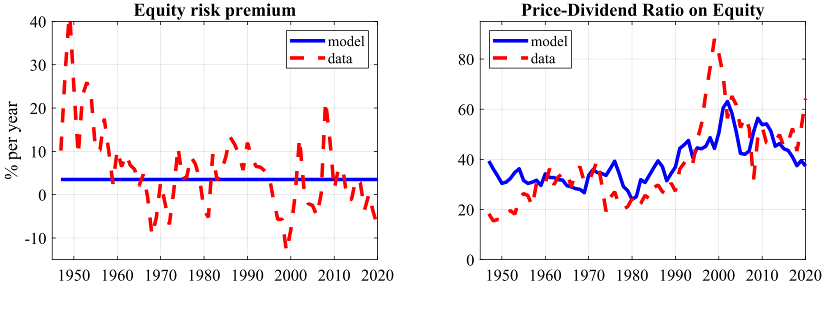


FIGURE E.4.—Equity Risk Premium and Price-Dividend Ratio. The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The price-dividend ratio is the price divided by the annualized dividend. Data are 1947–2020. Monthly stock dividends are seasonally adjusted.

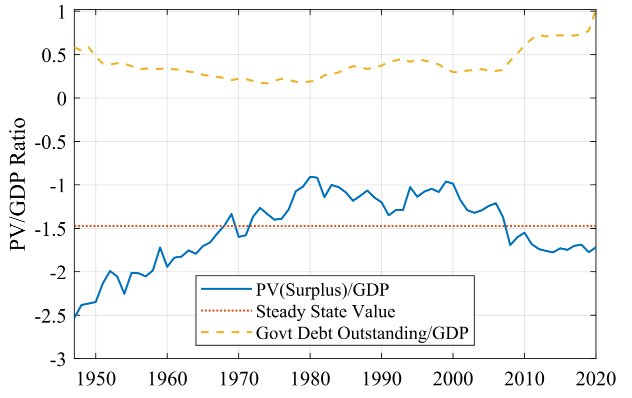


FIGURE E.5.—Present Value of Government Surpluses. The figure plots the model-implied present value of primary surpluses to GDP ratio (in blue) in the constant risk price model.

E.2. *The Full Model With Time-Varying Risk Price*

Next, we consider the full model:

$$m^{\$}_{t+1} = -y^{\$}_t(1) - \frac{1}{2}\mathbf{\Lambda}'_t\mathbf{\Lambda}_t - \mathbf{\Lambda}'_t\boldsymbol{\varepsilon}_{t+1},$$

with time-varying risk premium parameters:

$$\mathbf{\Lambda}_t = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 z_t.$$

We use moments we used in the first step, plus three additional sets of moments:

1a. No arbitrage restrictions on the 5-year bond: Given that the 5-year nominal yield is in the state space, we insist that the 5-year term nominal yield from the model matches the observed yield in the state space closely. In this second step, we also

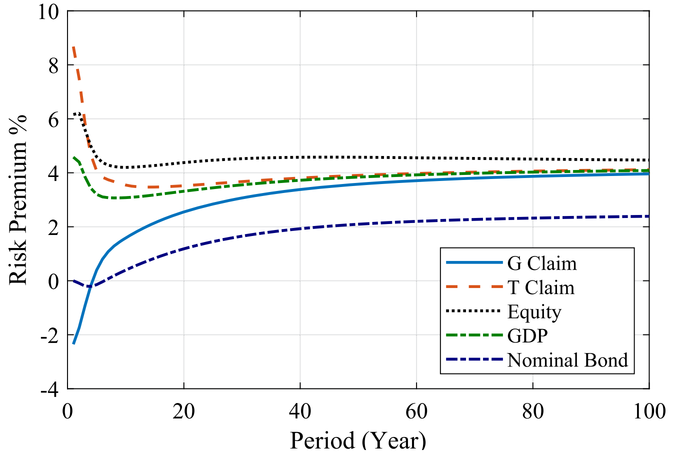


FIGURE E.6.—Term Structure of Risk Premia. The figure plots the term structure of risk premia for the government spending claim, the tax revenue claim, the equity claim, the GDP claim, and the nominal bond.

consider the conditional moment:

$$-\boldsymbol{B}^{\$}(5)/5 = \boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr}.$$

Together with equation (E.1), we obtain

$$\left(-A^{\$}(5) - \boldsymbol{B}^{\$}(5)'z_t\right)/5 = y_0^{\$}(5) + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\boldsymbol{z}_t$$

for arbitrary state variable z_t. Similarly, we aim to match the entire time series of nominal and real bond yields in the model with time-varying risk premia of the various maturities, rather than just the mean yields.

