Predictive Enforcement*

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Abstract

This paper examines the use of data-informed predictions to guide law enforcement efforts. Such "predictive" enforcement can lead to data being selectively and disproportionately collected from neighborhoods targeted for enforcement by the prediction. Predictive enforcement that fails to account for this endogenous "datafication" may lead to the over-policing of traditionally high-crime neighborhoods and performs poorly–in particular, in some cases as poorly as if no data were used. By incorporating the impact of enforcement on criminal incentives, we identify additional benefits of using data efficiently for deterrence.

Keywords: Data-driven enforcement, one-armed bandit with changing world, hidden Markov-chain, a bandit with endogenously-valued arm

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1 Introduction

Data-guided prediction algorithms are becoming increasingly prevalent in law enforcement. In major U.S. cities, predictive tools like Risk Terrain Modeling and Spatio-Temporal Modeling are now commonplace in police departments, helping to forecast crime "hot spots" and guide resource allocation (See Pearsall (2010); Perry, McInnis, Price, Smith, and Hollywood (2013); Brayne (2020)).[1] Beyond crime prevention, data analytics is also leveraged by credit card companies to detect fraud and by municipal governments to enforce regulations. A well-known anecdote involves New York City's inspection of "illegal conversions" (i.e., dividing a dwelling into smaller units for additional rental income), which pose significant fire hazards, among other problems. By linking various data sources and using them to predict likely violations, the City's analytics team achieved a five-fold improvement in inspection effectiveness.[2]

Despite the successes of *predictive enforcement*, it has also faced criticism. Concerns have been raised about the reliability of prediction algorithms and their tendency to disproportionately target certain communities, particularly those that are non-white and low-income. Critics argue that these algorithms, trained on historical data, may perpetuate existing biases. Additionally, a self-reinforcing cycle can emerge: increased police presence in high-crime areas leads to more arrests, justifying further police focus, and potentially creating a feedback loop that amplifies disparities.[3]

This paper develops a theoretical framework to analyze predictive enforcement, focusing on the inherent connection between enforcement actions and the collection of crime data, a process we term *datafication*. Crime data is seldom collected randomly; it is often "selected" by past enforcement and surveillance efforts. This creates a feedback loop where enforcement influences the information gathered, which in turn informs future enforcement decisions. This dynamics is particularly relevant for "victimless crimes" like drug offenses and prostitution, where detection relies primarily on police efforts.

[1]Police departments have used in-house algorithms as well as commercially developed algorithms such as PredPol, Palantir, Hunchlabs and IBM.

[2]"Before the big-data analysis, inspectors followed up the complaints they deemed most dire, but only in 13 percent of cases did they find conditions severe enough to warrant a vacate order. Now they were issuing vacate orders on more than 70 percent of the buildings they inspected. By indicating which buildings most needed their attention, big data improved their efficiency five-fold. ... Flowers and his kids looked like wizards with a crystal ball that let them see into the future and predict the most risky places. They took massive quantities of data lying around for years, largely unused after it was collected, and harnessed it in a novel way to extract real value." See Mayer-Schönberger and Cukier (2014).

[3]Lum and Isaac (2016) argue that this sort of feedback loop is a reason behind their finding that simulating the PredPol algorithm using Oakland's past drug arrests data leads to the concentration of patrolling in two traditionally crime-prone areas, even though the estimated drug activities (based on national health and drug use survey) are much more widely distributed over a large number of areas.

We utilize a bandit model to capture the interplay between enforcement and data collection. Time is continuous. At each instant $t \geq 0$ the policymaker (PM) allocates $y_t \in [0, 1]$ of enforcement resources (e.g., police patrols) to a single "neighborhood" at costs $cy_t > 0$. The term "neighborhood" broadly represents any observable characteristic the PM might target, such as location, demographics, or transaction types. The "cost" reflects the opportunity cost of diverting resources from alternative neighborhoods (or other potential targets). The harm from crime is mitigated with probability proportional to the allocated enforcement level y_t. Crime attempts themselves depend on an underlying *crime condition*, modeled as the arrival of crime opportunities. The arrival rate of these opportunities is uncertain, varying between a high rate $\lambda > 0$ in a "high-crime" state and zero in a "safe" state. The PM's primary challenge is to discern this unknown state, aided by crime data generated through past enforcement.

Our model builds upon the framework of Keller, Rady, and Cripps (2005) (henceforth, KRC) but introduces two key distinctions. First, unlike in KRC, the state is not fixed but instead transitions dynamically according to a continuous-time Markov chain. This "changing world" approach reflects the reality of fluctuating crime conditions, influenced by factors like foot traffic and evolving socioeconomic dynamics. Moreover, it serves a crucial analytical purpose: in a fixed-state model, the value of learning vanishes as the true state is eventually revealed. In contrast, our changing world model ensures the ongoing need for learning, allowing us to examine the long-term value of information acquisition, a crucial aspect of predictive enforcement.

Second, our model also endogenizes crime behavior based on criminals' expectations about enforcement. This makes the value of the "bandit arm" (enforcement activity) endogenous, as the intensity of enforcement (pulling a bandit arm) influences crime behavior, thus changing the PM's reward from enforcement itself. The bandit problem then becomes part of strategic interactions between policymakers and criminals. From an analytic standpoint, endogenizing crime behavior enables us to study the role of predictive enforcement in crime deterrence, a fundamental goal of law enforcement.

We analyze the **optimal predictive (OP)** policy and compare it with two benchmarks: the **non-predictive (NP)** policy, which disregards past crime data, and the **greedy predictive (GP)** policy, which uses predictions but ignores their impact on future data collection. These benchmarks represent traditional and current practices in predictive enforcement, with GP reflecting the common approach of using predictions for immediate goals without considering their long-term impact on data collection.

Our findings are as follows. First, we examine exogenous crime, where the probability of committing a crime upon receiving an opportunity is fixed at $x \in [0, 1]$. Under NP, the

PM chooses enforcement without using crime data, opting for full enforcement if the long-run crime rate exceeds the cost, or equivalently if the stationary probability π_0 of the high-crime state exceeds the *myopic cutoff*, $\hat{p}_M := c/(\lambda x)$, and no enforcement otherwise.

In both the GP and OP policies, the PM updates her belief p about the likelihood of a high-crime state based on crime data, particularly the timing of the last detected crime. She then adjusts enforcement levels accordingly. Under GP, the PM uses a cutoff policy, enforcing fully if the belief exceeds the myopic cutoff $\hat{p}_M$ and not enforcing otherwise. The OP policy also employs a cutoff structure, but its cutoff $\hat{p}$ may differ from the myopic one, depending on the crime rate λx relative to the cost c. If the crime rate is sufficiently low or sufficiently high, the OP policy eventually converges to either no enforcement or full enforcement, respectively, mirroring the outcomes of the NP and GP policies. This equivalence occurs because either very high or very low crime rate leaves little ambiguity about what the optimal policy should be, thus rendering prediction irrelevant.

When the crime rate is intermediate, prediction becomes valuable, depending on how it is used for enforcement. In this scenario, the optimal cutoff $\hat{p}$ is set below the myopic cutoff $\hat{p}_M$ to encourage more exploration, which GP, focused solely on exploitation, neglects. Consequently, GP tends to under-enforce compared to OP, particularly when the opportunity cost c is relatively high, namely when alternative neighborhoods have stably high crime rates. This observation aligns with criticisms of predictive enforcement disproportionately targeting such neighborhoods, even without inherent bias in the prediction algorithm. The issue stems from GP's insufficient exploration.

The success of predictive enforcement hinges on how predictions inform enforcement choices. If learning stems solely from enforcement actions, the GP policy, despite utilizing predictions, fails to outperform the NP policy in the long run. The inherent feedback loop between enforcement and exploration contributes to the suboptimality of the myopic GP approach. If the PM can also learn passively, such as through community crime reporting, GP can outperform NP. However, its performance approaches that of NP as passive learning diminishes.

In Section 4, we endogenize the crime probability $x \in [0, 1]$ by considering the potential criminals' incentives. The goal here is to study the deterrence effects of alternative enforcement policies. We assume there exists a "deterring" enforcement level $\overline{y} \in (0, 1)$ such that criminals commit crimes with certainty if they anticipate enforcement below this level, and refrain from committing crimes if they anticipate enforcement above it. At $\overline{y}$, criminals may choose any crime probability. We assume that potential criminals lack access to crime data and base their decisions solely on their prior expectations of enforcement. We then investigate a

Markov perfect equilibrium where the PM's strategy depends solely on her belief p about the likelihood of a high-crime state.

Under the NP regime, neither the criminal's strategy nor the PM's strategy utilizes predictions based on crime data. This results in a classic inspection game scenario: if the cost of enforcement c is not too high, both the PM and criminals adopt mixed strategies. The PM selects $y = \overline{y}$ that leaves criminals indifferent, while criminals randomize with crime probability $x = c/(\lambda\pi_0)$ that makes the PM indifferent across all enforcement levels, including $\overline{y}$.

Under both GP and OP, given any crime probability, the PM's optimal response is a cutoff policy, as established in the exogenous crime analysis. Criminals, in turn, form their expectations about the enforcement level based on the stationary distribution of the PM's belief updates and the associated enforcement actions, and they choose their optimal crime probability accordingly. The equilibrium involves the joint determination of the enforcement cutoff $\hat{p}$ and the crime probability x. We demonstrate that the equilibrium crime probability is lower under OP compared to GP, implying that OP outperforms GP, which, in turn, is payoff-equivalent to NP. The intuition is that the PM, under OP, has additional incentives for enforcement due to the information-gathering motive. This enhanced enforcement credibility leads to greater deterrence and, consequently, a lower crime rate in equilibrium.

Finally, we examine the scenario where criminals also have access to crime data and possess the same predictive capabilities as the PM. In this case, both the GP and OP regimes lead to the inspection game equilibrium for each prediction about the state, resulting in equivalent outcomes across all three regimes (NP, GP, and OP). This highlights that the value of predictive enforcement for the PM relies on having an informational advantage over potential criminals, which may diminish as criminals become more adept at utilizing data.

Related Literature. This paper contributes to the literature on enforcement, initiated by Becker (1968),[4] and inspection games, which study the interactions between enforcers and those subject to enforcement.[5] Our model offers a dynamic perspective on these models. While several recent papers explore dynamic inspection models, focusing on the optimal frequency of inspections (Dilmé and Garrett (2019), Varas, Marinovic, and Skrzypacz (2020); Wagner and Knoepfle (2021); Ball and Knoepfle (2023)), our work distinguishes itself by emphasizing the endogenous datafication process within a bandit framework. Specifically, we generalize the planner's problem in Keller, Rady, and Cripps (2005) by introducing a changing world and an endogenously-valued arm. The concept of endogenous datafication has also been explored in different contexts by Che and Hörner (2018) and Che, Kim, and Zhong (2020).

[4]See Polinsky and Shavell (2000) for survey.

[5]See Avenhaus, Von Stengel, and Zamir (2002) for a survey.

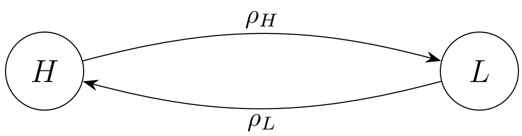
Ichihashi (2023) explores predictive policing from an information design perspective, demonstrating that withholding information about criminal types can enhance enforcement and deterrence. However, his model does not incorporate the dynamic feedback between enforcement and information acquisition, which is central to our analysis. Lum and Isaac (2016) and Ensign, Friedler, Neville, Scheidegger, and Venkatasubramanian (2018) examine this feedback loop, highlighting potential errors in inference and prediction of actual crime activities. In contrast, our model assumes a fully Bayesian policymaker who avoids such errors. Mohler, Short, Malinowski, Johnson, Tita, Bertozzi, and Brantingham (2015) study the performance of PredPol, a prominent predictive policing algorithm. Notably, our predictive policies (OP and GP) share a qualitative feature with PredPol's "earthquake model," which predicts an exponential decay in crime rates without new incidents. While PredPol directly assumes this through a self-exciting process, our model derives it endogenously from Bayesian updating about the changing state.

Our research also connects with the literature on racial profiling and algorithmic fairness.[6] While we do not explicitly incorporate fairness into the PM's objective, we draw implications for fairness from the failure to consider endogenous datafication.

2 Preliminaries

2.1 Setup

A **policymaker (PM)** performs law enforcement in a "neighborhood" with a unit mass of potential "criminals." Time is continuous, and at each instant $t \in [0, \infty)$, a crime opportunity arises to a single individual in that neighborhood at an uncertain Poisson rate $\tilde{\lambda}$ that depends on the underlying *state* ω_t, which is either "high" (H), indicating a crime-prone environment, or "low" (L), signifying a safe environment. In the low state, $\tilde{\lambda} = 0$, while in the high state, $\tilde{\lambda} = \lambda > 0$.



The state transitions according to a continuous-time Markov chain, switching from low to high at rate $\rho_L > 0$ and from high to low at rate $\rho_H > 0$. The *stationary* probability

[6]See Knowles, Persico, and Todd (2001); Bjerk (2007); Persico (2002, 2009); Curry and Klumpp (2009); Eeckhout, Persico, and Todd (2010) for the former, and Rambachan, Kleinberg, Ludwig, and Mullainathan (2020); Kleinberg, Ludwig, Mullainathan, and Sunstein (2018); Angwin, Larson, Mattu, and Kirchner (2022); Dressel and Farid (2021); Liang, Lu, and Mu (2022).

of the high state is $\pi_0 := \frac{\rho_L}{\rho_H+\rho_L}$, representing the long-run average proportion of time the neighborhood is crime-prone.

Upon receiving a crime opportunity, an individual commits a crime with probability $x > 0$, referred to as the crime level. We analyze two scenarios: exogenous crime, where the crime level is fixed, and endogenous crime, where it depends on the individual's incentive, influenced by the anticipated enforcement level.

At each time t, the PM chooses an enforcement level $y_t \in [0, 1]$ at a cost $c > 0$ per unit. This cost can represent direct expenses like hiring personnel or the opportunity cost of diverting resources from other areas under the PM's jurisdiction.

Two neighborhood model. To illustrate the nature of opportunity cost, consider a two-neighborhood model where neighborhood A has an uncertain crime condition, and neighborhood B has a known crime rate λ_B. If λ_B is high, B can be considered a traditionally high-crime area. The PM allocates a unit budget of resource allocation at "zero" cost between the two neighborhoods. A higher λ^B means a higher opportunity cost from pulling resources away from B, corresponding to a higher c in our model. Under-enforcing A relative to the optimal policy in our baseline model means excessive substitution of enforcement efforts from A to B in this two-neighborhood model, implying over-enforcement of B. □

The PM cannot directly observe the state ω_t, instead forming a belief p_t about the probability of the high-crime state at time t. Enforcement serves two purposes: first, it intervenes in crime attempts with probability y_t, mitigating harm. Second, successful intervention reveals the high-crime state, aiding future predictions.

We assume unmitigated harm costs \$1 to society.[7] Both enforcement functions align with common practices: addressing violations and gathering information for future predictions.

PM's payoff. The PM's payoff is the negative of her total discounted loss over an infinite horizon, arising from unmitigated harm and enforcement costs. Given belief p_t, enforcement level y_t, and crime probability x at time t, the PM's flow loss is $p_t\lambda x(1 - y_t) + cy_t$. The total discounted loss, with discount rate $r > 0$ and initial belief p, is

$$L(p) := \mathbb{E}\left[\int_0^\infty (p_t\lambda x(1 - y_t) + cy_t)e^{-rt}dt \,\middle|\, p_0 = p\right]. \tag{1}$$

We assume $\lambda x > c$ to avoid the trivial solution of no enforcement.

[7] Mitigation of harm may take the form of foiling a crime ex-ante or fulfilling justice ex-post by apprehending an offender after the commission of a crime.

Prediction. Predictive enforcement leverages predictions of future crime based on past crime data. In our model, this translates to forming a posterior belief p about the high-crime state using detected crime incidents. The belief updating process is formalized as follows: upon crime detection, the posterior jumps to 1, as crimes only occur in the high state. If no crime is detected, the posterior is updated using Bayes' rule, weighing the possibilities of a low state or a high state with undetected crime. Bayes rule suggests that p follows an ODE:[8]

$$\dot{p} = \underbrace{\rho_L(1-p) - \rho_H p}_{\text{natural rate}} \underbrace{- p(1-p)\lambda x y}_{\text{updating from ``no detection''}} =: f(p, y). \tag{2}$$

The first term, $\rho_L(1-p) - \rho_H p$, represents the "natural rate" of belief change due to the underlying Markov chain governing the state transitions. If the PM chooses $y_t = 0$ (no enforcement), learning is impossible, and the belief simply follows this natural rate, drifting towards the stationary probability π_0 over time.

If the PM chooses a positive enforcement level ($y_t > 0$), she may detect a crime, confirming the high state. However, not detecting a crime (despite enforcement) pushes her belief towards the low state. This downward pressure on the belief, captured by the second (underbraced) term in (2), is strongest when $y_t = 1$ (full enforcement). Let π_1 be the belief where $f(\pi_1, 1) = 0$. If the PM always chooses full enforcement, her belief drifts down if it's above π_1 and drifts up if it's below π_1, as long as no crime is detected. Naturally, π_1 is lower than the stationary probability π_0.

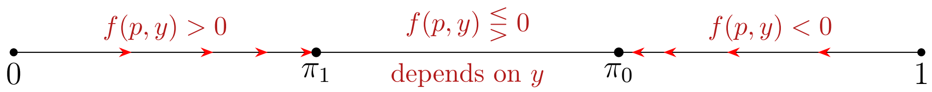


Figure 1: PM's belief updating given no detection.

Figure 1 illustrates the belief dynamics. Without crime detection, the belief drifts up if $p_t < \pi_1$ and down if $p_t > \pi_0$. The belief's evolution within the interval (π_1, π_0) depends on the enforcement level y. Higher enforcement generally leads the belief to settle closer to π_1, absent any crime detection.

[8] The ODE can be derived as follows. For any short duration $dt > 0$

$$\begin{aligned} p_{t+dt} =& \frac{p_t(1-\rho_H dt)(1-\lambda x y dt) + (1-p_t)\rho_L dt(1-\lambda x y dt)}{p_t(1-\rho_H dt)(1-\lambda x y dt) + (1-p_t)(1-\rho_L dt) + (1-p_t)\rho_L dt(1-\lambda x y dt) + p_t \rho_H dt} + o(dt) \\ =& \frac{p_t(1-\rho_H dt - \lambda x y dt) + (1-p_t)\rho_L dt}{1 - p_t \lambda x y dt} + o(dt). \end{aligned}$$

Subtracting p_t from this, dividing by dt, and letting $dt \to 0$, we obtain (2).

Remark 1 (**Fixed State Model**). In contrast to our model with a changing state, the KRC bandit model assumes a fixed state. Consequently, the natural rate is absent, and the belief always drifts downwards towards 0 in the absence of crime detection. In our model, the long-run belief remains bounded between π_1 and π_0.

Remark 2 (**Predictive Enforcement in Practice**). Our prediction rule, derived from the underlying uncertainty, shares a key feature with the PredPol algorithm's "earthquake model": the exponential decay of predicted crime risk with prolonged periods of no crime. While PredPol directly assumes this through a self-exciting process, our model derives it endogenously. Exploring the extent to which our model offers a Bayesian micro-foundation for such processes could be an interesting avenue for future research.

Admissibility. We focus on Markovian strategies where the PM's action depends solely on her current belief p. This is without loss of generality for exogenous crime and natural for endogenous crime, allowing us to suppress time dependence. However, the belief dynamics in equation (1) must be well-defined, imposing an admissibility restriction on the strategy. Essentially, if a belief p attracts drifts from both above and below, it must be a stationary point, implying $y(p) = z(p)$, where $z(p)$ satisfies $f(p, z(p)) = 0$. This ensures the belief process doesn't oscillate arbitrarily frequently around p. While admissibility may seem technical, it captures a substantive property: in a discrete-time approximation, $z(p)$ represents the fraction of time the PM chooses full enforcement (as opposed to no enforcement) in the "neighborhood" of p. The interior solution $z(\hat{p})$ in the continuous-time model corresponds to this relative fraction of time in the limit as the time length approaches zero.

2.2 Alternative enforcement regimes

We consider three scenarios regarding the PM's use of data for enforcement decisions, which serve as benchmarks to understand the value and implications of data-driven enforcement.