3a. No arbitrage restriction on the stock: Similarly, given the stock price-dividend ratio and dividend growth rate are in the state space, we insist that the stock return from the model matches the VAR-implied stock return closely. In this second step, we also consider the conditional moment:

$$\left(\boldsymbol{e}_{\Delta d} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^M \boldsymbol{e}_{pd}\right)' \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\Lambda}_1 = \left(\boldsymbol{e}_{\Delta d} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^M \boldsymbol{e}_{pd}\right)' \boldsymbol{\Psi} - \boldsymbol{e}'_{pd} - \boldsymbol{e}'_{y1}. \quad \text{(E.3)}$$

Together with equation (E.2), we obtain

$$\begin{aligned}
&\left(\boldsymbol{e}_{\Delta d} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^M \boldsymbol{e}_{pd}\right)' \boldsymbol{\Sigma}^{\frac{1}{2}} (\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1 \boldsymbol{z}_t) \\
&\quad = r_0^M + \pi_0 - y_0^{\$}(1) + \frac{1}{2}\text{Jensen} \\
&\qquad + \left(\left(\boldsymbol{e}_{\Delta d} + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^M \boldsymbol{e}_{pd}\right)' \boldsymbol{\Psi} - \boldsymbol{e}'_{pd} - \boldsymbol{e}'_{y1}\right) \boldsymbol{z}_t
\end{aligned}$$

for arbitrary state variable z_t. The left-hand side and the right-hand side describe the expected excess return of the equity from the asset pricing model and from the VAR dynamics, respectively.

8.(vi) We impose restrictions on the eigenvalues of the risk-neutral companion matrix $\boldsymbol{\Psi} - \boldsymbol{\Sigma}\boldsymbol{\Lambda}_1$. First, we require the eigenvalues to be below 0.995 to ensure all valuation ratios are well-defined in the model. Second, we require the imaginary parts of the eigenvalues to be below 0.5, which removes oscillatory behavior in the term structure of risk premia that is not supported by the data. Removing this restriction does not affect the fit for the asset prices nor the magnitude of the public debt valuation puzzle.

In a first pass, we fix the elements in $\boldsymbol{\Lambda}_0$ from the previous step and estimate the $\boldsymbol{\Lambda}_1$ parameters. That is, we fix the unconditional risk prices for inflation, interest spread, and GDP growth to obtain good starting values for the main estimation. In a second pass, we reestimate all risk price parameters as free variables, iterating on the moments until the value function converges.

The risk price estimates $\boldsymbol{\Lambda}_0$ and $\boldsymbol{\Lambda}_1$ are given by

$$\boldsymbol{\Lambda}_0 = \begin{bmatrix} 1.11 & 0 & -0.39 & 1.88 & 1.14 & 0 & 0.76 & 0 & 0 & 0 & 0 \end{bmatrix}',$$

$$\Lambda_1 = \begin{bmatrix} 34.89 & -24.38 & -45.74 & 18.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9.98 & -11.98 & -106.53 & -11.19 & 1.26 & -0.64 & -0.11 & -0.36 & 6.48 & -0.42 & 5.30 \\ -0.86 & 62.99 & 206.10 & -18.78 & 1.64 & 5.13 & 3.36 & 3.68 & 2.54 & 4.29 & -2.89 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -11.33 & -10.77 & -8.48 & -4.51 & 0.63 & -0.91 & -1.79 & -1.17 & 0.67 & 1.26 & -1.57 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Figure E.7 shows that the model matches the nominal term structure in the data closely. The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. In the estimation of the market prices of risk, we overweigh matching the 5-year bond yield since it is included in the VAR and the 30-year bond yield since the behavior of long-term bond yields is important for the results.

Figure E.8 shows that the model matches the real term structure in the data fairly well. The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real Treasury bond yields (Treasury Inflation Indexed securities). In the estimation of the market prices of risk, we overweigh matching the 30-year bond yield since the behavior of long-term bond yields is important for the results.

The top panels of Figure E.9 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well behaved, with very long-run nominal (real) yields stabilizing at around 6.12% (2.20%) per year.