Non-predictive enforcement (NP): The PM doesn't update her belief based on crime data, resulting in a constant enforcement policy dependent solely on her prior. We assume this prior follows the ground truth, namely, the long-run crime rate $\pi_0 \lambda x$, potentially formed from long-term memory or aggregate past records.[9]

The optimal non-predictive policy is to enforce fully if and only if the expected long-run harm from crime ($\pi_0 \lambda x$) exceeds the enforcement cost (c). Formally, the non-predictive

[9]For instance, the PM may initially run a full enforcement campaign, choosing $y = 1$ at each instant, for a duration T of time. Letting $N(T)$ denote the number of crimes, the empirical crime rate $\frac{N(T)}{T}$ converges to $\mathbb{E}[1_{\{\text{crime at } t\}}] = \pi_0 \lambda x$ almost surely, by Ergodic Theorem for (irreducible) Markov chains.

enforcement policy is defined as:

$$y_{\text{NP}}(p) = \begin{cases} 1 & \text{for } \pi_0 > \hat{p}_M \\ 0 & \text{for } \pi_0 < \hat{p}_M, \end{cases} \tag{3}$$

where $\hat{p}^M := c/(\lambda x)$ is the myopic cutoff. The policy is constant, taking either 0 (no enforcement) or 1 (full enforcement).

Greedy predictive enforcement (GP): In this benchmark, the PM updates her posterior belief p accurately based on crime history but acts myopically, disregarding the informational value of enforcement. The PM chooses the myopically optimal enforcement level:

$$y_{\text{GP}}(p) = \begin{cases} 1 & \text{for } p > \hat{p}_M \\ 0 & \text{for } p < \hat{p}_M, \end{cases} \tag{4}$$

where p is updated according to (2). While GP utilizes the full predictive power of crime data, it neglects the informational impact of enforcement decisions. The greedy nature of GP aligns with current enforcement practices, which typically do not consider the long-term informational value of enforcement actions.

Optimal predictive enforcement (OP): In this ideal scenario, the PM makes dynamically optimal enforcement choices. Like in GP, the PM forms her posterior belief accurately from crime data, but now she incorporates the informational value of enforcement into her decision-making, aiming for long-term optimality.

In the following analysis, NP and GP serve as benchmarks for comparison with OP. The goal is not to prove OP's superiority, which is inherent by design, but to understand the role of the "turned-off" elements in shaping the optimal policy and their contribution to its value.

3 Exogenous Crime

This section examines the scenario of exogenous crime, where the crime probability x is fixed within the interval $(0, 1]$. We begin by analyzing OP and then compare its outcomes with those of NP and GP.

3.1 Optimal Predictive Enforcement

This section focuses on the Optimal Predictive Enforcement (OP) scenario, where the PM dynamically optimizes enforcement decisions, considering both current crime predictions and the long-term informational value of enforcement. The problem is formulated as a Markov

Decision Problem (MDP) with the belief p as the state variable. The optimal policy, denoted by $p \to y(p)$, is time-independent due to the Markovian nature of the problem.

The analysis employs dynamic programming, focusing on the value function that maximizes the net present value of mitigated crimes minus enforcement costs.[10] The value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$rV(p) = \max_{y} (p\lambda x - c)\, y + p\lambda x y[V(1) - V(p)] + f(p, y)V'(p). \tag{5}$$

The HJB equation balances the cost of discounting future value (LHS) with the immediate benefits of enforcement $(p\lambda x - c)y$ and the value appreciation from Bayesian belief updates. The condition resembles that in KRC, with key differences arising from the changing state of the world. First, the law of motion governing belief evolution differs from that in KRC. Second, the changing state implies that learning remains essential even in the long run. In particular, the value function is not pinned down even for $p = 0$ or $p = 1$, since reaching these extreme states does not mean a permanent resolution of uncertainty. Instead, the value function at any belief depends on the PM's actions at other beliefs, leading to a fixed-point characterization.

The optimal policy, derived from the HJB equation, involves a cutoff structure: for some $\hat{p} \in [0, 1]$

$$y(p) = \begin{cases} 1, & \text{for } p > \hat{p} \\ 0, & \text{for } p < \hat{p}. \end{cases} \tag{6}$$

The following characterizes OP.

Theorem 1. *For each $c > 0$, there exist $\overline{\lambda} > \frac{c}{\pi_0} > \underline{\lambda} > c$ such that the optimal enforcement policy denoted $y^*(p)$ is of the form in (6) where the cutoff is*

Case 1: $\pi_0 \le \hat{p} < \hat{p}_M$ if $\lambda x \le \underline{\lambda}$ ("low" crime rate);

Case 2: $\hat{p} = \hat{p}_M \le \pi_1$ if $\lambda x \ge \overline{\lambda}$ ("high" crime rate);

Case 3: $\hat{p} \in (\pi_1, \pi_0)$ and $\hat{p} < \hat{p}_M$ if $\lambda x \in (\underline{\lambda}, \overline{\lambda})$ ("intermediate" crime rate).

The policy at the cutoff belief, $y^*(\hat{p})$, is arbitrary in Case 1 and Case 2, but in Case 3, $y^*(\hat{p}) = z(\hat{p})$, where $z(p)$ satisfies $f(p, z(p)) = 0$. Moreover, $\underline{\lambda}$ and $\overline{\lambda}$ are increasing in c. Finally, $\hat{p}$ is continuously decreasing in λx and increasing in c.

Proof. See Appendix A. □

[10] The equivalence between this value maximization and minimization of the loss function (1) is established in Lemma 13 in Appendix C.1 of Online Appendix.

As stated in Theorem 1, there are three cases, depending on where the optimal cutoff $\hat{p}$ lies relative to π_1 and π_0, depicted in Figure 1.

Case 1: low crime rates. In this scenario, the crime rate λx is so low that the optimal cutoff $\hat{p}$ exceeds π_0. If the initial belief is above $\hat{p}$, the PM initiates full enforcement until the belief drops below $\hat{p}$ due to a lack of crime detection. The belief eventually converges to the stationary probability π_0, and enforcement ceases permanently (see Figure 2). The loss of learning that occurs when the belief falls past $\hat{p}$ requires the optimal policy to lower the cutoff $\hat{p}$ below the myopic level $\hat{p}_M$, to maintain optimal tradeoff between exploitation and exploration. The long-run dynamics here contrast with the fixed-state bandit model, where full enforcement might persist indefinitely. The changing world necessitates eventual enforcement cessation as the belief stabilizes at π_0.

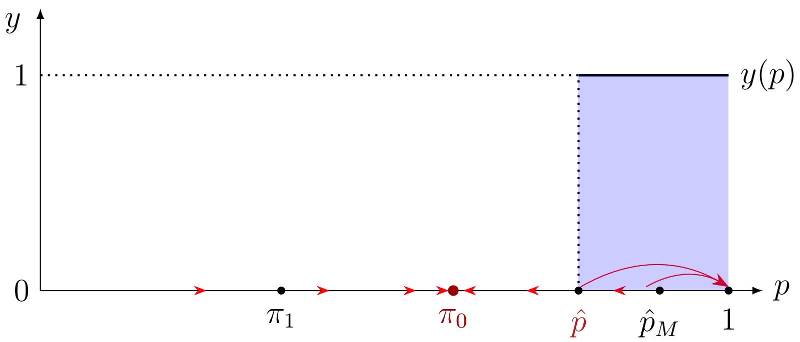


Figure 2: Low Crime Rates Case

Case 2: high crime rates. Here, the crime rate λx is high enough that the optimal cutoff $\hat{p}$ falls below π_1, the lowest possible stationary belief. The policymaker enforces fully even when the belief is below π_1. The belief eventually cycles within the region $[\pi_1, 1]$, jumping to 1 upon crime detection and drifting down to π_1 otherwise (see Figure 3). Enforcement is always full and never lets up, eliminating the need to trade off exploitation for exploration, resulting in $\hat{p} = \hat{p}_M$. This differs from the fixed-state model, where irreversible abandonment of exploration might occur. The changing world ensures persistent vigilance, maintaining enforcement even with prolonged crime absence.

Case 3: intermediate crime rates. In this case, the enforcement cutoff lies between π_1 and π_0. For beliefs above $\hat{p}$, the PM enforces fully. If no crime is detected, the belief drifts down, potentially reaching $\hat{p}$. At $\hat{p}$, partial enforcement at level $y = z(\hat{p})$ is employed, keeping

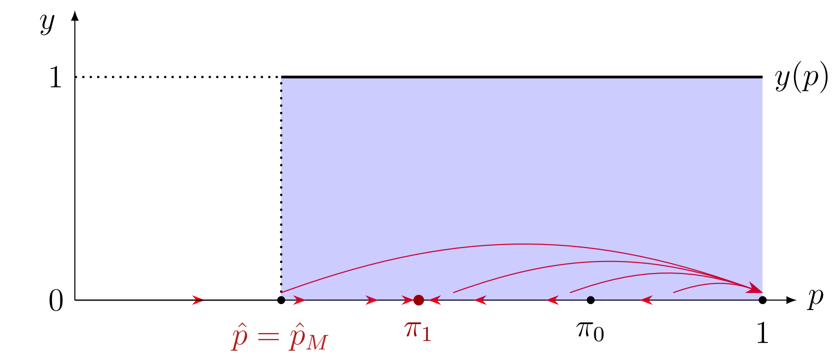


Figure 3: High crime rates case

the belief at $\hat{p}$ until crime detection triggers a jump to 1. See Figure 4. The system spends significant time at $\hat{p}$, and enforcement alternates between full and partial levels. This pattern deviates from the standard bandit model, where enforcement eventually becomes either full or nonexistent. The changing world leads to this "hedging" behavior, balancing the increasing likelihood of the "safe" state with the knowledge that the environment can become crime-prone again.

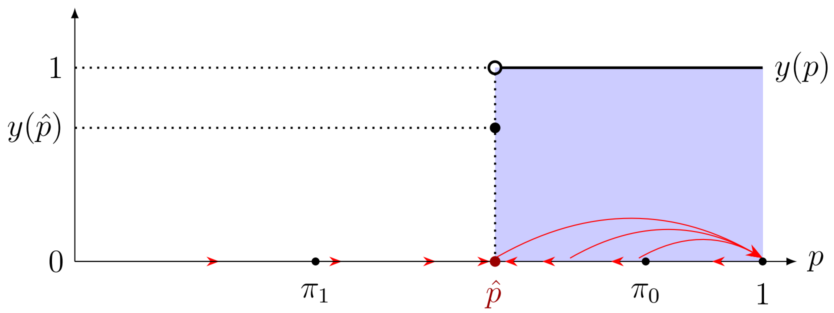


Figure 4: Intermediate crime rates case

Remark 3 (**The role of admissibility**)*.* The admissibility condition ensures the uniqueness of the optimal policy, even when the policymaker is indifferent at the cutoff belief $\hat{p}$. It pins down the interior enforcement level at $\hat{p}$, making the optimal policy well-defined and preventing any ambiguity in the model's predictions.

Remark 4 (**Long-run distribution**)*.* One can characterize the stationary distribution of the PM's belief and the enforcement under OP and GP. If the cutoff lies within (π_1, π_0), the enforcement has a two-point distribution with support $\{z(\hat{p}), 1\}$ under either policy. The long-run proportion of time the enforcement takes each level can be computed precisely; we do this in Section 4.2. Comparison of the long-run distribution between states H and L reveals the potential benefit of predictive enforcement. Under NP, the PM chooses the same enforcement level in both states, but under either GP or OP, the PM chooses, on average, a strictly higher enforcement level in $\omega = H$ than in $\omega = L$, when the cutoff lies within (π_1, π_0).

3.2 Comparison with Benchmarks

This section compares the optimal predictive (OP) policy with the non-predictive (NP) and greedy predictive (GP) benchmarks, focusing on scenarios where OP may strictly outperform them. The analysis centers on the long-run behavior of these policies, specifically within the *irreducible* set of beliefs that allow for transitions between any two beliefs in finite time. Specifically, when considering the optimal policy, we focus on the prior $p = \pi_0$ in Case 1, $p \in [\hat{p}, 1]$ in Case 2, and $p \in [\pi_1, 1]$ in Case 3. We simply call such a prior a **long-run prior**.

The initial focus is on cases where predictive enforcement, even when optimized, offers no long-run advantage.

Proposition 1. *In Case 1 and Case 2, $y^*(p) = y_{NP}(p) = y_{GP}(p)$ at any long-run prior p.*

The first proposition establishes that in scenarios with low or high crime rates (Figure 2 and Figure 3), the OP, NP, and GP converge to the same actions in the long run. This convergence occurs because the prediction problem is relatively simple in these cases, and past data doesn't significantly improve decision-making. The more interesting scenario is Case 3, where prediction is expected to be valuable, and the optimal policy might diverge from the benchmarks. There are two possibilities.

In Case 3-(a), $\hat{p}_M \geq \pi_0$. Under GP, the belief eventually converges to π_0, resulting in the eventual cessation of enforcement and exploration, mirroring the outcome under NP. Consequently, both NP and GP under-enforces relative to OP, which prescribes positive enforcement even in the long run.

In Case 3-(b), $\hat{p}_M < \pi_0$. In this case, NP prescribes full enforcement, whereas GP alternates between full enforcement (when $p > \hat{p}_M$) and partial enforcement (when $p = \hat{p}_M$). The comparison with the optimal policy reveals that GP tends to under-enforce relative to OP, as the optimal cutoff $\hat{p}$ is set below $\hat{p}_M$ to encourage more exploration (see Figure 5).[11] In contrast, NP results in over-enforcement compared to OP.

[11]One can show, and it is intuitive, that $z(\hat{p}) > z(\hat{p}_M)$, and that the belief stays longer at $z(\hat{p}_M)$ under GP than at $z(\hat{p})$ under OP.

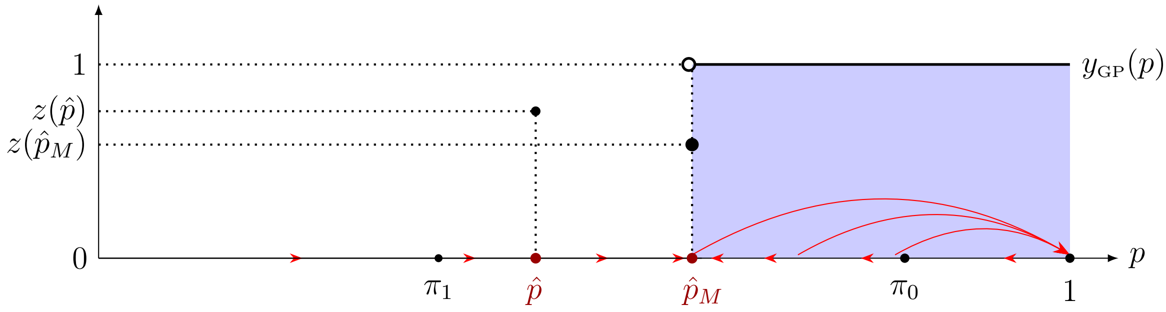


Figure 5: Enforcement Policy under GP: Case 3-(b)

Proposition 2. *In Case 3-(a), both NP and GP lead to no enforcement, which is too little compared with OP. In Case 3-(b), NP leads to full, and thus excessive, enforcement, whereas GP leads to insufficient enforcement, compared with OP.*

In summary, the NP policy might either under- or over-enforce relative to OP, while GP consistently under-enforces compared to OP. Notably, a shift from NP (representing the pre-AI regime) to GP (the current AI regime) always leads to reduced enforcement in Case 3. The interpretation in the context of the two-neighborhood model suggests that data-driven prediction, as currently practiced, tends to over-enforce in traditionally high-crime neighborhoods. This observation aligns with criticisms of predictive enforcement disproportionately targeting such neighborhoods, even in the absence of systemic bias in the prediction algorithm itself. The issue stems from insufficient exploration under the GP policy.

Two-neighborhood model interpretation: The findings from Proposition 2 have implications for the two-neighborhood model. Under-enforcement in the reference neighborhood in our model translates into substituting enforcement efforts away from A to B—and thus over-enforcement of the latter neighborhood—in the two neighborhoods model. This occurs when prediction is valuable (Case 3), and the alternative neighborhood has a relatively high crime rate. Thus, current data-driven prediction practices can lead to over-policing traditionally high-crime neighborhoods, even without inherent bias in the prediction algorithm. The shift from NP to GP further exacerbates this by increasing enforcement in the alternative neighborhood. □

Payoff comparison. An interesting question is how GP compares with NP in total expected loss. In particular, to what extent does data-driven prediction improve the efficiency of enforcement? Can the myopic use of data still be valuable, or equivalently, does all its value reside with the ability to harness the informational value of enforcement? While NP and GP lead to the same long-run outcome in Case 1, Case 2, and Case 3-(a), they don't in Case 3-(b). In the latter case, GP prescribes distinct enforcement levels based on the prediction, while NP obviously can't. Surprisingly, however, they entail the same losses in the long run:

Proposition 3 (Payoff comparison)**.** *The expected loss from GP is always identical to that from NP in the long run. The expected loss is strictly lower under OP in Case 3.*

Proof. It suffices to show the equivalence for Case 3-(b), in which $\pi_1 < \hat{p} < \hat{p}_M < \pi_0$. Recall from Proposition 2 that NP prescribes full enforcement in this case, so the intertemporal loss from NP is c/r. Meanwhile, GP implements enforcement that varies with p_t. To prove the equivalence between NP and GP, it suffices to show that that the intertemporal loss from GP is $\frac{c}{r}$ at any long-run prior p_0 in $[\hat{p}_M, 1]$ (the long-run support of p under GP). The equivalence follows since the total loss under GP is:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty (p_t\lambda x(1-y_t)+cy_t)e^{-rt}dt \middle| p_0 \in [\hat{p}_M, 1]\right] \\
&= \int_0^\infty \mathbb{E}\left[(p_t\lambda x(1-y_t)+cy_t)\middle| p_0 \in [\hat{p}_M, 1]\right] e^{-rt}dt \\
&= \int_0^\infty \mathbb{E}\left[1_{\{p_t>\hat{p}_M\}}\cdot c + 1_{\{p_t=\hat{p}_M\}}\cdot(\hat{p}_M\lambda x(1-z(\hat{p}_M))+cz(\hat{p}_M))\middle| p_0 \in [\hat{p}_M, 1]\right] e^{-rt}dt \\
&= \int_0^\infty ce^{-rt}dt = c/r,
\end{aligned}$$

where we used the fact that $y_{\text{GP}}(p_t) = 1_{\{p_t>\hat{p}_M\}} + 1_{\{p_t=\hat{p}_M\}}\cdot z(\hat{p}_M)$ and that $\hat{p}_M = \frac{c}{\lambda x}$. To compare the losses under GP and OP, observe first that given any common prior $p_0 \in [\hat{p}_M, 1]$, the total loss is weakly lower under OP than under GP by definition. The payoff comparison is strict since y_{OP} differs from y_{GP}. □

The intuition is explained as follows. Again consider Case 3-(b), the only case GP diverges from NP in the long run. In that case, GP differs from NP only when $p = \hat{p}_M$, where the GP prescribes partial enforcement $z(\hat{p}_M) < 1$, whereas NP always prescribes full enforcement. However, at $p = \hat{p}_M$, the PM is indifferent, so the enforcement is payoff-irrelevant! Hence, payoff-wise it is as if the PM is fully enforcing just as in NP.

The implication is rather striking. The equivalence suggests that prediction, and thus data, has no value if it is used in the myopically optimal manner. To see this at a deeper level, recall Case 3-(b). GP prescribes no enforcement when $p < \hat{p}_M$, whereas NP always prescribes full enforcement; and in this case, no enforcement is *strictly* better than full enforcement. Why doesn't GP then strictly outperform NP? The answer is: no such $p < \hat{p}_M$ is part of the support of the PM's beliefs. That is, prediction arises *only* when there is enforcement, but enforcement never occurs when it is undesirable under GP! This makes the prediction obtained under GP redundant.

Community Reporting. Equivalence between GP and NP rests crucially on the assumption that crime is detected only through enforcement. Although this assumption is valid for so-called victimless crimes, for other crimes, detection may also arise from community reporting of crimes. Appendix D extends our model to allow for such a possibility; we assume that the PM detects a crime with probability $w \in (0, 1)$ even when its enforcement fails to detect one. Proposition 13 demonstrates that the GP benefits from passive learning, while the NP does not. This suggests that GP outperforms NP when passive learning is present. The simple explanation is that passive learning enables the PM's belief to drop below the myopic cutoff $\hat{p}_M$, where GP recommends no enforcement—a strategy that's strictly optimal. Put simply, prediction occurs even when not enforcing the law is the best choice from a myopic perspective. However, as $w \to 0$, the GP's superiority over NP vanishes.[12] In this sense, the above equivalence should be viewed as a conceptual point highlighting the extent to which endogeneity of learning limits the value of predictive enforcement. It also suggests the importance of community reporting in rationalizing predictive enforcement.