The bottom left panel of Figure E.9 shows that the model matches the dynamics of the nominal bond risk premium, reasonably well. Bond risk premia decline in the latter part of the sample, possibly reflecting the arrival of foreign investors who value U.S. Treasuries highly. The bottom right panel shows a decomposition of the nominal bond yield on a 5-

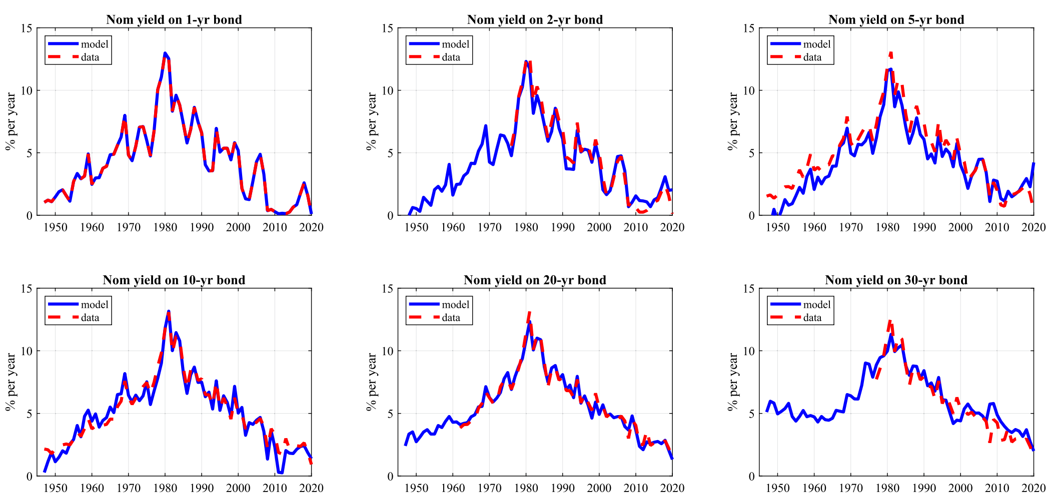


FIGURE E.7.—Dynamics of the Nominal Term Structure of Interest Rates. The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. Yields are measured at the end of the year. Data are from FRED and FRASER. The sample is annual, 1947–2020.

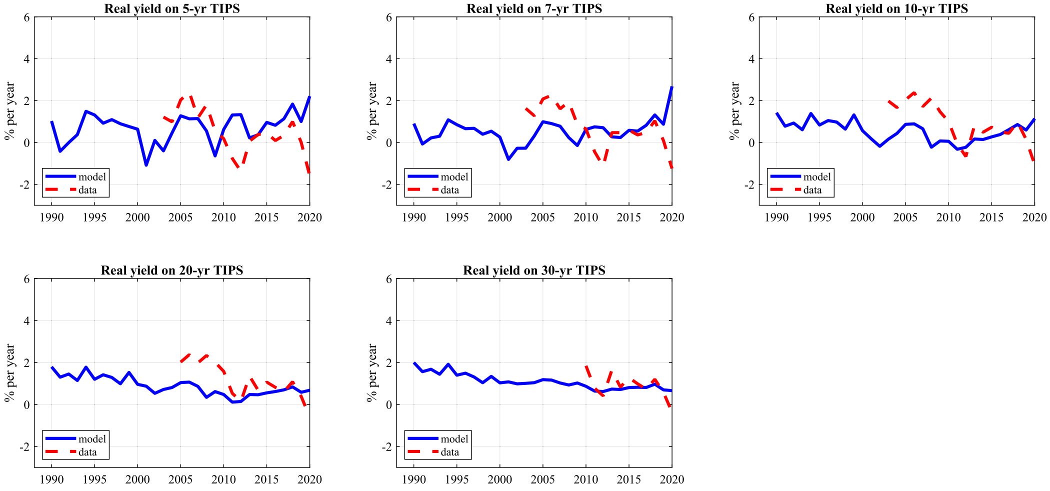


FIGURE E.8.—Dynamics of the Real Term Structure of Interest Rates. The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real bond yields. Data are from FRED and start in 2003. For ease of readability, we start the graph in 1990 but the model of course implies a real yield curve for the entire 1947–2020 period.

year bond. On average, the 4.14% nominal bond yield is comprised of a 0.72% real yield, a 3.16% expected inflation rate, and a 0.26% inflation risk premium. The graph shows that the importance of these components fluctuates over time.