4 Endogenous Crime and Deterrence

By assuming exogenous crime behavior, the previous section has focused on law enforcement's surveillance and remedial roles. However, an equally important goal of law enforcement is deterring crimes. To study the deterrence implications of alternative policies, we must endogenize the crime behavior.

We adopt a simple model in which a criminal commits a crime only if he anticipates the enforcement rate to be less than some "deterrence" threshold $\bar{y} \in (0, 1)$. More precisely, a criminal commits a crime with probability $x = 0$ if the expected enforcement level is $y > \bar{y}$, with probability $x = 1$ if $y < \bar{y}$, and with any probability $x \in [0, 1]$ if $y = \bar{y}$. This behavior could be micro-founded. Suppose a criminal benefits $b > 0$ from a crime but pays the penalty $\gamma \geq b$ when apprehended; the assumed behavior emerges with a threshold $\bar{y} := b/\gamma$.[13] To

[12]Insufficient crime reporting by victims and community is a real concern. The willingness by individuals to notify policy of crimes is in decline and at an all-time low in 2020 since 1993, in part due to mistrust of the police: in 2020, less than 43% of non-lethal crimes and less than 32% of property crimes were reported to the police (Xie, Ortiz Solis, and Chauhan (2024)).

[13]Note that criminals are myopic here. This is because there is a continuum of individuals who may receive a crime opportunity; any effect of a crime decision by an individual on future enforcement will be irrelevant

the extent that the PM can choose the criminal penalty, one can view the threshold $\bar{y}$ as a policy variable as well; in particular, a decrease in $\bar{y}$ captures the effect of raising the criminal penalty γ.

With the endogenous crime model, the PM's ability to commit to her enforcement policy becomes important. If the PM can commit to her enforcement policy, she may choose the enforcement level slightly above $\bar{y}$ to deter crime completely. More precisely, a simple **full-commitment policy** is either to deter crime fully by enforcing arbitrarily above $\bar{y}$ or to never enforce, whichever is cheaper, which leads to the expected loss of $\min\{c\bar{y}, \pi_0\lambda\}/r$.[14] This commitment policy virtually eliminates loss if the PM could lower $\bar{y}$ to an arbitrarily small number by raising the criminal penalty γ, as suggested by Becker (1968). Hence, with commitment, the expected loss $\min\{\pi_0\lambda, c\bar{y}\}/r$ would vanish in the limit as $\bar{y} \to 0$.

In practice, however, such a policy requires a strong commitment power, which policy-makers typically do not possess. Accordingly, henceforth, we assume the PM can't commit to a future enforcement level y, meaning she can only choose the policy in response to the likely crime rate. Effectively, this feature transforms the bandit problem into a dynamic game in which crime and enforcement have a mutual feedback relationship. As before, we consider three different regimes, NP, GP, and OP, regarding the PM's access to and use of crime data.

Unlike the PM, criminals do not access past crime data, meaning, as in the previous section, the crime level x is a constant (rather than a function of p), except now that it is determined endogenously by the criminals' equilibrium incentive. (Later, we will discuss the implications of criminals gaining access to the same data as PM.)

Consistent with the exogenous crime case, we will focus on a Markov perfect equilibrium in which the PM's policy depends only on her posterior, the payoff-relevant state. Since criminals choose a constant probability $x \in [0, 1]$, the PM's best response follows the characterization of the previous section. The only difference is that criminals' equilibrium choice of x will depend on the PM's enforcement policy, thereby on the enforcement regimes. Given the endogeneity of x and the lack of commitment power by the PM, the OP is not guaranteed to be optimal or even to outperform the other two regimes.

We begin our analysis with the NP regime.

4.1 Non-predictive enforcement

In this regime, neither the criminals nor the PM updates the belief based on past history. Hence, the crime probability x and the enforcement level y are constant, determined solely

since the probability of receiving a crime opportunity in the future is zero.

[14] No enforcement results in $x = 1$, which entails the long-term loss of $\pi_0\lambda/r$.

based on the long-run prior π_0. The outcome then depends on whether λ is large relative to c for enforcement to be worthwhile for the PM.

Suppose first $\lambda\pi_0 \leq c$. Then, the crime opportunity arrives so rarely that even if the criminals choose $x = 1$, the PM does not find it optimal to allocate any positive y. So $(x, y) = (1, 0)$ is the only equilibrium. In the equilibrium, the PM's long-run loss equals $\lambda\pi_0/r$.

Suppose next $\lambda\pi_0 > c$. In this case, a classic *inspection game* ensues. It is well known that there is no pure strategy equilibrium. If criminals choose $x = 1$, then PM will choose $y = 1$. But then criminals would deviate to $x = 0$. The PM will, in turn, deviate to $y = 0$. Hence, the only equilibrium is in mixed strategies. In equilibrium, a criminal chooses x so that

$$\pi_0\lambda x = c, \tag{7}$$

which keeps the PM indifferent in her enforcement. In turn, the PM chooses $y = \bar{y}$, incentivizing criminals to randomize. The structure of this equilibrium is explained by the PM's lack of commitment power. Unable to commit to a deterring enforcement level, the PM requires crimes to occur on the path to exert an enforcement effort. The lack of commitment power also explains why the Beckerian prescription of lowering $\bar{y}$ (via raising the criminal penalty) does not work: as $\bar{y}$ falls, the enforcement falls, but the crime rate $\pi_0\lambda x$ remains unchanged, as shown in (7). Indeed, the PM's expected loss equals c/r regardless of $\bar{y}$ since the PM is indifferent to full enforcement.

We summarize the results.

Proposition 4. *NP admits a unique equilibrium in which:*

- *if* $\lambda \leq \frac{c}{\pi_0}$, *then* $(x_{NP}, y_{NP}) = (1, 0)$;
- *if* $\lambda > \frac{c}{\pi_0}$, *then* $(x_{NP}, y_{NP}) = (\frac{c}{\pi_0\lambda}, \bar{y})$.

Under NP, the expected loss for the PM is $\min\{\pi_0\lambda, c\}/r$ *and does not depend on* $\bar{y}$.

4.2 Criminals' belief formation under predictive policies

We next turn to GP and OP. Given an equilibrium crime level x, our analysis from Section 3 shows that the PM's best response is a cutoff policy in these two regimes. Fix such a policy with any cutoff $\hat{p}$: $y_{\hat{p}}(p) := 1_{\{p>\hat{p}\}} + z(\hat{p}) \cdot 1_{\{p=\hat{p}\}}$.

To understand criminals' incentives under such a policy, we must first study their beliefs about the enforcement level $\mathbb{E}_p[y_{\hat{p}}(p)|\omega = H, x]$. The expectation is conditional on state $\omega = H$

(which the criminals know upon receiving the crime opportunity) and the "equilibrium" crime level x (which influences the belief about the PM's posterior and, henceforth, the enforcement level). Since the enforcement level depends on the PM's belief p, the criminal must, in turn, form a belief about the relative likelihoods of alternative p's the PM may have. In the long run, the belief is given by the stationary distribution of the PM's belief.[15] We thus digress briefly on how the distribution is characterized. While the characterization is of independent interest, the reader interested in the belief formation may skip the segment and go to Section 4.2.2

4.2.1 Stationary distribution of the PM's belief.

To begin, fix a cutoff policy with cutoff $\hat{p} \in (\pi_1, \pi_0)$ and an equilibrium crime level x; the expected enforcement is rather trivial when $\hat{p}$ lies outside that interval. Let $\Phi : [\hat{p}, 1] \to [0, 1]$ denote the stationary distribution of the PM's belief p under that policy.[16] The distribution admits density $\phi(p)$ for each $p \in (\hat{p}, 1)$ and atom $m(\hat{p})$ at $\hat{p}$, which is intuitive since the belief spends "real" time at $\hat{p}$. In Appendix E, we show that ϕ and m respectively satisfy

$$\phi(p) = \phi(\hat{p}) \exp\left(\int_{\hat{p}}^{p} r(s)ds\right),$$

where $r(s) := -\frac{s\lambda x + f'(s,1)}{f(s,1)}$, and

$$m(\hat{p})z(\hat{p})\hat{p}\lambda x = -\phi(\hat{p})f(\hat{p}, 1).$$

The stationary distribution is derived from the "balance condition:" for each $p \in (\hat{p}, 1)$, the flow from $[\hat{p}, p]$ into $(p, 1]$ (namely, the flows corresponding to the jumps to 1) must balance the flow in the opposite direction (namely, the drift of belief across p from above).

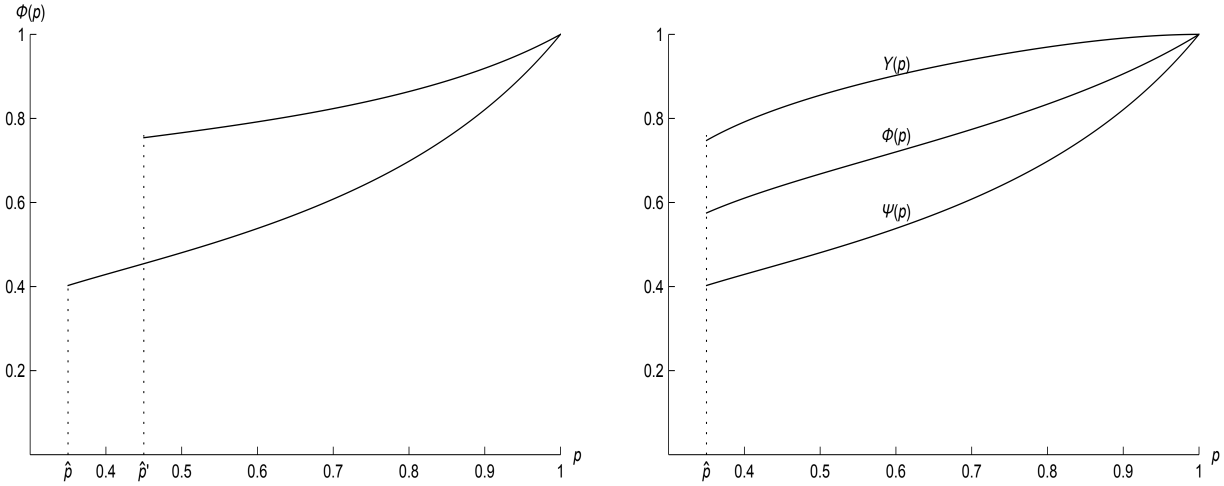
Figure 6(a) depicts the Φ under two cutoffs $\hat{p}$ and $\hat{p}' > \hat{p}$.

The policy with a lower cutoff involves more exploration. Hence, the stationary distribution corresponding to $\hat{p}$ is a mean-preserving spread of the one corresponding to $\hat{p}'$.[17]

[15] The formal justification can be given by the logic of "a Poisson arrival sees time average" (PASTA); see Wolff (1982). Intuitively, for a Poisson-arriving agent, his belief that p falls into some interval $(p', p'') \subset [\hat{p}, 1]$ equals the fraction of time p lies in that interval, once proper conditioning is made on the fact that state $\omega = H$.

[16] The stationary distribution exists and is unique. Note first that the stochastic process of the PM's belief is a Markov process: once any p (within the support) is reached, the future process is identical and depends only on p (regardless of the prior history). It is also positive-recurrent, with the process returning to the same belief in finite expected time. The regenerative nature of the process means that it admits a unique limit distribution forming a unique stationary distribution (see Asmussen (2003), p. 170, Theorem 1.2).

[17] This observation is proved in Lemma 15 in Appendix E.



(a) CDF Φ's under two cutoffs $\hat{p}$ and $\hat{p}' > \hat{p}$

(b) CDF's Φ, Υ, Ψ

Figure 6: Invariant distributions of posterior beliefs

Using the (unconditional) stationary distribution Φ, we can derive the long-run distribution of the beliefs in each state. The distributions of p conditional on states H and L are respectively

$$\Psi(p) := \frac{\hat{p}m(\hat{p}) + \int_{\hat{p}}^{p} \tilde{p}\phi(\tilde{p})d\tilde{p}}{\pi_0} \text{ and } \Upsilon(p) := \frac{(1-\hat{p})m(\hat{p}) + \int_{\hat{p}}^{p}(1-\tilde{p})\phi(\tilde{p})d\tilde{p}}{1-\pi_0}$$

if $p \geq \hat{p}$, and zero otherwise. Naturally, Ψ first-order stochastically dominates Φ, which in turn first-order stochastically dominates Υ.[18] Figure 6(b) depicts Φ, Ψ, and Υ, for a given cutoff policy. Observe that Ψ puts strictly lower mass at $\hat{p}$ than does Υ. To the extent that a cutoff policy enforces strictly less at $\hat{p}$ than at $p > \hat{p}$, the policy enforces more in state H than in L, revealing the sense of how the predictive policy can make enforcement effective (both in GP and OP).

The relevant belief the criminals use to evaluate the enforcement level is $\Psi(p)$, as they know that the state is H when making their crime decisions. We now turn to that decision.

4.2.2 Criminals' belief formation.

We now characterize criminals' beliefs about the enforcement level, under a predictive policy with cutoff $\hat{p}$. The answer is trivial if the cutoff lies outside the interval (π_1, π_0): as seen before, if $\hat{p} \geq \pi_0$, the long-run enforcement level is zero, and if $\hat{p} \leq \pi_1$, the PM chooses $y(p) = 1$ for

[18] This can be easily verified using the fact that $\hat{p} < \pi_0$.

all $p \geq \hat{p}$.

Hence, we assume $\hat{p} \in (\pi_1, \pi_0)$. In this case, the enforcement is binary: $y(p) = 1$ for $p > \hat{p}$ and $y(p) = z(p) \in (0,1)$ for $p = \hat{p}$. The preceding analysis suggests that criminals form the expectation of the enforcement using Ψ as the distribution. Let $\mu(\hat{p}) := \Psi(\hat{p})$ denotes the mass assigned to $\hat{p}$. Then, the conditional enforcement level that the criminals face is given by:

$$\mathbb{E}_{p\sim\Psi}[y_{\hat{p}}(p)] = \mu(\hat{p})z(\hat{p}) + (1 - \mu(\hat{p})) \cdot 1. \tag{8}$$

In Appendix E, we establish that $\mu(\hat{p})$ and $z(\hat{p})$ are respectively increasing and decreasing in $\hat{p}$. It then follows from (8) that the perceived enforcement level $\mathbb{E}_{p\sim\Psi}[y_{\hat{p}}(p)]$ is decreasing in $\hat{p}$. This is intuitive: a lower cutoff policy involves a higher expected enforcement level. We now show that for any x, there exists a unique cutoff $\hat{p} = \overline{p}(\lambda x)$ such that the associated cutoff policy leads to the deterring level of enforcement $\overline{y}$.

Lemma 1. *Given any $x \in (0,1]$, there exists a cutoff $\overline{p}(\lambda x) \in (\pi_1, \pi_0)$ such that the associated cutoff policy $y_{\hat{p}}$ with $\hat{p} = \overline{p}(\lambda x)$ gives rise to a conditional enforcement level of $\overline{y}$; i.e.,*

$$\mathbb{E}_{p\sim\Psi}[y_{\hat{p}}(p)] = \overline{y}. \tag{9}$$

Also, the "deterring" cutoff $\overline{p}(\lambda x)$ is continuous in x and increases in $\overline{y}$, with $\lim_{\overline{y}\to 0} \overline{p}(\lambda x) = \pi_0$ and $\lim_{\overline{y}\to 1} \overline{p}(\lambda x) = \pi_1$ for all x.

Proof. See Appendix E. □

This characterization is useful for our equilibrium analysis for GP and OP. Suppose, for instance, that the equilibrium has an interior crime level $x \in (0,1)$ in either the GP or OP regime. Then, criminals must be indifferent, meaning they anticipate the enforcement level of $\overline{y}$ upon receiving a crime opportunity. Lemma 1 then means that, for such an equilibrium to sustain, the PM must find it optimal to choose a cutoff $\overline{p}(\lambda x)$. This, in turn, pins down the equilibrium crime level x.

4.3 Greedy predictive enforcement

We are now ready to study the PM's behavior under GP. To this end, it is convenient to define $\lambda^M \in (\frac{c}{\pi_0}, \frac{c}{\pi_1})$ to be the smallest λ that satisfies

$$\hat{p}_M(\lambda) = \frac{c}{\lambda} = \overline{p}(\lambda). \tag{10}$$

If $\lambda = \lambda^M$, then given $x = 1$, the GP policy (with its myopic cutoff $\hat{p}_M(\lambda)$) satisfies (9) and implements the deterring enforcement level $\overline{y}$. Such a λ^M is well-defined.[19]

Suppose first $\lambda \leq \lambda^M$. Then, $\hat{p}_M(\lambda x) = \frac{c}{\lambda x} > \overline{p}(\lambda x)$ for all $x < 1$. In other words, the GP policy does not generate sufficient enforcement to deter crime. Hence, in equilibrium, criminals choose $x_{\text{GP}} = 1$, and the PM responds with the corresponding myopic cutoff $\hat{p}_M(\lambda) = \frac{c}{\lambda}$.

Suppose next $\lambda > \lambda^M$. In this case, we have a mixed strategy equilibrium in which criminals commit crimes with probability $x \in (0, 1)$. For them to randomize, the conditional expected enforcement level must be $\overline{y}$. This, in turn, requires the PM to find it optimal to choose $\overline{p}(\lambda x)$. Under GP, this requires that

$$\hat{p}_M(\lambda x) = \frac{c}{\lambda x} = \overline{p}(\lambda x), \tag{11}$$

which determines the equilibrium crime level x.

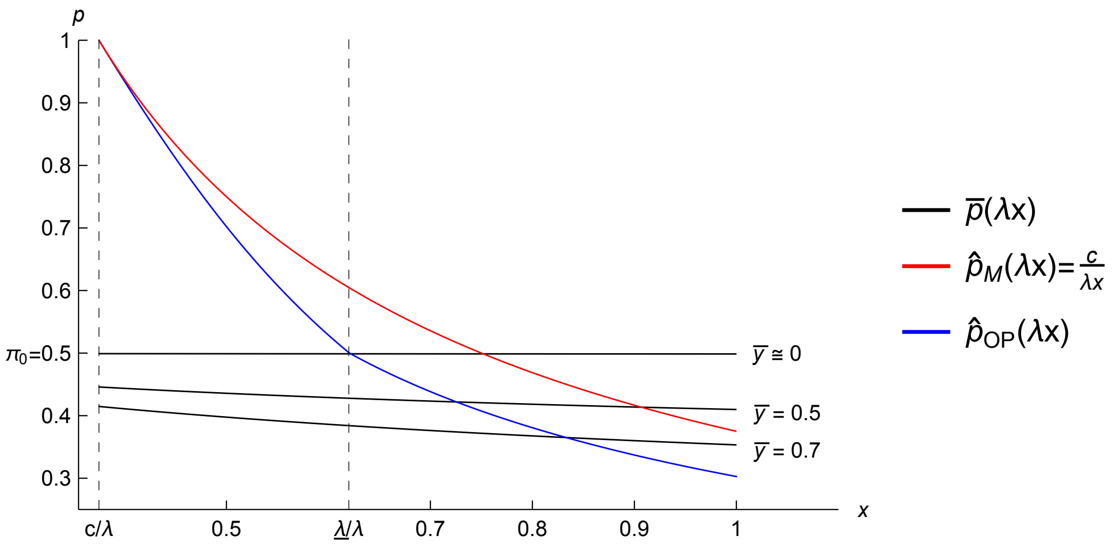


Figure 7: Equilibria under OP and GP with $\lambda = 4$, $c = \frac{3}{2}$, $r = 2$, and $\rho_L = \rho_H = 1$.

Figure 7 illustrates how the crime level is determined in equilibria for alternative deterrence levels $\overline{y}$. Fix a $\overline{y}$. The upper black curve depicts the deterring cutoffs $\overline{p}(\lambda x)$ over all possible x.[20] Meanwhile, the myopic cutoff $\hat{p}_M(\lambda x) = \frac{c}{\lambda x}$ is depicted by the red curve. The equilibrium condition (11) is satisfied at the intersection of the two curves.

[19]To see the existence of such λ^M, let us fix $x = 1$ and consider a cutoff policy with $\hat{p} = \hat{p}_M(\lambda) = \frac{c}{\lambda}$. With $\lambda = \frac{c}{\pi_0}$, we have $\hat{p}_M(\lambda) = \pi_0$ so $\mathbb{E}_{p\sim\Psi}[y_{\hat{p}}(p)] = 0 < \overline{y}$, implying $\overline{p}(\lambda) < \hat{p}_M(\lambda)$. With $\lambda = \frac{c}{\pi_1}$, we have $\hat{p}_M(\lambda) = \pi_1$ so $\mathbb{E}_{p\sim\Psi}[y_{\hat{p}}(p)] = 1 > \overline{y}$, implying $\hat{p}_M(\lambda) > \overline{p}(\lambda)$. Thus, there exists $\lambda \in (\frac{c}{\pi_0}, \frac{c}{\pi_1})$ such that $\hat{p}_M(\lambda) = \overline{p}(\lambda)$, by the continuity of $\hat{p}_M(\cdot)$ and $\overline{p}(\cdot)$, which further guarantees the existence of a smallest such λ.

[20] The curve $\overline{p}(\cdot)$ is negatively sloped for all numerical analyses we performed, and this is intuitive. Suppose

Note from (10) that $x = \lambda^M/\lambda$ solves (11). We thus have the following result:

Proposition 5. *GP admits an equilibrium in which:*

- *if* $\lambda \leq \lambda^M$, *then* $x_{GP} = 1 \geq \min\{1, \frac{c}{\pi_0 \lambda}\} = x_{NP}$, *and* $y_{GP}(\cdot)$ *is a cutoff with* $\hat{p} = \frac{c}{\lambda}$;
- *if* $\lambda > \lambda^M$, *then* $x_{GP} = \frac{\lambda^M}{\lambda} \in (0,1)$ *and* $y_{GP}(\cdot)$ *is a cutoff with* $\hat{p} = \frac{c}{\lambda^M}$.

Under GP, the expected loss for the PM is $\min\{\pi_0\lambda, c\}/r$ *and does not depend on* $\overline{y}$.

The payoff consequence (the last statement) follows from a simple observation. When $\pi_0\lambda \leq c$, the PM chooses zero enforcement in the long run, so the total loss is simply the loss from crime, $\pi_0\lambda/r$. If $\lambda \geq c/\pi_0$, the PM adopts a nontrivial cutoff policy, with a myopic cutoff given $x_{\text{GP}} = \min\{\frac{\lambda^M}{\lambda}, 1\}$. Since the PM is indifferent at the myopic cutoff, her payoff is the same as if she always enforces fully, so her total loss is c/r in this case. Note that the payoff of the PM does not depend on the deterring enforcement level $\overline{y}$.[21] While surprising at first glance, this irrelevance again follows from the PM's lack of commitment power.

Similar to the exogenous crime case, the equilibrium enforcement differs between NP and GP. If $\pi_0\lambda > c$, then the PM chooses $\overline{y}$ under NP; under GP, she alternates between full and partial enforcement which is on average less than $\overline{y}$.[22] At the same time, the crime level is generally higher under GP than under NP. This is because GP has a lower cutoff than π_0, which requires a higher crime rate to incentivize the PM to enforce under GP than under NP. As it turns out, however, these two opposing effects exactly cancel. A comparison with Proposition 4 establishes payoff equivalence:

Corollary 1 (Payoff equivalence between NP and GP). *The expected loss for the PM is the same between NP and GP.*

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x goes up. Then, the belief drifts faster to $\hat{p}$ (as $-f(p,1)$ increases), whereas the jump rate at $\hat{p}$ remains constant, since $z(\hat{p})$ falls such that $xz(\hat{p})$ remains constant. This means that $m(\hat{p})/\phi(\hat{p})$ must rise. As long as $\phi(\hat{p})$ does not fall drastically, then the overall enforcement must fall. For the conditional enforcement to remain at $\overline{y}$, the cutoff must be lowered to maintain the enforcement at $\overline{y}$. However, the behavior of $\phi(\hat{p})$ as x rises is ambiguous, so we are not able to prove that $\overline{p}(\cdot)$ is always negatively-sloped. Nevertheless, the equilibrium and welfare characterization below does not depend on this ambiguity.

[21] This is true even though as the deterring enforcement level $\overline{y}$ falls, presumably because the criminal penalty has increased, the equilibrium crime goes down, as illustrated in Figure 7. The reason is that the cutoff increases to such an extent that at the cutoff belief, the overall crime rate $\hat{p}_M(\lambda x_{\text{GP}})\lambda x_{\text{GP}}$ remains the same, namely at c.

[22] This can be seen as follows. If $\hat{p}_M(\lambda) < \overline{p}(\lambda)$, the expected enforcement *conditional on* $\omega = H$ is strictly less than $\overline{y}$. Since the expected enforcement is even less conditional on $\omega = L$, the unconditional expectation is even less than $\overline{y}$. If $\hat{p}_M(\lambda) \geq \overline{p}(\lambda)$, the expected enforcement conditional on $\omega = H$ equals $\overline{y}$, but it is strictly less than $\overline{y}$ *conditional on* $\omega = L$, so the unconditional expectation is even less than $\overline{y}$.

The intuition for this equivalence is the same as before: the information is obtained *only when enforcement is myopically optimal* and hence is redundant. To be specific, suppose $\pi_0\lambda > c$, or else the long-run enforcement decision is identical between the two regimes. In that case, the PM chooses full enforcement under NP, but posterior-dependent enforcement under GP, enforcing fully if $p > \hat{p}_M(\lambda x_{\text{GP}})$ and partially at $z(\hat{p}_M) < 1$ if $p = \hat{p}_M(\lambda x_{\text{GP}})$. Yet, whenever the enforcement is partial, the PM is indifferent, so she experiences the same expected loss.

4.4 "Optimal" predictive enforcement

We now study the Markov perfect equilibrium under OP.[23] From our analysis in Section 3, for any crime level x, the PM's optimal response is a cutoff policy, and the resulting (long-run) outcome depends on the optimal cutoff $\hat{p}(\lambda x)$. (Note we now make the dependence of $\hat{p}$ on λx explicit.)

To characterize the equilibrium, define $\lambda^* \in (\underline{\lambda}, \lambda^M)$ to be the smallest λ that satisfies

$$\hat{p}(\lambda) = \overline{p}(\lambda). \tag{12}$$

That is, if $\lambda = \lambda^*$, then, given $x = 1$, an optimal cutoff $\hat{p}$ yields a deterring enforcement $\overline{y}$. Such a λ^* is well defined.[24]

Suppose first $\lambda \leq \lambda^*$. Then, we have $\hat{p}(\lambda x) > \overline{p}(\lambda x)$ for all $x < 1$, meaning the enforcement will be insufficient to deter crimes. Hence, in equilibrium, criminals choose $x = 1$, and the PM optimally responds with a cutoff $\hat{p}(\lambda)$.

Suppose next $\lambda > \lambda^*$. Then, criminals must randomize in equilibrium. This requires that the associated enforcement level equals $\overline{y}$, which, in turn, requires that

$$\hat{p}(\lambda x) = \overline{p}(\lambda x), \tag{13}$$

which pins down the equilibrium crime level x_{OP}. Figure 7 depicts x_{OP} as an intersection of the optimal cutoff facing x (the blue curve) and the deterring-cutoff level $\overline{p}(x)$ (the black curve). It follows from (12) that $x = \lambda^*/\lambda$ solves this equation. Hence, in equilibrium, criminals choose $x_{\text{OP}} = \lambda^*/\lambda$ and the PM chooses a cutoff $\hat{p}(\lambda^*) = \overline{p}(\lambda^*)$, which yields the marginally

[23] As mentioned earlier, the moniker "optimal" only refers to the optimality of the PM's response to the anticipated crime level x.

[24] To see the existence of such λ^*, let us fix $x = 1$ and consider a cutoff policy with $\hat{p} = \hat{p}(\lambda)$. If $\lambda = \lambda^M$, then we have $\hat{p}(\lambda^M) < \frac{c}{\lambda^M}$, which implies by the definition of λ^M that $\mathbb{E}_{p\sim\Psi}[y_{\hat{p}}(p)] > \bar{y}$. If $\lambda = \underline{\lambda}$, then $\hat{p}(\lambda) = \pi_0$ so $\mathbb{E}_{p\sim\Psi}[y_{\hat{p}}(p)] = 0 < \bar{y}$. Then, there exists λ such that $\hat{p}(\lambda) = \overline{p}(\lambda)$, by the continuity of $\hat{p}(\cdot)$ and $\overline{p}(\cdot)$, which further guarantees the existence of a smallest such λ.

deterring enforcement $\overline{y}$ in expectation, thus justifying criminals' randomization. The OP equilibrium is characterized as follows:

Proposition 6. *The OP regime admits an equilibrium in which:*

- *if $\lambda \leq \lambda^*$, then $x_{OP} = x_{GP} = 1$, and $y_{OP}(\cdot)$ is a cutoff with $\hat{p} = \hat{p}(\lambda)$ which is less than the GP cutoff $\hat{p}_M(\lambda) = \frac{c}{\lambda}$;*
- *if $\lambda > \lambda^*$, then $x_{OP} = \frac{\lambda^*}{\lambda} < \min\{\frac{\lambda^M}{\lambda}, 1\} = x_{GP}$, and $y_{OP}(p)$ is a cutoff with $\hat{p} = \overline{p}(\lambda^*)$.*

The PM incurs a weakly (resp. strictly) lower expected loss under OP than under GP or NP (if $\lambda > \underline{\lambda}$).

It is instructive to compare the equilibrium outcomes of OP and GP. Figure 7 illustrates the comparison in a typical scenario. To begin, recall from Section 3 that, facing the same crime level x, OP prescribes higher enforcement (i.e., a lower cutoff) than GP. This is seen in Figure 7 by the fact that $\hat{p}(\lambda x)$ lies strictly below $\hat{p}_M(\lambda x)$ for each x. At the same time, the deterring cutoff $\overline{p}(\lambda x)$ is typically negatively sloped (recall Footnote 20). Consequently, OP leads to a higher cutoff and a lower crime level than GP.

Why does the crime level fall under OP? The answer is traced to the informational motive OP harnesses, which makes the PM more aggressive under OP than under GP, all else equal. In response, criminals adjust their crime level below what they would choose under GP. In short, the informational motive of the PM acts as a commitment to deter crime, at least partially. This explains why the expected loss is smaller under OP than the other two regimes. Specifically, if $\lambda \leq \underline{\lambda}$, all three regimes lead to no enforcement resulting in the (same) expected loss of $\pi_0\lambda/r$. If $\lambda > \underline{\lambda}$, the OP entails an expected loss strictly smaller than c/r.

The harnessing of the (partial) commitment also means that the Beckerian prescription is valuable under OP, unlike under NP or GP. To see this, assume $\lambda > \lambda^*$, and suppose the PM lowers $\overline{y}$ by raising the criminal penalty. Then, as depicted in Figure 7, the deterring cutoff shifts up. Since $\hat{p}(\cdot)$ is decreasing (recall Theorem 1), λ^* must fall. Consequently, the crime probability x_{OP} falls; intuitively, as the deterring enforcement level falls, it takes a lower crime rate to motivate the PM to enforce. Given a decreased x_{OP}, the PM's expected loss would fall if her enforcement policy were fixed; since she optimally adjusts her enforcement policy against the lowered x_{OP}, she must be even better off when $\overline{y}$ decreases.

Now let $\overline{y} \to 0$. As noted in Lemma 1, $\overline{p}(\lambda x) \to \pi_0$ for all x. Consequently, $\lambda^* \to \underline{\lambda}$, so the crime level converges to $\underline{\lambda}/\lambda$. Further, as $\overline{p}(\lambda x) \to \pi_0$, for all x, $z(\overline{p}(x)) \to 0$, and once the posterior reaches the cutoff $\hat{p}^*$, it stays there arbitrarily long. Consequently, Ψ converges to a degenerate distribution at π_0. Since the PM does virtually no enforcement at $\overline{p}(x_{\mathrm{OP}}) \approx \pi_0$

in the limit, her expected loss converges to $\pi_0\lambda(\underline{\lambda}/\lambda)/r = \pi_0\underline{\lambda}/r$. By contrast, as $\overline{y} \to 1$, $\overline{p}(x) \to \pi_1$, so the OP involves full enforcement in the limit, with expected loss tending to c/r, the same as NP and GP.

Corollary 2. *As $\overline{y}$ falls, the PM's expected loss under OP decreases. As $\overline{y} \to 1$, the loss converges to $\min\{\pi_0\lambda, c\}/r$, the same as that under NP and GP. As $\overline{y} \to 0$, the loss tends to $\min\{\pi_0\lambda, \pi_0\underline{\lambda}\}/r$.*

While the expected loss falls as $\overline{y}$ decreases, it does not vanish in the limit. This is because the commitment power enabled by OP is not complete. Even though the informational motive by the PM makes her enforcement more credible under OP than under NP or GP, as $\overline{y}$ decreases, a positive crime level is still required to make the enforcement credible even in the limit as $\overline{y} \to 0$. This contrasts with the simple full-commitment policy under which the expected loss of the PM vanishes as $\overline{y} \to 0$.

4.5 Predictive criminals

So far, we have assumed that criminals cannot act predictively in the way the PM can. This assumption is realistic in many regulatory contexts where enforcement agencies collect crime data privately and have an option not to publicly disclose that data. But in some other contexts, for instance, in the case of common crimes (e.g., street robbery, and drug-related offenses), the crime data may be widely available and updated reasonably frequently, and news media could also make the information more widely available. Even in other cases, potential offenders may become as savvy as enforcement agencies in making use of data.

In this subsection, we assume that criminals, not just the PM, access past crime data, which allows them to infer the PM's posterior belief p and respond accordingly to her decision at each p. Independently of the plausibility of this scenario, considering it will help us understand the role of the informational advantage PM has over potential criminals.

In the case of NP, since PM has no information other than π_0, criminals access no more information, so the analysis remains unchanged. However, unlike the case of non-predictive crimes, GP and OP are equivalent irrespective of λ. To see it, let $(y(p), x(p))$ denote the enforcement and crime levels in equilibrium for any posterior p in the support. In both regimes, it is a Markov perfect equilibrium for the PM to enforce at $y(p) = \overline{y}$ if and only if $p \geq \hat{p} = \frac{c}{\lambda}$. In response, potential criminals choose

$$x(p) = \frac{c}{p\lambda},$$

which keeps the PM indifferent, for each $p \geq \hat{p}$. Essentially, criminals and the PM play the "inspection game" equilibrium for each $p \geq \hat{p}$. Consequently, the expected total loss is the same across all three regimes.

Proposition 7. *If potential criminals base their decision on the posterior that PM accesses, all three regimes result in the same total expected loss.*

Proof. See Appendix B. □

This result, together with Corollary 1 and Proposition 6, suggests that the PM's informational advantage is important for her to realize the value of predictive enforcement.

5 Conclusion

The paper presents a model of predictive enforcement that highlights the dynamic relationship between enforcement actions and data collection. Our analysis demonstrates the potential of predictive methods to enhance enforcement decisions and deter crime. However, it also underscores the importance of how predictions are used to guide enforcement. A myopic approach that focuses solely on immediate enforcement objectives without considering the long-term informational implications can lead to suboptimal outcomes, potentially performing as poorly as if no data or predictions were used at all. This is particularly true in cases where prediction relies heavily on active surveillance or enforcement, as with victimless crimes. Furthermore, such a myopic approach can result in the over-policing of traditionally high-crime neighborhoods. Our findings thus offer a theoretical explanation for the concerns about discriminatory outcomes in predictive policing, even in the absence of inherent bias in the prediction algorithms. The paper also reveals that the enforcement agency's pursuit of information can bolster the credibility of enforcement efforts and contribute to increased deterrence.

Our model is intentionally simplified to highlight the fundamental dynamics at play and the key insights they generate. However, there are several promising avenues for future research. The model could be extended to encompass multiple neighborhoods, each with its own uncertain crime conditions. In the context of exogenous crime, the current one-armed bandit analysis might potentially be generalized using techniques similar to Gittens indices. However, such a generalization may prove more challenging for the endogenous crime case. Another worthwhile extension would involve incorporating nonlinear, such as convex, enforcement costs. This would likely lead to a gradual decrease in enforcement after each crime detection and could also result in GP outperforming NP, similar to the scenario with community reporting. The feedback loop between enforcement and learning would still adversely affect

GP, but its effects might be more nuanced. Another exciting direction for future research lies in bridging the gap between theory and practice in predictive enforcement. Empirical research in this area is still nascent, and our model could provide valuable theoretical perspectives to inform empirical investigations. In particular, it would be interesting to examine how the specific approach to balancing exploration and exploitation impacts data selection and enforcement outcomes, both in terms of efficiency and fairness.

While our analysis has focused primarily on efficiency as the sole objective, it's important to acknowledge that the use of predictive algorithms in law enforcement raises significant concerns about fairness and potential bias. Future research could explore modifications to enforcement strategies, such as GP, that explicitly incorporate fairness constraints across different target groups or neighborhoods. Our findings suggest that such an approach might not only promote equitable outcomes but could also yield unintended efficiency benefits. However, the specific outcomes would depend on the precise algorithmic implementation of fairness considerations. The exploration of fairness, along with the other extensions discussed previously, represents a fertile ground for future research, promising to deepen our understanding of the complex interplay between prediction, enforcement, deterrence and societal well-being.

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A Proof of Theorem 1

A.1 Preliminary Results for the Analysis of HJB Equation

This section provides the analysis of HJB equation (5) via a couple of lemmas. The proofs for all the results in this section, except for Proposition 8 and Proposition 9, are contained in the Online Appendix.

Given the function $f(p, y)$ that governs the evolution of PM's belief, let

$$g(p) := f(p, 1) = \rho_L(1-p) - \rho_H p - p(1-p)\lambda x \text{ and } h(p) := f(p, 0) = \rho_L(1-p) - \rho_H p.$$

Observe that $h(\pi_0) = 0$ and that $h(p)$ is strictly decreasing so $h(p) > (<)0$ for $p < (>)\pi_0$. Meanwhile, π_1 is a unique root of $g(p)$ in the interval $[0, 1]$ such that $g(p) > (<)0$ for $p < (>)\pi_1$.

Lemma 2. *$\pi_1 < \pi_0$ while π_1 decreases from π_0 to 0 as λx increases from 0 to ∞.*

Given the cutoff policy and the definitions of $g(p)$ and $h(p)$, (5) becomes

$$rV(p) = \begin{cases} h(p)V'(p) & \text{for } p < \hat{p} \quad (14) \\ p\lambda x - c + p\lambda x\,[V(1) - V(p)] + g(p)V'(p) & \text{for } p \geq \hat{p}. \quad (15) \end{cases}$$

Let us parametrize the ODE in (15) as follows:

$$D[K] \qquad rV(p) = p\lambda x - c + p\lambda x\,[K - V(p)] + g(p)V'(p) \text{ with } V(1) = K. \tag{16}$$

The following result provides a set of useful properties for a (unique) solution to $D[K]$:

Lemma 3. *For any given $\varepsilon > 0$, $D[K]$ admits a unique solution over $[\pi_1 + \varepsilon, 1]$, denoted $V(p;K)$, which satisfies the following properties:*

(i) $V(p;K)$ and $V'(p;K)$ are continuous in λ, x, and c as well as p and K;

(ii) With $K = \underline{K} := \frac{(\lambda x - c)(r+\rho_L) - c\rho_H}{r(r+\rho_L+\rho_H)}$, $V(p;K)$ is linear with the slope equal to $\frac{\lambda x}{r+\rho_L+\rho_H}$;

(iii) $V'(p;K)$ is strictly decreasing in K while $V(p;K)$ is strictly increasing in K;

(iv) $V''(p;K)$ is strictly increasing in K while $V''(p;K) \gtreqless 0$ if $K \gtreqless \underline{K}$;

A.2 Conditions for the Solution to HJB Equation

This section provides a set of conditions for the solution to HJB equation (5) via a series of lemmas. Let $W(p,y)$ denote the maximand of (5) and define

$$\Delta(p) := \frac{1}{x}\frac{\partial W(p,y)}{\partial y} = p\lambda - \frac{c}{x} + p\lambda\,[V(1) - V(p) - (1-p)V'(p)]\,. \tag{17}$$

First of all, we require the value matching and smooth pasting: that is, $V(\hat{p}_-) = V(\hat{p}_+)$ and $V'(\hat{p}_-) = V'(\hat{p}_+)$. These conditions are related to the function Δ as follows:

Lemma 4. *(i) If the value matching holds, then the smooth pasting is equivalent to $\Delta(\hat{p}_+) = 0$ unless $\hat{p} = \pi_0$;*

(ii) If the smooth pasting holds, then the value matching is equivalent to $\Delta(\hat{p}_+) = 0$.