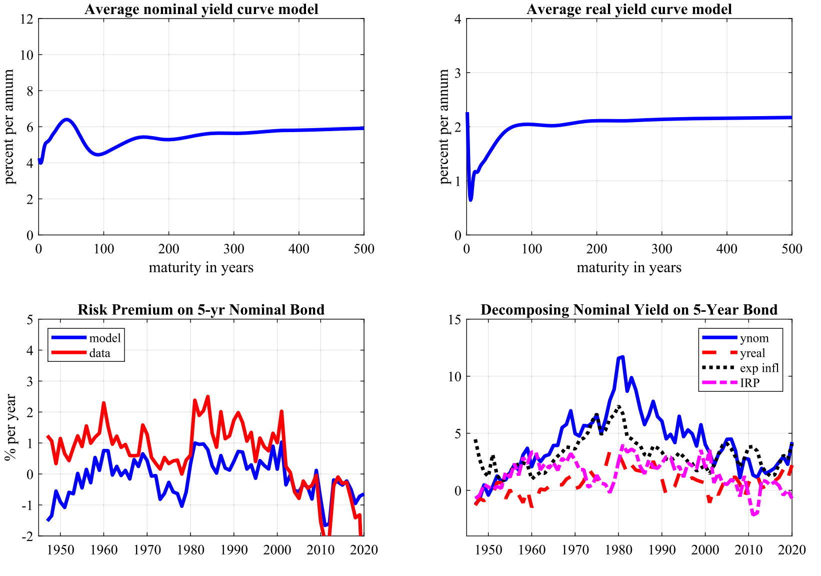


FIGURE E.9.—Long-term Yields and Bond Risk Premia. The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 to 500 years. The bottom left panel plots the nominal bond risk premium on the 5-year bond in model and data. The nominal bond risk premium is measured as the 5-year bond yield minus the expected 1-year bond yield over the next 5 years. The bottom right panel decomposes the model's 5-year nominal bond yield into the 5-year real bond yield, the five-year expected inflation, and the 5-year inflation risk premium.

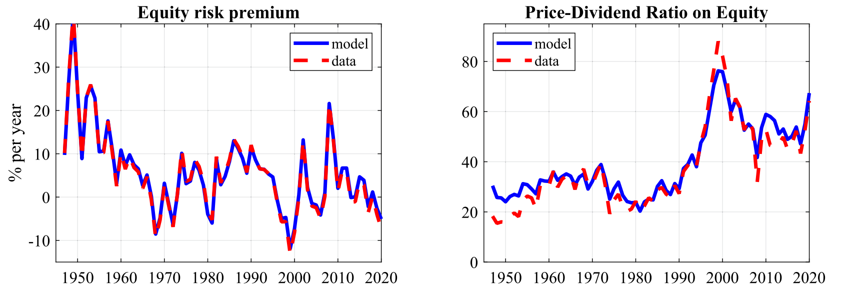


FIGURE E.10.—Equity Risk Premium and Price-Dividend Ratio. The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The price-dividend ratio is the price divided by the annualized dividend. Data are 1947–2020. Monthly stock dividends are seasonally adjusted.

Figure E.10 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panel. The risk premium in the data is the expected equity excess return implied by the VAR. Its dynamics are sensible, with low risk premia in the dot-com boom of 1999–2000 and high risk premia in the Great Financial Crisis of 2008–2009. The model matches these equity risk premium dynamics tightly. The figure's right panel shows a good fit for equity price-dividend levels. Hence, the model fits both the behavior of expected returns and stock price levels.

E.3. *Extension: Global VAR*

As a robustness check, we explore adding additional global state variables, inspired by Andrews, Colacito, Croce, and Gavazzoni (2023). We construct three time series: global GDP growth, global inflation, and global stock returns. All three series are the GDP-weighted averages of the corresponding domestic series for the following 17 countries: Australia, Belgium, Switzerland, Germany, Denmark, Spain, Finland, France, UK, Italy, Japan, Netherlands, Norway, Sweden, Canada, Ireland, and Portugal. Since we want data going back to 1947, we use the global macrofinance database of Jordà-Schularick-Taylor

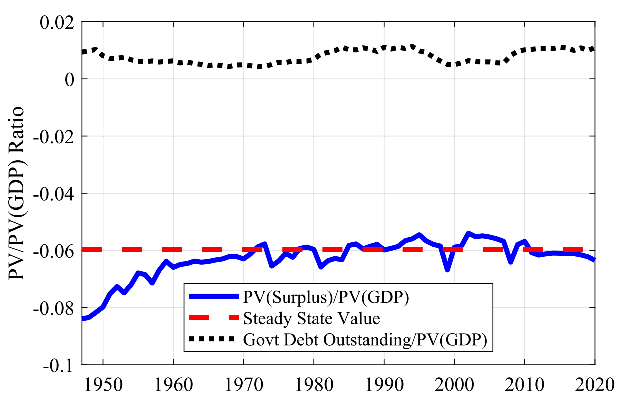


FIGURE E.11.—PV(Surpluses)/PV(GDP) Ratio. The figure plots the model-implied present value of primary surpluses to the present value of the GDP ratio (in blue) in the full time-varying risk price model. The black line is the debt/PV(GDP) ratio.