The optimality of cutoff policy $y(p) = 1_{\{p \geq \hat{p}\}}$ also requires that $\frac{\partial W(p,y)}{\partial y}$ be 'single-crossing' at $p = \hat{p}$: that is, $\frac{\partial W(p,y)}{\partial y} \leq (\geq) 0$ if $p \leq (\geq) \hat{p}$. Given that $\Delta(p) = \frac{1}{x}\frac{\partial W(p,y)}{\partial y} = 0$ at $p = \hat{p}$ (as shown in Lemma 4), this requires $\Delta'(\hat{p}_-) \geq 0$ and $\Delta'(\hat{p}_+) \geq 0$, which leads to:

Lemma 5. *Letting*

$$\sigma(p) := \frac{h(p)c}{p^2(1-p)\lambda x(r+\rho_L+\rho_H)},$$

we must have

$$\Delta'(\hat{p}_-) = \frac{\hat{p}(1-\hat{p})\lambda(r+\rho_L+\rho_H)}{h(\hat{p})}\left[\sigma(\hat{p}) - V'(\hat{p})\right] \geq 0 \text{ for } \hat{p} \neq \pi_0 \tag{18}$$

and

$$\Delta'(\hat{p}_+) = \frac{\hat{p}(1-\hat{p})\lambda(r+\rho_L+\rho_H)}{g(\hat{p})}\left[\sigma(\hat{p}) - V'(\hat{p})\right] \geq 0 \text{ for } \hat{p} \neq \pi_1. \tag{19}$$

As it turns out, the constraints in (18) and (19) are binding in the case $\hat{p} \in (\pi_1, \pi_0)$ (that is, Case 3). To see it, observe that if $\hat{p} \in (\pi_1, \pi_0)$, then we have $g(\hat{p}) < 0 < h(\hat{p})$. Thus, the inequalities in (18) and (19) can hold simultaneously only if $V'(\hat{p}) = \sigma(\hat{p})$:

Lemma 6. *If $\hat{p} \in (\pi_1, \pi_0)$, then $\Delta'(\hat{p}) = 0$ or equivalently*

$$V'(\hat{p}) = \sigma(\hat{p}) = \frac{h(\hat{p})c}{\hat{p}^2(1-\hat{p})\lambda x(r+\rho_L+\rho_H)}. \tag{20}$$

Let

$$\overline{\lambda} := \begin{cases} \frac{c(\rho_L+\rho_H-c)}{(\rho_L-c)} & \text{if } \rho_L - c > 0 \\ \infty & \text{otherwise.} \end{cases} \tag{21}$$

Lemma 7. *(i) There exists a unique $\underline{p} < \pi_0$ such that*

$$\sigma(\underline{p}) = \frac{\lambda x}{r+\rho_L+\rho_H}. \tag{22}$$

(ii) The following inequalities are equivalent: (a) $\lambda x \geq \overline{\lambda}$; (b) $\pi_1 \geq \hat{p}_M$; (c) $\pi_1 \geq \underline{p}$; (d) $\underline{p} \geq \hat{p}_M$.

The next two lemmas provide some conditions for Case 1 and Case 2:

Lemma 8. *$V(p) = V'(p) = 0$ for all $p \in [0, \hat{p}]$ if and only if $\hat{p} \geq \pi_0$.*

Lemma 9. *$V(p)$ is linear on $[\hat{p}, 1]$ with $\hat{p} = \hat{p}_M$ and V taking the form in Lemma 3-(ii) if and only if $\hat{p} \leq \pi_1$. This case arises if and only if $\lambda x \geq \overline{\lambda}$.*

In all the cases not covered in Lemma 8 and Lemma 9, V is strictly convex:

Lemma 10. *$V''(p) > 0$ if $p \leq \hat{p} < \pi_0$ or $p \geq \hat{p} > \pi_1$.*

A.3 Analysis of HJB Equation for $p \geq \hat{p}$

Solving for the HJB equation requires finding the value function that solves (14) and (15) and makes the policy function $y(p) = 1_{\{p \geq \hat{p}\}}$ optimal. To this end, we look for a couple $(\hat{p}, \hat{K})$ such that the solution to $D[\hat{K}]$ in (16) satisfies $\Delta(\hat{p}_+) = 0$ as well as the necessary conditions on $V'(\hat{p}_+)$ identified in Lemma 6, Lemma 8, or Lemma 9 depending on (λ, x, c).

A.3.1 Case 2: $\lambda x \geq \overline{\lambda}$

Proposition 8. *For $\lambda x \geq \overline{\lambda}$, there exists a unique pair $(\hat{p}, \hat{K})$ with $\hat{p} = \hat{p}_M \leq \pi_1$ and $\hat{K} = \underline{K}$ such that the solution to $D[\hat{K}]$ takes the form in Lemma 3-(ii) and satisfies $\Delta(\hat{p}_+) = 0$, where $\hat{p} = \hat{p}_M = \pi_1$ if and only if $\lambda x = \overline{\lambda}$.*

Proof. The result follows from Lemma 10. That $\Delta(\hat{p}_+) = 0$ follows from (17), using the fact that $\hat{p} = \hat{p}_M$ and V is linear on $[\hat{p}, 1]$. The last part of the statement follows from Lemma 7-(ii). □

A.3.2 Case 3: $\lambda x \in (\underline{\lambda}, \overline{\lambda})$

In this section, we identify $\underline{\lambda} < \overline{\lambda}$ such that Case 3 arises if and only if $\lambda x \in (\underline{\lambda}, \overline{\lambda}$. To do so, we show that if $\lambda x \in (\underline{\lambda}, \overline{\lambda})$, then there is a solution to $D[\hat{K}]$ with some pair $\hat{K}$ and $\hat{p} \in (\pi_1, \pi_0)$ that satisfies (20) and $\Delta(\hat{p}_+) = 0$.

First, we establish a couple of lemmas, whose proofs are in the Online Appendix.

Lemma 11. *With $K = \underline{K}$, $D[K]$ has a solution over the interval $[\underline{p}, 1]$ that satisfies $V'(p; K) = \frac{\lambda x}{r+\rho_L+\rho_H} = \sigma(\underline{p}), \forall p \in [\underline{p}, 1]$.*[25] *Also, for any (λ, x, c) satisfying $\lambda x < \overline{\lambda}$, we have $\underline{p} \in (\pi_1, \min\{\hat{p}_M, \pi_0\})$ that is decreasing in λx and increasing in c.*

Lemma 12. *Consider any (λ_1, x_1, c_1) and (λ_2, x_2, c_2) with $\lambda_i x_i < \overline{\lambda}_i$ for $i = 1, 2$ such that $\lambda_1 x_1 \leq \lambda_2 x_2$ and $c_1 \geq c_2$ with at least one inequality being strict.*[26] *Let $\underline{p}_i$, $\underline{K}_i$, and $V_i(\cdot; K)$ denote $\underline{p}$, $\underline{K}$, and the solution to $D[K]$, respectively, corresponding to (λ_i, x_i, c_i). For any $K \geq \underline{K}_1$ and $p \geq \underline{p}_1$, we have $V_1'(p; K) < V_2'(p; K)$ and $V_1(p; K) > V_2(p; K)$.*

[25] Recall the definitions of $\underline{p}$ and $\underline{K}$ from (22) and Lemma 3-(ii), respectively.
[26] Here, $\overline{\lambda}_i$ is the value of $\overline{\lambda}$ corresponding to (λ_i, x_i, c_i).

Proposition 9. *There is some* $\underline{\lambda} \in (c, \frac{c}{\pi_0})$ *such that for* $\lambda x \in (\underline{\lambda}, \overline{\lambda})$*, there exists a unique pair* $\hat{K} > \underline{K}$ *and* $\hat{p} \in (\pi_1, \pi_0)$ *such that* $D[\hat{K}]$ *has a solution over* $[\hat{p}, 1]$ *satisfying*

$$V'(\hat{p}; \hat{K}) = \sigma(\hat{p}) \text{ and } \Delta(\hat{p}; \hat{K}) = 0. \tag{23}$$

Also, $\hat{p}$ *continuously increases in* c *and decreases in* λx *from* π_0 *to* π_1 *as* λx *increases from* $\underline{\lambda}$ *to* $\overline{\lambda}$.

Proof. To begin, let us plug $p = 1$ into (16) to obtain

$$rK = \lambda x - c - \rho_H V'(1; K) \text{ or } V'(1; K) = \frac{\lambda x - c - rK}{\rho_H}. \tag{24}$$

Define $\overline{K}$ to be such that $V'(1; \overline{K}) = 0$ in (24), that is,

$$\overline{K} := \frac{\lambda x - c}{r} > \frac{(\lambda x - c)(r + \rho_L) - c\rho_H}{r(r + \rho_L + \rho_H)} = \underline{K}.^{27} \tag{25}$$

Step 1: For any (λ, x, c) *satisfying* $\lambda x < \overline{\lambda}$*, there exists* $K_0 \in (\underline{K}, \overline{K})$ *such that for any* $K \in [\underline{K}, K_0]$*, there is a unique* $p(K) \in [\underline{p}, \pi_0]$ *that satisfies* $V'(p(K); K) = \sigma(p(K))$ *and is continuously increasing in* K*. Also,* $p(K_0) = \pi_0$ *and hence* $V'(\pi_0; K_0) = \sigma(\pi_0) = 0$ *while* $p(\underline{K}) = \underline{p}$ *and* $V'(\underline{p}; \underline{K}) = \sigma(\underline{p})$.

First, $V'(1; \overline{K}) = 0$ implies $V'(p; \overline{K}) < 0$ for all $p < 1$ since $V'(\cdot; \overline{K})$ is increasing by Lemma 3-(iv). Thus, $V'(\pi_0; \overline{K}) < \sigma(\pi_0) = 0$. Observe next that we have

$$V'(\pi_0; \underline{K}) = V'(\underline{p}; \underline{K}) = \sigma(\underline{p}) > \sigma(\pi_0),$$

where the equalities hold due to Lemma 11 and the inequality due to the fact that $\pi_0 > \underline{p}$ and σ is decreasing. Since $V'(\pi_0; \underline{K}) > \sigma(\pi_0) > V'(\pi_0; \overline{K})$ and since V' is continuously and strictly decreasing in K, there exists a unique $K_0 \in (\underline{K}, \overline{K})$ such that $V'(\pi_0; K_0) = \sigma(\pi_0) = 0$, meaning $p(K_0) = \pi_0$.

For any $K \in (\underline{K}, K_0)$, we have

$$\sigma(\underline{p}) = V'(\underline{p}; \underline{K}) > V'(\underline{p}; K) \text{ and } \sigma(\pi_0) = V'(\pi_0; K_0) < V'(\pi_0; K), \tag{26}$$

where the first equality follows from Lemma 11 while the inequalities holds due to Lemma 3-(iii). These two inequalities, together with the monotonicity of σ and V' with respect to p,

[27] Recall the definition of $\underline{K}$ from Lemma 3-(ii).

imply that there must be a unique $p(K) \in (\underline{p}, \pi_0)$ such that $V'(p(K); K) = \sigma(p(K))$. Note that (26) also implies $p(\underline{K}) = \underline{p}$ and $p(K_0) = \pi_0$.

That $p(K)$ is continuously increasing in K is straightforward from the fact that V' continuously increases in p and decreases in K while σ continuously decreases in p.

Step 2: $\Delta(p(K); K) < 0$ *for any* $K \leq \underline{K}$.

By Lemma 3-(ii), Lemma 3-(iii), and Step 1, we have for any $K \leq \underline{K}$ and $p > p(\underline{K}) = \underline{p}$,

$$V'(p; K) \geq V'(p; \underline{K}) = V'(\underline{p}; \underline{K}) = \sigma(\underline{p}) > \sigma(p),$$

which implies that $p(K) \leq \underline{p}$ since $V'(p(K); K) = \sigma(p(K))$. Thus, $\hat{p} \leq \underline{p} < \frac{c}{\lambda x}$. Then,

$$\begin{aligned}\Delta(p(K); K) &= p(K)\lambda - \tfrac{c}{x} + p(K)\lambda \left[V(1; K) - V(p(K); K) - (1 - p(K))V'(p(K); K)\right] \\ &< \lambda p(K) \left[V(1; K) - V(p(K); K) - (1 - p(K))V'(p(K); K)\right] \leq 0,\end{aligned}$$

where the second inequality follows from Lemma 3-(iv).

Step 3: $\Delta(p(K); K)$ *is strictly increasing in* $K \geq \underline{K}$.

Note first that using $g(p) = h(p) - \lambda x p(1 - p)$, we can rewrite (15) as

$$rV(p) = h(p)V'(p) + x\Delta(p) \text{ for } p \geq \hat{p}. \tag{27}$$

By (27) and the definition of $p(K)$ from Step 3, we have

$$\begin{aligned}x\Delta(p(K); K) &= rV(p(K); K) - h(p(K))V'(p(K); K) \\ &= rV(p(K); K) - h(p(K))\sigma(p(K)).\end{aligned} \tag{28}$$

To prove the desired result, consider any $K_2 > K_1 \geq \underline{K}$ so that $p(K_2) > p(K_1)$. Observe that $h(p(K_1))\sigma(p(K_1)) > h(p(K_2))\sigma(p(K_2))$ since both h and σ are decreasing. Observe also that, by Lemma 3-(iv), $V'(p; K_1) \geq V'(p(K_1); K_1) = \sigma(p(K_1)) \geq 0, \forall p \in [p(K_1), p(K_2)]$, which implies that $V(p(K_2); K_1) \geq V(p(K_1); K_1)$. By these observations and (28),

$$\begin{aligned}x\Delta(p(K_1); K_1) &= rV(p(K_1); K_1) - h(p(K_1))\sigma(p(K_1)) \\ &< rV(p(K_2); K_1) - h(p(K_2))\sigma(p(K_2)) \\ &< rV(p(K_2); K_2) - h(p(K_2))\sigma(p(K_2)) = x\Delta(p(K_2); K_2),\end{aligned}$$

where the second inequality follows from Lemma 3-(iii).

Step 4: Any $\lambda x \in [\frac{c}{\pi_0}, \overline{\lambda})$ *admits a unique pair* $(\hat{p}, \hat{K})$ *with* $\hat{K} > \underline{K}$ *and* $\hat{p} \in (\pi_1, \pi_0)$ *satisfying* (23)

Observe first that $p(K_0) = \pi_0 \geq \frac{c}{\lambda x}$ or $\lambda p(K_0) \geq \frac{c}{x}$, so

$$\Delta(p(K_0); K_0) \geq p(K_0)\lambda\left[V(1; K_0) - V(\pi_0; K_0) - (1-\pi_0)V'(\pi_0; K_0)\right] > 0,$$

where the inequality is due to the strict convexity of V as shown in Lemma 3-(iv) with $K_0 > \underline{K}$. By this inequality and Step 2, we have $\Delta(p(\underline{K}); \underline{K}) < 0 < \Delta(p(K_0); K_0)$. Given this, the continuity and strict monotonicity of $\Delta(p(K), K)$ with respect to K imply that there exists a unique $\hat{K} \in (\underline{K}, K_0)$ such that $\Delta(p(\hat{K}), \hat{K}) = 0$ and $p(\hat{K}) \in (\underline{p}, \pi_0)$. Note also that $V'(p(\hat{K})) = \sigma(p(\hat{K}))$ by definition of $p(K)$. Given Step 2, the strict monotonicity of $\Delta(p(K); K)$ for $K \geq \underline{K}$ implies that a pair $(\hat{p}, \hat{K})$ satisfying (23) is unique. That $\hat{p} > \pi_1$ follows from the fact that $\hat{p} = p(\hat{K}) > \underline{p}$ and $\underline{p} > \pi_1$ if $\lambda x < \overline{\lambda}$ by Lemma 7-(ii).

Step 5: Consider any (λ_i, x_i, c_i), $i = 1, 2$, *with* $\lambda_i x_i < \overline{\lambda}_i$ *such that* $\lambda_1 x_1 \leq \lambda_2 x_2$ *and* $c_1 \geq c_2$ *with at least one inequality being strict. If* (λ_1, x_1, c_1) *admits a pair* $(\hat{p}_1, \hat{K}_1)$ *with* $\hat{p}_1 \in (\pi_1, \pi_0)$ *satisfying* (23), *so does* (λ_2, x_2, c_2) *with* $\hat{p}_2 < \hat{p}_1$.[28]

Let us use the subscript i to denote functions or variables associated with (λ_i, x_i, c_i).

Consider a pair $(\hat{p}_1, \hat{K}_1)$ solving (23) under (λ_i, x_i, c_i). Observe first that by Lemma 12, $V_2'(\hat{p}_1; \hat{K}_1) > V_1'(\hat{p}_1; \hat{K}_1) = \sigma_1(\hat{p}_1) > \sigma_2(\hat{p}_1)$ (where the second inequality holds since σ is decreasing in λx and increasing in c). Also, $V_2'(\hat{p}_1; \overline{K}_2) < 0$ (recall the argument from Step 1). Thus, one can find $K \in (\hat{K}_1, \overline{K}_2)$ such that $V_2'(\hat{p}_1; K) = \sigma_2(\hat{p}_1)$, which means $\hat{p}_1 = p_2(K)$. Note that $\hat{p}_1 > \underline{p}_1 > \underline{p}_2$, which implies $\hat{K}_2 > \underline{K}_2$ since $p_2(\underline{K}_2) = \underline{p}_2$ and $p_2(\cdot)$ is increasing. Next, we show that $\Delta_2(p_2(K); K) = \Delta_2(\hat{p}_1; K) > 0$. Consider first the case in which $V_2(\hat{p}_1; K) > V_1(\hat{p}_1; \hat{K}_1)$. Using (28) and the fact that $\sigma_1(\hat{p}_1) > \sigma_2(\hat{p}_1)$, we obtain

$$0 = x_1\Delta_1(\hat{p}_1; \hat{K}_1) = rV_1(\hat{p}_1; \hat{K}_1) - h(\hat{p}_1)\sigma_1(\hat{p}_1) < rV_2(\hat{p}_1; K) - h(\hat{p}_1)\sigma_2(\hat{p}_1) = x_2\Delta_2(\hat{p}_1; K).$$

Consider next the case in which $V_2(\hat{p}_1; K) \leq V_1(\hat{p}_1; \hat{K}_1)$. Using (17), we again obtain

$$\begin{aligned}0 = x_1\Delta_1(\hat{p}_1; \hat{K}_1) &= \hat{p}_1\lambda_1 x_1 - c_1 + \hat{p}_1\lambda_1 x_1\left[V_1(1; \hat{K}_1) - V_1(\hat{p}_1; \hat{K}_1) - (1-\hat{p}_1)V_1'(\hat{p}_1; \hat{K}_1)\right] \\ &< \hat{p}_1\lambda_2 x_2 - c_2 + \hat{p}_1\lambda_1 x_1\left[V_2(1; K) - V_2(\hat{p}_1; K) - (1-\hat{p}_1)V_2'(\hat{p}_1; K)\right] \\ &< \hat{p}_1\lambda_2 x_2 - c_2 + \hat{p}_1\lambda_2 x_2\left[V_2(1; K) - V_2(\hat{p}_1; K) - (1-\hat{p}_1)V_2'(\hat{p}_1; K)\right] = x_2\Delta_2(\hat{p}_1; K),\end{aligned}$$

where the first inequality holds due to the fact that $V_1(1; \hat{K}_1) = \hat{K}_1 < K = V_2(1; K)$ and

[28] Note that the dependence of π_1 on (λ_i, x_i, c_i) is suppressed here for simplicity.

$V_1'(\hat{p}_1; \hat{K}_1) = \sigma_1(\hat{p}_1) > \sigma_2(\hat{p}_1) = V_2'(\hat{p}_1; K)$. The second inequality holds since the expression in the square brackets is positive due to the convexity of $V_2(\cdot; K)$. Recalling $\hat{p}_1 = p_2(K)$, we have thus proven $\Delta_2(\hat{p}_1; K) = \Delta_2(p_2(K); K) > 0 > \Delta_2(p_2(\underline{K}_2); \underline{K}_2)$, which implies by Step 3 that there exists a unique $\hat{K}_2 \in (\underline{K}_2, K)$ such that $\Delta(p_2(\hat{K}_2); \hat{K}_2) = 0$. Moreover, $\hat{p}_2 = p_2(\hat{K}_2) < p_2(K) = \hat{p}_1$ by the monotonicity of $p_2(\cdot)$.

Step 6: With $\lambda x = c$, there exists no pair $(\hat{p}, \hat{K})$ satisfying (23).

Given Step 2 and Step 3, it suffices to show that with $\lambda x = c$, $\Delta(p(K_0); K_0) < 0$, since it will imply $\Delta(p(K); K) < 0, \forall K \in [\underline{K}, K_0]$. Recall that $p(K_0) = \pi_0$. Then, we have $\sigma(p(K_0)) = 0 = V'(p(K_0); K_0)$. Using this and (16), we also have $V(p(K_0); K_0) = \frac{\pi_0 \lambda x(1+K_0)-c}{r+p(K_0)\lambda x}$. Plugging these into the definition of Δ and rearranging, we obtain

$$\Delta(p(K_0); K_0) = \frac{r\left(\pi_0 \lambda x(1+K_0) - c\right)}{x\left(r + \pi_0 \lambda x\right)} < \frac{r\left(\lambda x\left(1 + \frac{\lambda x - c}{r}\right) - c\right)}{x\left(r + \pi_0 \lambda x\right)} = \frac{(\lambda x - c)(r + \lambda x)}{x\left(r + \pi_0 \lambda x\right)} = 0,$$

where the first inequality holds since $\pi_0 < 1$ and $K_0 < \overline{K} = \frac{\lambda x - c}{r}$.

Step 7: There is $\underline{\lambda} \in (c, \frac{c}{\pi_0})$ such that any $\lambda x \in (\underline{\lambda}, \overline{\lambda})$ admits a unique pair $(\hat{p}, \hat{K})$ with $\hat{p} \in (\pi_0, \pi_1)$ satisfying (23).

This statement follows immediately from combining Step 2 to Step 6.

The proofs for the next two steps are in the Online Appendix.

Step 8: With $\lambda x = \underline{\lambda}$, $(\hat{p}, \hat{K})$ satisfies (23) if and only if $(\hat{p}, \hat{K}) = (\pi_0, K_0)$. Also, $\hat{p}$ increases to π_0 as λx decreases to $\underline{\lambda}$.

Step 9: $\hat{p}$ decreases to π_1 as λx increases to $\overline{\lambda}$. □

A.3.3 Case 1: $\lambda x \in (c, \underline{\lambda}]$

Proposition 10. *For any (λ, x, c) with $\lambda x \in (c, \underline{\lambda}]$, there exists a unique pair $\hat{K} > \underline{K}$ and $\hat{p} \in [\pi_0, 1)$ such that $D[\hat{K}]$ has a solution over $[\hat{p}, 1]$ satisfying*

$$V'(\hat{p}_+; \hat{K}) = 0 = \Delta(\hat{p}_+; \hat{K}). \tag{29}$$

Also, $\hat{p}$ continuously decreases in λx and increases in c.

A.4 Comparative Statics for $\hat{p}$ and Verification

Proposition 11. *Consider $\overline{\lambda}$, $\underline{\lambda}$, and $\hat{p}$ identified in Proposition 8 to Proposition 10.*

(i) $\hat{p} \le \hat{p}_M = \frac{c}{\lambda x}$*, where the inequality is strict if and only if* $\hat{p} > \pi_1$*.*

(ii) $\overline{\lambda}$ *and* $\underline{\lambda}$ *are increasing in c;*

(iii) $\hat{p}$ *is continuously decreasing in* λx *and continuously increasing in c.*

With the solution to $D[\hat{K}]$ obtained in Appendix A.3 for the range $[\hat{p}, 1]$, we can extend it to the interval $[0, \hat{p}]$ by solving (14) with the boundary condition $V(\hat{p}_-) = V(\hat{p}_+)$. Given the value function thus obtained, the policy function $y(p) = 1_{\{p \ge \hat{p}\}}$ is optimal:

Proposition 12. *Given the value function V obtained so far, the policy function* $y(p) = 1_{\{p \ge \hat{p}\}}$ *maximizes* $W(p, y)$ *(which is the maximand in (5)).*

B Proof of Proposition 7

First of all, the PM's equilibrium payoff under NP is equal to $\frac{\min\{c, \pi_0 \lambda\}}{r}$.

Let us analyze the case of OP to construct an equilibrium that yields the same payoff for PM as under NP. Given the criminal's response $x(p)$, the PM's problem is to solve

$$rL(p) = \min_y \Big\{ p\lambda x(p)(1-y) + cy + p\lambda x(p) y \left[L(1) - L(p)\right] + \Big(\rho_L(1-p) - \rho_H p - p(1-p)\lambda x(p) y\Big) L'(p) \Big\}. \quad (30)$$

By extending Lemma 13 to the current case, solving (30) is equivalent to solving

$$rV(p) = \max_y \Big\{ \left(p\lambda x(p) - c\right) y + p\lambda x(p) y \left[V(1) - V(p)\right] + \Big(\rho_L(1-p) - \rho_H p - p(1-p)\lambda x(p) y\Big) V'(p) \Big\}. \quad (31)$$

We look for an equilibrium in which PM employs the following "cutoff" policy: $y(p)$ is positive if and only if $p \ge \hat{p}$ for some $\hat{p}$. Given the criminals' behavior, neither $y(p) \in (0, \bar{y})$ nor $y(p) > \bar{y}$ can be optimal for any belief $p \ge \hat{p}$. Thus, it must be that $y(p) = \bar{y} 1_{\{p \ge \hat{p}\}}$. The criminals' response $x(p)$ should be given such that PM finds it optimal to choose $y = 0$ if $p < \hat{p}$ and choose any $y \in [0, 1]$ if $p \ge \hat{p}$. Thus, by differentiating the maximand in (31) with y, we must have

$$(p\lambda x(p) - c) + p\lambda x(p) \left[V(1) - V(p) - (1-p)V'(p)\right] \le (=) 0 \text{ (if } p \ge \hat{p}). \quad (32)$$

Consider the following strategy for PM and criminals: with $\hat{p} = \frac{c}{\lambda}$, $y(p) = \bar{y}1_{\{p \geq \hat{p}\}}$ while $x(p) = 1$ if $p < \hat{p}$ and $x(p) = \frac{c}{p\lambda}$ if $p \geq \hat{p}$. Plugging this into (31), it is straightforward to verify that $V(p) = 0, \forall p \in [0, 1]$. Given $V(p)$ and $x(p)$, the LHS of (32) is zero if $p \geq \hat{p}$ and negative otherwise, as desired. Clearly, the criminals' response to the PM's policy is also optimal at each $p \in [0, 1]$.

To evaluate the PM's total loss in the long run, consider first the case with $\hat{p} = \frac{c}{\lambda} \geq \pi_0$. In this case, the long-run belief absorbs into π_0 for sure. Since $\pi_0 \leq \hat{p}$, $x(\pi_0) = 1$ and $y(\pi_0) = 0$. Thus, at $p = \pi_0$, (30) becomes

$$rL(\pi_0) = \pi_0\lambda + \big(\rho_L(1 - \pi_0) - \rho_H\pi_0\big)L'(\pi_0) = \pi_0\lambda$$

or $L(\pi_0) = \frac{\pi_0\lambda}{r}$.

Consider next the case with $\hat{p} = \frac{c}{\lambda} < \pi_0$. In this case, the belief cycles within the interval $[\max\{\hat{p}, \pi_1\}, 1]$. Recall that for $p \geq \hat{p}$, any $y \in [0, 1]$ is optimal for PM while $x(p) = \frac{c}{p\lambda}$. Thus, for any $p \geq \hat{p}$, we can plug $x(p) = \frac{c}{p\lambda}$ and $y = 0$ into (30) to obtain

$$rL(p) = c + \big(\rho_L(1 - p) - \rho_H p\big)L'(p).$$

It is straightforward to see that this ODE has a (unique) solution, $L(p) = \frac{c}{r}$ for all $p \geq \hat{p}$. In sum, we have proven that PM's (expected) total loss is equal to $\frac{\min\{c, \pi_0\lambda\}}{r}$.

Note that the PM's policy under OP is also her myopic best response. Thus, the equilibrium we just described for OP constitutes an equilibrium for GP as well, completing the proof.

Online Appendix for:
Predictive Enforcement

C Analysis for Section 3

C.1 Omitted Proofs for Lemmas

Lemma 13. *Any policy $y(\cdot)$ solves (5) if and only if it minimizes the loss function in (1).*

Proof. Let $L_y(p)$ denote the total discounted loss of PM with belief $p_t = p$ who chooses any enforcement policy $y : [0,1] \to [0,1]$. For a short duration $dt > 0$, we can write

$$L_y(p) = \big[p\lambda x(1 - y(p)) + cy(p)\big]dt + p\lambda xy(p)dt(1 - rdt)L_y(1) \\ + (1 - rdt)(1 - p\lambda xy(p)dt)L_y(p_{t+dt}) + o(dt).$$

By taking a limit as $dt \to 0$ and applying (2), we obtain:

$$rL_y(p) = p\lambda x(1 - y(p)) + cy(p) + p\lambda xy(p)(L_y(1) - L_y(p)) + L_y'(p)f(p, y(p)).$$

Hence, we obtain the following HJB equation:

$$rL(p) = \min_{y\in[0,1]} p\lambda x(1 - y) + cy + p\lambda xy(L(1) - L(p)) + L'(p)f(p, y). \tag{SA.1}$$

Let us next derive an ODE for the total (expected) discounted sum of both deterred and undeterred crimes, denoted $C(p)$, when the probability of $\omega = H$ at t is p. Since whether a crime occurs is independent of the PM's policy, we have for any $y \in [0,1]$,

$$\begin{aligned} C(p) =& p\lambda xdt + p\lambda xydt(1 - rdt)C(1) + (1 - rdt)(1 - p\lambda xydt)C(p_{t+dt}) \\ =& p\lambda xdt + p\lambda xydt(1 - rdt)C(1) + (1 - rdt)(1 - p\lambda xydt)\left(C(p) + f(p, y)C'(p)dt\right) \\ \implies rC(p) =& p\lambda x + p\lambda xy\left(C(1) - C(p)\right) + f(p, y)C'(p). \end{aligned}$$

Thus,

$$p\lambda xy\left(C(1) - C(p)\right) + f(p, y)C'(p) = f(p, 0)C'(p). \tag{SA.2}$$

Given any policy $y : [0,1] \to [0,1]$, let us define V_y analogously to L_y: that is, V_y is the total discounted value of deterred crimes minus enforcement costs associated with y. Note that for any policy y, we have $V_y(p) + L_y(p) = C(p)$ at each $p \in [0,1]$ since the terms of enforcement cost in V_y and L_y cancel out, so $V_y(\cdot)$ and $L_y(\cdot)$ add up to the total discounted sum of deterred and undeterred crimes.

Let y^* be the optimal policy that solves (5). Fix any $p \in [0,1]$ and let $z^* = y^*(p)$. Then, we obtain for any policy $y : [0,1] \to [0,1]$

$$\begin{aligned} rL_{y^*}(p) &= p\lambda x(1-z^*) + cz^* + p\lambda x z^* \left[L_{y^*}(1) - L_{y^*}(p)\right] + f(p,z^*)L'_{y^*}(p) \\ &= p\lambda x(1-z^*) + cz^* + p\lambda x z^* \left[C(1) - V_{y^*}(1) - (C(p) - V_{y^*}(p))\right] + f(p,z^*)(C'(p) - V'_{y^*}(p)) \\ &= p\lambda x + f(p,0)C'(p) - \Big[p\lambda x z^* - cz^* + p\lambda x z^*(V_{y^*}(1) - V_{y^*}(p)) + f(p,z^*)V'_{y^*}(p)\Big] \\ &\le p\lambda x + f(p,0)C'(p) - \Big[p\lambda x y(p) - cy(p) + p\lambda x y(p)(V_y(1) - V_y(p)) + f(p,y(p))V'_y(p)\Big] \\ &= rL_y(p), \end{aligned}$$

where the third equality follows from (SA.2) while the inequality from the optimality of y^* for (5). The last equality holds for the same reason that the first three equalities hold. Hence, y^* solves (SA.1).

The proof that any optimal policy y^* solving (SA.1) solves (5) as well is analogous and hence omitted. □

Proof of Lemma 3. The differential equation $D[K]$ can be rewritten as

$$V'(p) = \frac{r + p\lambda x}{g(p)} V(p) + \frac{c - p\lambda x(1+K)}{g(p)},$$

which clearly has a unique solution for $p \in [\pi_1 + \varepsilon, 1]$ that satisfies the desired continuity, proving Part (i).

Part (ii) is established by finding a linear solution to $D[K]$ constructively. To do so, let $V(p) = K + a(p-1)$ for $p \in [\hat{p}, 1]$. Then, the RHS of (16) becomes

$$\begin{aligned} &p\lambda x - c + p\lambda x a(1-p) + [\rho_L(1-p) - \rho_H p - p(1-p)\lambda x]a \\ =& -c + \rho_L a + [\lambda x - (\rho_L + \rho_H)a]p, \end{aligned}$$

which must equal the LHS of (16), $rV(p) = r(K + a(p-1)) = r(K-a) + rap$. We must thus have $\rho_L a - c = r(K-a)$ and $\lambda x - (\rho_L + \rho_H)a = ra$, which can be solved to yield $a = \frac{\lambda x}{r + \rho_L + \rho_H}$ and $K = \frac{(\lambda x - c)(r+\rho_L) - c\rho_H}{r(r+\rho_L+\rho_H)}$, as desired.

To prove Part (iii), let us consider any $K_2 > K_1$. For simplicity, let $V_i(\cdot)$ denote $V(\cdot; K_i)$ for $i = 1, 2$. To begin, observe that we have $V'_1(1) > V'_2(1)$ by (24). Suppose now for contradiction that there is some $p < 1$ such that $V_1'(p) \leq V_2'(p)$. One can then find some $p' < 1$ such that $V_1'(p') = V_2'(p')$ and $V_1'(p) > V_2'(p), \forall p > p'$. Note first that

$$V_1(p') = V_1(1) - \int_{p'}^{1} V_1'(p)dp < V_2(1) - \int_{p'}^{1} V_2'(p)dp = V_2(p') \tag{SA.3}$$

since $V_1(1) = K_1 < K_2 = V_2(1)$ and $V_1'(p) < V_2'(p), \forall p \in [p', 1]$. We then rewrite (16) with $p = p'$, $V = V_i$, and $K = V_i(1)$ as

$$g(p')V_i'(p') + p'\lambda x - c = rV_i(p') - p'\lambda x[V_i(1) - V_i(p')].$$

With $V_1'(p') = V_2'(p')$, we have the LHS, and thus RHS, of this equation unchanged across $i = 1, 2$. However,

$$\begin{aligned} rV_1(p') - p'\lambda x[V_1(1) - V_1(p')] &= rV_1(p') - p'\lambda x \int_{p'}^{1} V_1'(p)dp \\ &< rV_2(p') - p'\lambda x \int_{p'}^{1} V_2'(p)dp \\ &= rV_2(p') - p'\lambda x[V_2(1) - V_2(p')], \end{aligned}$$

where the inequality holds since $V_2(p') > V_1(p')$ from (SA.3) and $V_2'(p) < V_1'(p), \forall p > p'$. We have thus obtained a contradiction. That $V(p; K)$ is strictly increasing in K can be established by an analogous argument, so its proof is omitted.

For Part (iv), let us differentiate both sides of (16) with p to obtain

$$rV'(p; K) = \lambda x(1 + K) - \lambda x V(p; K) + (g'(p) - p\lambda x) V'(p; K) + g(p)V''(p; K),$$

which can be combined with (16) to remove $V(p)$ and obtain

$$V''(p; K) = \frac{[r^2 + r(\lambda x + \rho_L + \rho_H) + \lambda x \rho_L] V'(p; K) - \lambda x(c + r + rK)}{(r + \lambda p x)g(p)}. \tag{SA.4}$$

Since the numerator is strictly decreasing in K and the denominator is negative (recall $g(p) < 0$ for $p > \pi_1$), $V''(p; K)$ is strictly increasing in K. Combining this with Part (ii) proves the rest of Part (iv). □

Proof of Lemma 4. To prove Part (i), suppose that the value matching holds. Then, by (14) and (27),

$$h(\hat{p})V'(\hat{p}_+) + x\Delta(\hat{p}_+) = rV(\hat{p}_+) = rV(\hat{p}_-) = h(\hat{p})V'(\hat{p}_-)$$

or

$$h(\hat{p})V'(\hat{p}_+) + x\Delta(\hat{p}_+) = h(\hat{p})V'(\hat{p}_-).$$

Thus, $\Delta(\hat{p}_+) = 0$ is equivalent to $V'(\hat{p}_+) = V'(\hat{p}_-)$ unless $\hat{p} = \pi_0$. The proof of Part (ii) is analogous and hence omitted. □

Proof of Lemma 5. Let us first differentiate both sides of (14) with p to obtain $rV'(p) = h'(p)V'(p) + h(p)V''(p)$ or $V''(p) = \frac{(r+\rho_L+\rho_H)V'(p)}{h(p)}$ for $p \neq p_0$. Substituting this into the differentiation of $\Delta(p)$ in (17), we obtain for $p \leq \hat{p}$ with $p \neq \pi_0$,

$$\begin{aligned}
\Delta'(p) &= \frac{\Delta(p)}{p} + \frac{c}{px} - p(1-p)\lambda V''(p) \\
&= \frac{\Delta(p)}{p} + \frac{c}{px} - \frac{p(1-p)\lambda(r+\rho_L+\rho_H)}{h(p)}V'(p), \\
&= \frac{\Delta(p)}{p} + \frac{p(1-p)\lambda(r+\rho_L+\rho_H)}{h(p)}\left[\sigma(p) - V'(p)\right],
\end{aligned} \tag{SA.5}$$

which yields (18) since $\Delta(\hat{p}_-) = 0$

Let us next differentiate both sides of (15) to obtain

$$rV'(p) = \lambda x + \lambda x[V(1) - V(p)] - p\lambda x V'(p) + g'(p)V'(p) + g(p)V''(p),$$

which can be rewritten as

$$\begin{aligned}
g(p)V''(p) &= (r - g'(p) + p\lambda x)V'(p) + \lambda x[V(p) - (1 + V(1))] \\
&= (r+\rho_L+\rho_H)V'(p) + (1-p)\lambda x V'(p) + \lambda x[V(p) - (1+V(1))] \\
&= (r+\rho_L+\rho_H)V'(p) - \frac{c}{p} - \frac{\Delta(p)x}{p}.
\end{aligned} \tag{SA.6}$$

Substituting this into the differentiation of $\Delta(p)$, we obtain for $p \geq \hat{p}$ with $p \neq \pi_1$,

$$\Delta'(p) = \frac{\Delta(p)}{p} + \frac{c}{px} - p(1-p)\lambda V''(p)$$

$$
\begin{aligned}
&= \frac{\Delta(p)}{p} + \frac{c}{px} - p(1-p)\lambda \frac{(r+\rho_L+\rho_H)V'(p) - \frac{c}{p} - \frac{\Delta(p)x}{p}}{g(p)} \\
&= \left(1 + \frac{p(1-p)\lambda x}{g(p)}\right)\left(\frac{\Delta(p)}{p} + \frac{c}{px}\right) - \frac{p(1-p)\lambda(r+\rho_L+\rho_H)}{g(p)} V'(p) \\
&= \frac{h(p)}{g(p)}\left(\frac{\Delta(p)}{p} + \frac{c}{px}\right) - \frac{p(1-p)\lambda(r+\rho_L+\rho_H)}{g(p)} V'(p) \\
&= \frac{h(p)}{g(p)}\left(\frac{\Delta(p)}{p}\right) + \frac{p(1-p)\lambda(r+\rho_L+\rho_H)}{g(p)} \left[\sigma(p) - V'(p)\right], \qquad \text{(SA.7)}
\end{aligned}
$$

which yields (19) since $\Delta(\hat{p}_+) = 0$. □

Proof of Lemma 7. Part (i) follows from the continuity of σ and this claim:

Claim 1. $\sigma(p)$ strictly decreases from ∞ to 0 as p increases from 0 to π_0.