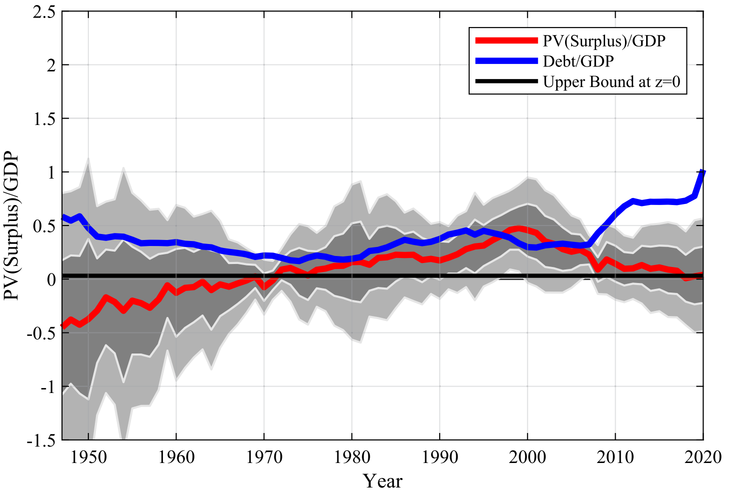


FIGURE E.12.—Upper Bounds on U.S. fiscal backing with Global VAR.

Macrohistory Database. We add these three series to the VAR, after the U.S. state variables, but before the U.S. fiscal cash flows. We refer to this system as the global VAR.

In a first exercise, we redo our upper bound exercise of Section 5 with the global VAR. Figure E.12 plots the result. The graph shows that our results on the bond valuation puzzle are robust to including these international state variables. The average wedge is 32.08% of GDP in the model with global state variables compared to 32.07% in the benchmark model without global state variables. This exercise shows that global inflation, growth, and equity market dynamics do not change the U.S. fiscal surplus dynamics much.

In a second exercise, we redo our asset pricing exercise of Section 6 with the global VAR. We assume that the market prices of global GDP growth risk and global inflation risk are nonzero but constant. This adds two nonzero elements to $\mathbf{\Lambda}_0$. We insist that the global equity risk premium dynamics match the dynamics in the VAR. That is, we free up the corresponding element in the constant market price of risk vector $\mathbf{\Lambda}_0$, as well as the elements of the row corresponding to the global stock return in the time-varying market price of risk matrix $\mathbf{\Lambda}_1$. We set the latter elements to match the expected global stock

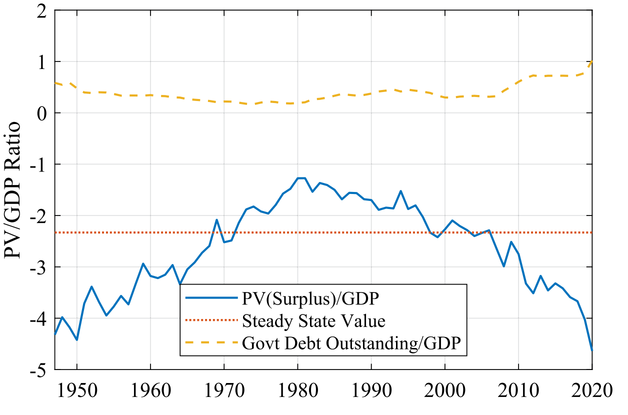


FIGURE E.13.—Present Value of Surpluses with Global VAR.

TABLE E.1

RMSE OF KEY ASSET PRICING MOMENTS.

	Risk Premium Equity	Excess Returns 5-yr Bond	Yield Curve (Nominal)	Yield Curve (Real)	PD ratio
Global VAR	2.05	7.13	0.61	1.23	6.79
Benchmark VAR	0.70	7.30	0.69	1.14	4.76

return dynamics to those implied by the VAR. Figure E.13 shows that the bond valuation puzzle remains. The present value of surpluses is similar to that in the domestic benchmark model, implying a valuation gap of around 300% of GDP relative to the market value of government debt. One note of caution is that United States and global inflation have a time-series correlation of 84%, which complicates the separate identification of the market prices of domestic and inflation risk. Table E.1 summarizes the RMSE for key moments for the model with global factors and the benchmark model. We find that this extended asset pricing model delivers a similar fit to our benchmark model. While the fit for the bond yield moments is similar, the fit for the domestic equity moments deteriorates somewhat.

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