Proof. Note first that $\frac{h(p)}{p^2(1-p)}$ is strictly decreasing on $[0, \pi_0]$:

$$
\begin{aligned}
\frac{d}{dp}\left(\frac{h(p)}{p^2(1-p)}\right) &= \frac{-2p^2(\rho_H+\rho_L) + p(\rho_H + 4\rho_L) - 2\rho_L}{(1-p)^2p^3} \\
&= \frac{-2(\rho_H+\rho_L)(p-\pi_0)^2 + \frac{2(\rho_L)^2}{\rho_H+\rho_L} + p\rho_H - 2\rho_L}{(1-p)^2p^3} < 0,
\end{aligned}
$$

since the numerator is maximized at $p = \pi_0$ within the range $[0, \pi_0]$ and the maximized value is equal to $\frac{-\rho_L\rho_H}{\rho_H+\rho_L} < 0$. Thus, $\sigma(p)$ is strictly decreasing. Then, the result follows from observing that $\sigma(0) = \infty$ and $\sigma(\pi_0) = 0$ since $h(0) > 0 = h(\pi_0)$. □

For Part (ii), let us first prove the following claim:

Claim 2. $\lambda x \geq \overline{\lambda}$ is equivalent to

$$
\lambda x(\rho_L - c) + c(c - \rho_L - \rho_H) \geq 0. \qquad \text{(SA.8)}
$$

Proof. Notice that if $\rho_L - c > 0$, then (SA.8) is just a rearrangement of $\lambda x \geq \overline{\lambda}$. It thus suffices to show that (SA.8) implies $\rho_L > c$. Otherwise, we would have

$$
\text{LHS of (SA.8)} \leq c(\rho_L - c) + c(c - \rho_L - \rho_H) = -c\rho_H < 0,
$$

where the first inequality holds since $\lambda x > c$. □

Thanks to Claim 2, the equivalence between (a) and (b) will follow from showing that

$g(\hat{p}_M) \geq 0$ is equivalent to (SA.8):

$$\begin{aligned}
&g(\hat{p}_M) = \rho_L - (\rho_L + \rho_H + \lambda x)\hat{p}_M + \lambda x(\hat{p}_M)^2 \geq 0 \\
\Leftrightarrow &\rho_L - (\rho_L + \rho_H) \cdot \frac{c}{\lambda x} - c + \frac{c^2}{\lambda x} \geq 0 \\
\Leftrightarrow &\lambda x(\rho_L - c) + c(c - \rho_L - \rho_H) \geq 0.
\end{aligned}$$

To prove the equivalence between (b) and (c), observe that since $0 = g(\pi_1) = h(\pi_1) - \pi_1(1-\pi_1)\lambda x$, we have

$$\begin{aligned}
\sigma(\underline{p}) - \sigma(\pi_1) = \frac{\lambda x}{r + \rho_L + \rho_H} - \sigma(\pi_1) &= \frac{\lambda x}{r + \rho_L + \rho_H} - \frac{h(\pi_1)c}{\pi_1^2(1-\pi_1)\lambda x(r + \rho_L + \rho_H)} \\
&= \frac{c}{r + \rho_L + \rho_H}\left(\frac{\lambda x}{c} - \frac{1}{\pi_1}\right) \geq 0. \qquad \text{(SA.9)}
\end{aligned}$$

Since σ is decreasing, we have $\pi_1 \geq \underline{p}$ if and only if $\pi_1 \geq \hat{p}_M$.

For the equivalence between (d) and (a), observe

$$\sigma(\hat{p}_M) - \sigma(\underline{p}) = \frac{\lambda x[\lambda x(\rho_L - c) + c(c - \rho_L - \rho_H)]}{c(\lambda x - c)\,(r + \rho_L + \rho_H)}.$$

Thus, given that σ is decreasing, $\underline{p} \geq \hat{p}_M$ is equivalent to $\lambda x(\rho_L - c) + c(c - \rho_L - \rho_H) \geq 0$, which is equivalent to (a) due to Claim 2. □

Proof of Lemma 8. To prove the *if* direction, observe first that by (14), $V(p) = 0$ implies $V'(p) = 0$ for $p \in [0, \hat{p}]$. By the same equation, $h(\pi_0) = 0$ implies $V(\pi_0) = 0$. Suppose for contradiction that there is some $p < \pi_0$ such that $V(p) < 0$. Then, we must have some $\check{p} \in (p, \pi_0)$ such that $V(\check{p}) < 0$ and $V'(\check{p}) > 0$. Since $h(\check{p}) > 0$ and thus $h(\check{p})V'(\check{p}) > 0$, (14) implies $V(\check{p}) > 0$, a contradiction. Suppose next that there is $p < \pi_0$ such that $V(p) > 0$. Then, we must have some $\check{p} \in (p, \pi_0)$ such that $V(\check{p}) > 0$ and $V'(\check{p}) < 0$. Since $h(\check{p}) > 0$ and thus $h(\check{p})V'(\check{p}) < 0$, (14) implies $V(\check{p}) < 0$, a contradiction. The argument so far establishes $V(p) = 0$ for all $p \leq \pi_0$. An analogous argument can be used to establish the same result for the range $[\pi_0, \hat{p}]$.

To prove the *only if* direction, suppose that $V(p) = 0$ for all $p \in [0, \hat{p}]$. Then, $V'(\hat{p}_+) = V'(\hat{p}_-) = 0$, which by (19) requires that $\frac{h(\hat{p})}{g(\hat{p})} \geq 0$. This cannot hold if $\hat{p} \in (\pi_1, \pi_0)$ since then $h(\hat{p}) > 0 > g(\hat{p})$. If $\hat{p} \leq \pi_1$, then $V'(p_+) = \frac{\lambda x}{r + \rho_L + \rho_H} > 0$ (as will be shown in Lemma 9), a contradiction. The remaining possibility is $\hat{p} \geq \pi_0$, as desired. □

Proof Lemma 9. To prove the *if* direction of the first statement, let us differentiate both sides of (15) to obtain

$$rV'(p) = g'(p)V'(p) + g(p)V''(p) - \lambda x V(p) - p\lambda x V'(p) + \lambda x(1 + V(1)).$$

We can combine this equation and (15) to remove $V(p)$ and obtain

$$V'(p) = \frac{\lambda x(c + r + rV(1)) + (r + p\lambda x)g(p)V''(p)}{r^2 + r(\lambda x + \rho_L + \rho_H) + \lambda x \rho_L}. \tag{SA.10}$$

Let us denote $a = \frac{\lambda x(c+r+rV(1))}{r^2+r(\lambda x+\rho_L+\rho_H)+\lambda x\rho_L}$. We show that $V'(p) = a, \forall p \geq \hat{p}$. Note first that this is true for $p = \pi_1$ since $g(\pi_1) = 0$ and (SA.10) imply $V'(\pi_1) = a$. To first prove the linearity of V over $(\pi_1, 1]$, suppose not, i.e., $V'(p) \neq a$ for some $p \in (\pi_1, 1]$. Consider first the case $V'(p) > a$. We must then have some $\check{p} \in (\pi_1, p)$ such that $V''(\check{p}) > 0$ and $V'(\check{p}) > a$. Since $g(\check{p}) < 0$ and thus $g(\check{p})V''(\check{p}) < 0$, the equation (SA.10) implies $V'(\check{p}) < \frac{\lambda x(c+r+rV(1))}{r^2+r(\lambda x+\rho_L+\rho_H)+\lambda x\rho_L} = a$, a contradiction. Consider next the case $V'(p) < a$. Then, we must have some $\check{p} \in (\pi_1, p)$ such that $V''(\check{p}) < 0$ and $V'(\check{p}) < a$. Since $g(\check{p}) < 0$ and thus $g(\check{p})V''(\check{p}) > 0$, the equation (SA.10) implies $V'(\check{p}) > \frac{\lambda x(c+r+rV(1))}{r^2+r(\lambda x+\rho_L+\rho_H)+\lambda x\rho_L} = a$, a contradiction. The argument so far establishes that $V'(p) = a, \forall p \in [\pi_1, 1]$. An analogous argument can be used to show that $V(p)$ is linear on $[\hat{p}, \pi_1]$ as well.

According to Lemma 3-(iv), V can be linear on $[\pi_1, 1]$ only if $K = \underline{K}$, in which case Lemma 3-(ii) applies. To show $\hat{p} = \frac{c}{\lambda x}$, observe that the linearity of V over $[\hat{p}, 1]$ implies $V(1) - V(p) - V'(p)(1 - p) = 0$ for all $p \geq \hat{p}$, which, by (17), means

$$\Delta(p) = p\lambda - \frac{c}{x}. \tag{SA.11}$$

Hence, by Lemma 4, the smooth pasting requires $\Delta(\hat{p}) = 0$, implying $\hat{p} = \frac{c}{\lambda x} = \hat{p}_M$.

To prove the *only if* direction of the first statement, let us substitute $\hat{p} = \hat{p}_M = \frac{c}{\lambda x}$, $\Delta(\hat{p}) = 0$, and $V'(p_-) = V'(p_+) = \frac{\lambda x}{r+\rho_L+\rho_H}$ into (SA.5) to obtain

$$\Delta'(\hat{p}) \geq 0 \tag{SA.12}$$

$$\Leftrightarrow \frac{[\lambda x(\rho_L - c) + c(c - \rho_L - \rho_H)]}{\lambda x \rho_L - c\,(\rho_L + \rho_H)} \geq 0$$

$$\Leftrightarrow \lambda x(\rho_L - c) + c(c - \rho_L - \rho_H) \geq 0, \tag{SA.13}$$

where the second equivalence holds since $\hat{p} = \frac{c}{\lambda x} < \pi_0 = \frac{\rho_L}{\rho_L+\rho_H}$ implies $\lambda x \rho_L - c\,(\rho_L + \rho_H) > 0$. Then, $\hat{p} = \frac{c}{\lambda x} \geq \pi_1$ follows from combining (SA.13) with Lemma 7-(ii) and Claim 2.

The *only if* direction of the second statement follows directly from the first statement, provided that $\hat{p} = \hat{p}_M \le \pi_1$ implies $\lambda x \ge \overline{\lambda}$ according to Lemma 7-(ii).

To prove the *if* direction, it suffices to show that (SA.8) implies $\hat{p} \le \pi_1$. Suppose for a contradiction that $\hat{p} > \pi_1$ and hence $\hat{p} > \max\{\hat{p}_M, \underline{p}\}$ due to Lemma 7-(ii). We consider two cases depending on $\hat{p} \in (\pi_1, \pi_0)$ or $\hat{p} \ge \pi_0$.

Consider first the case $\hat{p} \in (\pi_1, \pi_0)$. Recall from Lemma 3-(ii) that for any $\varepsilon > 0$, $V(p; \underline{K}) = \frac{\lambda x}{r+\rho_L+\rho_H} = \sigma(\underline{p}), \forall p \in [\pi_1 + \varepsilon, 1]$. Using this, we argue that $V(1) \ge \underline{K}$. Else if $V(1) < \underline{K}$, then we would have for all $p \ge \hat{p}$

$$V'(\hat{p}) = V'(\hat{p}; V(1)) > V'(\hat{p}; \underline{K}) = \sigma(\underline{p}) > \sigma(\hat{p}),$$

where the first inequality follows from Lemma 3-(iii) while the second inequality holds since σ is decreasing. This inequality contradicts the condition in (18) that requires $V'(\hat{p}) \le \sigma(\hat{p})$ since $h(\hat{p}) > 0$.

That $V(1) \ge \underline{K}$ implies V is (weakly) convex over $[\hat{p}, 1]$ due to Lemma 3-(iv). Then, we have a contradiction since, by (17),

$$\Delta(\hat{p}) = \hat{p}\lambda - \frac{c}{x} + \hat{p}\lambda \left[V(1) - V(\hat{p}) - (1-\hat{p})V'(\hat{p})\right] \ge \hat{p}\lambda - \frac{c}{x} > 0,$$

where the first inequality follows from the convexity of V while the second from $\hat{p} > \hat{p}_M$.

Consider next the case $\hat{p} \ge \pi_0$. By Lemma 8 (together with the value-matching and smooth pasting), we have $V(\hat{p}) = V'(\hat{p}) = 0$. Then, by (17),

$$\Delta(\hat{p}) = \hat{p}\lambda - \frac{c}{x} + \hat{p}\lambda V(1) \ge \hat{p}\lambda - \frac{c}{x} > 0,$$

a contradiction. □

Proof of Lemma 10. Let us first consider the case $p \le \hat{p} < \pi_0$. Note that V is obtained from solving (14) in this case. Note also that $V'(\hat{p}) > 0$: if $\hat{p} \in (\pi_1, \pi_0)$, then this follows from (20); if $\hat{p} \le \pi_1$, then we have $V'(\hat{p}) = \frac{\lambda x}{r+\rho_L+\rho_H} > 0$ by Lemma 9. Given $h(\hat{p}) > 0$, we have $V(\hat{p}) > 0$ by (14). Also, (14) has a unique solution by the standard result in differential equations. Indeed,

$$V(p) = \left(\frac{h(p)}{h(\hat{p})}\right)^{-\frac{r}{\rho_L+\rho_H}} V(\hat{p}) \tag{SA.14}$$

solves (14), as one can easily check. It is then straightforward to check that this solution satisfies $V''(p) > 0$.

Let us next consider the case $p \ge \hat{p} > \pi_1$. In this case, V is obtained from solving (15).

By Lemma 3-(iv), $V''(p;K) \gtreqless 0$ everywhere over the interval $[\hat{p},1]$ depending on $K \gtreqless \underline{K}$. The case $K = \underline{K}$ (i.e., $V''(p) = 0, \forall p \in [\hat{p},1]$) can be easily ruled out since then Lemma 9 would imply $\hat{p} \leq \pi_1$. It thus suffices to show that $V''(p) < 0, \forall p \in [\hat{p},1]$ cannot hold.

To do so, we first observe that since $\lambda x \geq \overline{\lambda}$ would imply $\hat{p} \leq \pi_1$ by Lemma 9, we must have $\lambda x < \overline{\lambda}$, which implies $\underline{p} < \hat{p}_M$ due to Lemma 7-(ii). Let us consider two subcases depending on whether $\hat{p} \geq \pi_0$ or $\hat{p} \in (\pi_1, \pi_0)$. In the former case, Lemma 8 together with the value matching and smooth pasting implies $V(\hat{p}) = V'(\hat{p}) = 0$. Thus, if $V''(p) < 0, \forall p \in [\hat{p},1]$, then $V(p) < 0$ for $p > \hat{p}$, which cannot be true.

To deal with the case $\hat{p} \in (\pi_1, \pi_0)$, we first prove the following claim:

Claim 3. If $\hat{p} \in (\pi_1, \pi_0)$ and $V''(p) < 0, \forall p \in [\hat{p},1]$, then we must have $\hat{p} < \underline{p}$.

Proof. By Lemma 3-(iv), $V''(p) < 0, \forall p \in [\hat{p},1]$ implies $V(1) < \underline{K}$, which in turn implies $V'(p) = V'(p;V(1)) > V'(p;\underline{K}), \forall p \geq \hat{p}$ due to Lemma 3-(iii). Suppose $\underline{p} \leq \hat{p}$ for contradiction. Then, for any $p \geq \hat{p}$,

$$V'(p) > V'(p;\underline{K}) = \sigma(\underline{p}) \geq \sigma(\hat{p}),$$

which contradicts Lemma 6. □

Suppose now for contradiction that $V''(p) < 0, \forall p \geq \hat{p}$. Then, by $\underline{p} < \hat{p}_M$ and Claim 3, we have $\hat{p} < \hat{p}_M = \frac{c}{\lambda x}$. Thus, the strict concavity of V yields

$$\Delta(\hat{p}) = \hat{p}\lambda - \frac{c}{x} + \hat{p}\lambda\left[V(1) - V(\hat{p}) - (1-\hat{p})V'(\hat{p})\right] < \hat{p}\lambda - \frac{c}{x} < 0,$$

a contradiction. Hence, we conclude that $V''(p) > 0, \forall p \geq \hat{p}$. □

Proof of Lemma 11. The first statement can be established by using the same argument as in the proof of Lemma 9.

To prove the second statement, note that by Lemma 7-(ii), $\lambda x < \overline{\lambda}$ is equivalent to $\pi_1 < \underline{p} < \hat{p}_M$. Thus, $\underline{p} \in (\pi_1, \min\{\hat{p}_M, \pi_0\})$. The monotonicity of $\underline{p}$ with respect to λ follows from the fact that σ is decreasing in p and λx and increasing in c while $\frac{\lambda x}{r+\rho_L+\rho_H}$ is increasing in λx. □

Proof of Lemma 12. Observe first that $\underline{p}_2 < \underline{p}_1$ since $\underline{p}$ is decreasing in λx and increasing in c as shown in Lemma 11.

To simplify notations, let $V_i(\cdot)$ denote $V_i(\cdot;K)$. Let us first establish $V_1'(p) < V_2'(p), \forall p \geq \underline{p}_1$. To do so, observe first that $V_1'(1) < V_2'(1)$ due to (24), $\lambda_1 x_1 \leq \lambda_2 x_2$, and $c_1 \geq c_2$ (with at least one inequality being strict). Suppose for contradiction that there is some $p \in [\underline{p}_1, 1]$

such that $V_1'(p) \geq V_2'(p)$. Then, there must exist $\check{p} \in [p, 1]$ such that $V_1'(\check{p}) = V_2'(\check{p})$ and $V_1'(p) < V_2'(p), \forall p > \check{p}$. Given this and $V_1(1) = V_2(1) = K$,

$$V_1(\check{p}) = V_1(1) - \int_{\check{p}}^{1} V_1'(p)dp > V_2(1) - \int_{\check{p}}^{1} V_2'(p)dp = V_2(\check{p}). \tag{SA.15}$$

Using (16) and the fact that $V_1'(\check{p}) = V_2'(\check{p})$, we can write

$$\begin{aligned}
&r(V_1(\check{p}) - V_2(\check{p})) \\
&= \check{p}(\lambda_1 x_1 - \lambda_2 x_2) - (c_1 - c_2) + \check{p}\lambda_1 x_1 [K - V_1(\check{p})] - \check{p}\lambda_2 x_2 [K - V_2(\check{p})] - (\lambda_1 x_1 - \lambda_2 x_2)\check{p}(1-\check{p})V_1'(\check{p}) \\
&\leq \check{p}(\lambda_1 x_1 - \lambda_2 x_2) [V_1(1) - V_1(\check{p})] - (\lambda_1 x_1 - \lambda_2 x_2)\check{p}(1-\check{p})V_1'(\check{p}) \\
&= \check{p}(\lambda_1 x_1 - \lambda_2 x_2) [V_1(1) - V_1(\check{p}) - (1-\check{p})V_1'(\check{p})] \leq 0,
\end{aligned}$$

where the first inequality follows from the fact that $K = V_1(1)$ and $V_1(\check{p}) > V_2(\check{p})$ while the second inequality from the convexity of V_1 for $K \geq \underline{K}_1$ as established in Lemma 3-(iv). This inequality contradicts the inequality in (SA.15).

Then, $V_1(p) > V_2(p), \forall p \in [\underline{p}_1, 1)$ follows from the same argument as in (SA.15), using the fact that $V_1'(p) < V_2'(p), \forall p \in [\underline{p}_1, 1]$. □

C.2 Ommitted Parts of the Proof for Proposition 9

Proof of Step 8. Let (p^-, K^-) denote the limit of $(\hat{p}, \hat{K})$ satisfying (23) as λx decreases to $\underline{\lambda}$. By the continuity of V and V' with respect to p, K, and (λ, x, c), the pair (p^-, K^-) is a unique solution at $\lambda x = \underline{\lambda}$.[29]

To show that $p^- = \pi_0$, suppose for a contradiction that $p^- < \pi_0$.[30] Then, we must have $K^- < K_0$ since $p^- = p(K^-)$ and $\pi_0 = p(K_0)$ while $p(\cdot)$ is increasing. This implies $\Delta(p(K_0); K_0) > 0 = \Delta(p(K^-); K^-)$ since $\Delta(p(\cdot); \cdot)$ is increasing. Now let us choose some (λ_1, x_1, c) such that $\lambda_1 x_1 = \underline{\lambda} - \varepsilon$ with small $\varepsilon > 0$. Let $V_1(\cdot; \cdot)$, $\Delta_1(\cdot; \cdot)$, $\sigma_1(\cdot)$, and $p_1(\cdot)$ denote the corresponding functions. By the continuity of these functions with respect to (λ, x, c), we can make $\varepsilon > 0$ sufficiently small that $\Delta_1(p_1(K_0); K_0) > 0$. Let K' be such that $\sigma_1(p^-) = V_1'(p^-; K')$, i.e., $p_1(K') = p^-$. Then, we have $K' < K^-$ since $\sigma_1(p^-) > \sigma(p^-) = V'(p^-; K^-) > V_1'(p^-; K^-)$ (where the second inequality holds due to Lemma 12) and since $V_1'(p^-; \cdot)$ is decreasing. Also, using $\lambda_1 x_1 < \underline{\lambda}$ and derivations analogous to those in Step 5, we can show that $\Delta_1(p^-; K') < \Delta(p^-; K^-) = 0$. Given that $\Delta_1(p_1(K_0); K_0) > 0 > \Delta_1(p_1(K'); K')$, we can find some $K_1 \in (K', K_0)$ such that $p_1(K_1) < \pi_0$ and $\Delta_1(p_1(K_1); K_1) =$

[29] The uniqueness follows from Step 1 and 4.

[30] Note that we cannot have $p^- > \pi_0$ since p^- is the limit of $\hat{p} < \pi_0$.

0 while $\sigma_1(p_1(K_1)) = V_1'(p_1(K_1); K_1)$, which contradicts the definition of $\underline{\lambda}$ (since $\lambda_1 x_1 < \underline{\lambda}$). ☐

Proof of Step 9. The monotonicity of $\hat{p}$ is immediate from Step 5. To prove the convergence, let (p^+, K^+) denote the limit of $(\hat{p}, \hat{K})$ satisfying (23) as λx increases to $\overline{\lambda}$. Clearly, $p^+ \geq \pi_1$. Suppose $p^+ > \pi_1$ for a contradiction. Since, at $\lambda x = \overline{\lambda}$, we have $\pi_1 = \underline{p} = \frac{c}{\lambda x}$ by Lemma 2, we can choose λx sufficiently close to $\overline{\lambda}$ (or sufficiently large if $\overline{\lambda} = \infty$) that π_1 is close to $\frac{c}{\lambda x}$. Letting $(\hat{p}, \hat{K})$ denote a pair satisfying (23) for such λx, we have $\hat{p} > \pi_1$, which implies that $\hat{p} > \frac{c}{\lambda x}$ and $\hat{p} > \underline{p}$ since $\hat{p}$ is close to $p^+ > \pi_1$. Then, $\Delta(\hat{p}; \hat{K}) = 0$ requires $V(1; \hat{K}) - V(\hat{p}; \hat{K}) - (1 - \hat{p})V'(\hat{p}; \hat{K}) < 0$, which implies $\hat{K} < \underline{K}$ by Lemma 3-(iv). Then, by Lemma 3-(iii) and Claim 1, $V'(\hat{p}; \hat{K}) > V'(\hat{p}; \underline{K}) = \frac{\lambda x}{r + \rho_L + \rho_H} = \sigma(\underline{p}) > \sigma(\hat{p})$, a contradiction. ☐

C.3 Proof of Proposition 10

The proof proceeds in a few steps and will employ the notations and results from the proof of Proposition 9. For instance, we continue to use the notations K_0 and $\overline{K}$.

Step 1: No pair $(\hat{p}, \hat{K})$ with $\hat{K} < K_0$ or $\hat{K} \geq \overline{K}$ can satisfy (29).

Recall first that $V'(\pi_0; K_0) = 0$ by the definition of K_0. We can then observe that if $\hat{K} < K_0$, then, for all $p \geq \pi_0$,

$$V'(p; \hat{K}) \geq V'(\pi_0; \hat{K}) > V'(\pi_0; K_0) = 0$$

since V' is decreasing in K and increasing in p. Observe also that if $\hat{K} \geq \overline{K}$, then $V'(p; \hat{K}) < V'(1; \hat{K}) \leq V'(1; \overline{K}) = 0, \forall p < 1$.

Step 2: For any $K \in [K_0, \overline{K})$, there is a unique $p(K) \in [\pi_0, 1)$ that satisfies $V'(p(K); K) = 0$ and is continuously increasing in K.

For any $K \in [K_0, \overline{K})$,

$$V'(\pi_0; K) \leq V'(\pi_0; K_0) = 0 < V'(1; K),$$

where the second inequality holds since $V'(1; K) = \frac{\lambda x - c - rK}{\rho_H} > 0$ for $K < \overline{K} = \frac{\lambda x - c}{r}$. Thus, there exists a unique $p(K) \in (\pi_0, 1)$ that satisfies the desired property, since $V'(\cdot; K)$ is strictly increasing.

The monotonicity of $p(K)$ follows easily from the fact that V' is increasing in p and decreasing in K.

Step 3: For each $\lambda x \in (c, \underline{\lambda}]$, there exists a unique pair $(\hat{p}, \hat{K})$ satisfying (29) with $\hat{p} \in [\pi_0, 1)$ and $\hat{K} \in [K_0, \overline{K})$.

Consider any $\lambda_1 x_1 \in (c, \underline{\lambda}]$ and $\lambda_2 x_2 = \underline{\lambda}$. Let $V_i(p; K)$, $\sigma_i(p)$, $\Delta_i(p; K)$, and $p_i(K)$ denote the functions associated with (λ_i, x_i, c). Let K_i and $\overline{K}_i$ denote K_0 and $\overline{K}$ corresponding to (λ_i, x_i, c). Let us first observe that $K_2 > K_1$. By Lemma 12, $V_2'(\pi_0; K_1) > V_1'(\pi_0; K_1) = 0 = V_2'(\pi_0; K_2)$, which implies $K_1 < K_2$ since V_2' is decreasing in K by Lemma 3-(iii).

By Step 8 in the proof of Proposition 9, we have $p_2(K_2) = \pi_0$ and $\Delta_2(p_2(K_2); K_2) = 0$. Thus, using (27), we obtain

$$\begin{aligned} 0 = \Delta_2(p_2(K_2); K_2) &= \frac{rV_2(\pi_0; K_2)}{x} \\ &\geq \frac{rV_1(\pi_0; K_2)}{x} > \frac{rV_1(\pi_0; K_1)}{x} = \Delta_1(p_1(K_1); K_1), \end{aligned} \tag{SA.16}$$

where the first inequality follows from Lemma 12 while the second inequality from Lemma 3-(iv) and $K_2 > K_1$.

Observe next that $p_1(\overline{K}_1) = 1$ since $V_1'(1; \overline{K}_1) = 0$. Thus, using (17), we obtain

$$\Delta_1(p_1(\overline{K}_1); \overline{K}_1) = \Delta_1(1; \overline{K}_1) = \lambda_1 - \frac{c}{x} > 0.$$

Combining this with (SA.16) and using the strict monotonicity of $\Delta_1(p_1(\cdot); \cdot)$, we can conclude that there is a unique $\hat{K} \in [K_1, \overline{K}_1)$ such that $\Delta_1(p_1(\hat{K}); \hat{K}) = 0$ and $p_1(\hat{K}) \in [\pi_0, 1)$.

Step 4: For the pair $(\hat{p}, \hat{K})$ satisfying (29), $\hat{p}$ is decreasing in λx and increasing in c.

The proof of this step is analogous to that of Step 5 in the proof of Proposition 9, and hence omitted.

C.4 Proof of Proposition 11

Part (i) follows from Proposition 8 in the case $\hat{p} \leq \pi_1$, where $\hat{p} = \frac{c}{\lambda x}$. To deal with the case $\hat{p} > \pi_1$, rewrite $\Delta(\hat{p}) = 0$ as

$$\hat{p}\lambda - \tfrac{c}{x} = -\lambda \hat{p}\left[V(1) - V(\hat{p}) - (1-\hat{p})V'(\hat{p})\right]. \tag{SA.17}$$

From the proofs of Proposition 9 and Proposition 10, we know that for $p \geq \hat{p}$, $V(p) = V(p; K)$ for some $K > \underline{K}$. Thus, by Lemma 3-(iv), V is strictly convex, which implies that the expression inside the square bracket in (SA.17) is strictly positive, so $\hat{p} < \frac{c}{\lambda x}$.

For Part (ii), the monotonicity of $\overline{\lambda}$ is immediate from its definition while the monotonicity of $\underline{\lambda}$ follows from Step 5 in the proof of Proposition 9.

For Part (iii), let us focus on establishing the comparative statics with λx since doing so with c is analogous. To do so, consider any $\lambda_2 x_2 > \lambda_1 x_1$. Let π_{0i}, π_{1i} and $\hat{p}_i$ denote the values of π_0, π_1 and $\hat{p}$, respectively, under $\lambda_i x_i, i = 1, 2$. Note that $\pi_{01} = \pi_{02}$ and $\pi_{11} > \pi_{12}$.

The desired result $\hat{p}_1 > \hat{p}_2$ follows immediately from Proposition 8 to Proposition 10 if $\lambda_2 x_2 > \lambda_1 x_1 \geq \overline{\lambda}$ or if $\overline{\lambda} > \lambda_2 x_2 > \lambda_1 x_1 > \underline{\lambda}$ or if $\lambda_1 < \lambda_2 \leq \underline{\lambda}$. In the case $\lambda_1 x_1 < \overline{\lambda} \leq \lambda_2 x_2$, we have $\hat{p}_1 > \pi_{11} > \pi_{12} \geq \hat{p}_2$ as desired. In the case $\lambda_1 x_1 \leq \underline{\lambda} < \lambda_2 x_2 < \overline{\lambda}$, we have $\hat{p}_1 \geq \pi_{01} = \pi_{02} > \hat{p}_2$.

C.5 Proof of Proposition 12

Establishing the optimality of the policy function $y(p) = 1_{\{p \geq \hat{p}\}}$ amounts to showing $\Delta(p) \geq (\leq) 0$ for $p \geq (\leq) \hat{p}$. The proof of this result proceeds in a few steps. To begin, we reproduce (SA.5) and (SA.7):

$$\Delta'(p) = \begin{cases} \dfrac{\Delta(p)}{p} + \dfrac{p(1-p)\lambda(r+\rho_L+\rho_H)}{h(p)}\left[\sigma(p) - V'(p)\right] & \text{for } p \leq \hat{p} \qquad \text{(SA.18)} \\ \dfrac{h(p)}{g(p)}\left(\dfrac{\Delta(p)}{p}\right) + \dfrac{p(1-p)\lambda(r+\rho_L+\rho_H)}{g(p)}\left[\sigma(p) - V'(p)\right] & \text{for } p \geq \hat{p}. \qquad \text{(SA.19)} \end{cases}$$

Step 1: If $\Delta(p) \geq 0$ at $p = \max\{\pi_0, \hat{p}\}$, then $\Delta(p) \geq 0, \forall p \geq \max\{\hat{p}, \pi_0\}$.

Note that $g(p) < 0$ and $h(p) \leq 0$ for $p \geq \max\{\hat{p}, \pi_0\}$. Hence, since $V'(p) \geq 0$, (SA.19) implies that $\Delta'(p) > 0$ whenever $\Delta(p) \geq 0$.[31] If $\Delta(p) \geq 0$ at $p = \max\{\pi_0, \hat{p}\}$, this should imply that $\Delta(p) \geq 0$ for all $p > \max\{\pi_0, \hat{p}\}$.

Step 2: With $\hat{p} > \pi_0$, $\Delta(p) \leq (\geq) 0$ for $p < (>) \hat{p}$.

In this case, we have $\max\{\hat{p}, \pi_0\} = \hat{p}$, so $\Delta(p) = 0$ at $p = \max\{\hat{p}, \pi_0\}$, which implies by Step 1 that $\Delta(p) \geq 0, \forall p \geq \hat{p}$.

To show that $\Delta(p) \leq 0$ for $p \leq \hat{p}$, note that $V'(p) = 0$ for all $p \leq \hat{p}$, and that $h(\hat{p}) < 0$ and $g(\hat{p}) < 0$. Consequently, by (SA.18), (SA.19), and $\Delta(\hat{p}) = 0$,

$$\Delta'(\hat{p}_-) = \frac{\hat{p}(1-\hat{p})\lambda(r+\rho_L+\rho_H)}{h(\hat{p})}\sigma(\hat{p}) > 0,$$

and

$$\Delta'(\hat{p}_+) = \frac{\hat{p}(1-\hat{p})\lambda(r+\rho_L+\rho_H)}{g(\hat{p})}\sigma(\hat{p}) > 0.$$

[31] That $V'(p) \geq 0$ can be easily verified using the results from Proposition 8 to Proposition 10.

This means that there exists $\varepsilon > 0$ such that for $p \in (\hat{p}-\varepsilon, \hat{p})$, $\Delta(p) < 0$, and for $p \in (\hat{p}, \hat{p}+\varepsilon)$, $\Delta(p) > 0$.

If $\Delta(p) > 0$ for any $p < \hat{p}$, we must have $\check{p} \in [p, \hat{p}]$ such that $\Delta'(\check{p}) < 0$ and $\Delta(\check{p}) = 0$. But we have

$$\Delta'(\check{p}) = \frac{\check{p}(1-\check{p})\lambda(r+\rho_L+\rho_H)}{h(\check{p})}\sigma(\check{p}) > 0,$$

a contradiction.

Step 3: With $\hat{p} \in (\pi_1, \pi_0)$, $\Delta(p) \geq 0$ for $p \geq \hat{p}$.

The result will follow from Step 1 once we prove $\Delta(p) \geq 0$ for $p \in [\hat{p}, \pi_0]$. Let us thus fix any $p \in [\hat{p}, \pi_0]$. Rewrite (SA.19) for the part of $p \geq \hat{p}$ as

$$\Delta'(p) = \frac{h(p)}{g(p)}\left(\frac{\Delta(p)}{p}\right) - \frac{\delta(p)}{g(p)p}, \tag{SA.20}$$

where

$$\delta(p) := p^2(1-p)\lambda(r+\alpha)V'(p) - \frac{c}{x}h(p)$$

with $\alpha := \rho_L + \rho_H$. Differentiating this, we get

$$\delta'(p) = (2p-3p^2)\lambda(r+\alpha)V'(p) + \frac{c\alpha}{x} + p^2(1-p)\lambda(r+\alpha)V''(p). \tag{SA.21}$$

By (SA.6), we have $V''(p) = \frac{(r+\alpha)V'(p)-\frac{c}{p}-\frac{\Delta(p)x}{p}}{g(p)}$ and substitute this into (SA.21) to obtain

$$\begin{aligned}
\delta'(p) =&(2p-3p^2)\lambda(r+\alpha)V'(p) + \frac{c\alpha}{x} + p^2(1-p)\lambda(r+\alpha)\frac{(r+\alpha)V'(p) - \frac{c}{p} - \frac{\Delta(p)x}{p}}{g(p)} \\
=&(2p-3p^2)\lambda(r+\alpha)V'(p) + \frac{c\alpha}{x} - p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)} - p(1-p)\lambda(r+\alpha)\frac{c}{g(p)} \\
&+ \frac{p^2(1-p)\lambda(r+\alpha)^2V'(p)}{g(p)} \\
=&(2p-3p^2)\lambda(r+\alpha)V'(p) + \frac{c\alpha}{x} - p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)} - p(1-p)\lambda(r+\alpha)\frac{c}{g(p)} \\
&+ (r+\alpha)\frac{p^2(1-p)\lambda(r+\alpha)V'(p) - \frac{c}{x}h(p)}{g(p)} + (r+\alpha)\frac{ch(p)}{xg(p)} \\
=&(2p-3p^2)\lambda(r+\alpha)V'(p) + \frac{c\alpha}{x} - p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)} + (r+\alpha)\frac{\delta(p)}{g(p)} + \frac{c(r+\alpha)}{x} \\
=&2p(1-p)\lambda(r+\alpha)V'(p) - p^2\lambda(r+\alpha)V'(p)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c\alpha}{x}-p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)}+(r+\alpha)\frac{\delta(p)}{g(p)}+\frac{c(r+\alpha)}{x}\\
=&2p(1-p)\lambda(r+\alpha)V'(p)-\frac{p^2(1-p)\lambda(r+\alpha)V'(p)-h(p)\frac{c}{x}}{1-p}-\frac{h(p)c}{(1-p)x}\\
&+\frac{c\alpha}{x}-p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)}+(r+\alpha)\frac{\delta(p)}{g(p)}+\frac{c(r+\alpha)}{x}\\
=&2p(1-p)\lambda(r+\alpha)V'(p)-\frac{\delta(p)}{1-p}-\frac{h(p)c}{(1-p)x}\\
&+\frac{c\alpha}{x}-p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)}+(r+\alpha)\frac{\delta(p)}{g(p)}+\frac{c(r+\alpha)}{x}\\
=&2p(1-p)\lambda(r+\alpha)V'(p)-\frac{\delta(p)}{1-p}-\frac{c\alpha}{x}+\frac{\rho_H c}{(1-p)x}\\
&+\frac{c\alpha}{x}-p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)}+(r+\alpha)\frac{\delta(p)}{g(p)}+\frac{c(r+\alpha)}{x}\\
=&2p(1-p)\lambda(r+\alpha)V'(p)-\delta(p)\left(\frac{1}{1-p}-\frac{r+\alpha}{g(p)}\right)+\frac{\rho_H c}{(1-p)x}\\
&-p(1-p)\lambda(r+\alpha)\frac{\Delta(p)x}{g(p)}+\frac{c(r+\alpha)}{x}. \qquad \text{(SA.22)}
\end{aligned}
$$

By Proposition 9, we have $\Delta(\hat{p})=0$ and $V'(\hat{p})=\sigma(\hat{p})\geq 0$ or equivalently $\delta(\hat{p})=0$. It thus follows that $\delta'(\hat{p}_+)>0$.

Thus, using (SA.20) with $\Delta(\hat{p}_+)=\Delta'(\hat{p}_+)=\delta(\hat{p}_+)=0$, we obtain

$$\Delta''(\hat{p}_+)=-\frac{\delta'(\hat{p}_+)}{g(\hat{p})\hat{p}}>0,$$

from which it follows that $\Delta(p')>0$ for all $p'\in(\hat{p},\hat{p}+\varepsilon]$, for some $\varepsilon>0$. We now prove the following claim:

Claim 4. If $\Delta(p'')<0$ for some $p''\in[\hat{p},\pi_0]$, then there exists $\check{p}\in(\hat{p},p'')$ such that $\Delta(\check{p})\geq 0$, $\delta(\check{p})=0$ and $\delta'(\check{p})\leq 0$

Proof. Let $p_0=\sup\{p'\in(\hat{p}+\varepsilon,p''):\Delta(p)>0,\forall p<p'\}$. Then, we must have $\Delta(p_0)\leq 0$. Given this, it cannot be the case that $\delta(p)>0,\forall p\in[\hat{p},p_0]$, since it would imply that whenever $\Delta(p)=0$, we have $\Delta'(p)>0$ by (SA.20) and $g(p)<0$, which in turn implies $\Delta(p_0)>0$. Thus, we must have some $p_1\in(\hat{p},p_0]$ with $\delta(p_1)\leq 0$. Since $\delta(\hat{p}_+)>0$, this implies that there must be some $\check{p}\in(\hat{p},p_1]$ such that $\delta(\check{p})=0$ and $\delta'(\check{p})\leq 0$. □

Suppose for contradiction that $\Delta(p'')<0$ for some $p''\in[\hat{p},\pi_0]$. By the above claim, one can find $\check{p}\in(\hat{p},p'')$ that $\Delta(\check{p})\geq 0$, $\delta(\check{p})=0$ and $\delta'(\check{p})\leq 0$. Substituting this into (SA.22) with $p = \check{p}$, we obtain

$$\delta'(\check{p}) = 2\check{p}(1-\check{p})\lambda(r+\alpha)V'(\check{p}) + \frac{\rho_H c}{(1-\check{p})x} - \check{p}(1-\check{p})\lambda(r+\alpha)\frac{\Delta(\check{p})x}{g(\check{p})} + \frac{c(r+\alpha)}{x} > 0$$

since $V'(\check{p}) \geq 0$, $\Delta(\check{p}) \geq 0$, and $g(\check{p}) < 0$. This contradicts $\delta'(\check{p}) \leq 0$.

Step 4: With $\hat{p} \in (\pi_1, \pi_0)$, $\Delta(p) \leq 0$ for $p \leq \hat{p}$.

Let us differentiate both sides of (14) and rearrange them to obtain

$$V''(p) = (r+\alpha)\frac{V'(p)}{h(p)}. \tag{SA.23}$$

We plug this into (SA.21) to obtain

$$\begin{aligned}
\delta'(p) =& (2p-3p^2)\lambda(r+\alpha)V'(p) + \frac{c\alpha}{x} + p^2(1-p)\lambda(r+\alpha)^2\frac{V'(p)}{h(p)} \\
=& (2p-3p^2)\lambda(r+\alpha)V'(p) + \frac{(r+\alpha)}{h(p)}\left(p^2(1-p)(r+\alpha)\lambda V'(p) - \frac{c}{x}h(p)\right) + \frac{c(r+2\alpha)}{x} \\
=& 2p(1-p)\lambda(r+\alpha)V'(p) - \frac{p^2(1-p)\lambda(r+\alpha)V'(p) - \frac{c}{x}h(p)}{1-p} - \frac{c}{(1-p)x}h(p) \\
&+ \frac{(r+\alpha)}{h(p)}\delta(p) + \frac{c(r+2\alpha)}{x} \\
=& 2p(1-p)\lambda(r+\alpha)V'(p) - \frac{\delta(p)}{1-p} - \frac{c\alpha}{x} + \frac{\rho_H c}{(1-p)x} + \frac{(r+\alpha)}{h(p)}\delta(p) + \frac{c(r+2\alpha)}{x} \\
=& 2p(1-p)\lambda(r+\alpha)V'(p) - \frac{\delta(p)}{1-p} + \frac{\rho_H c}{(1-p)x} + \frac{(r+\alpha)}{h(p)}\delta(p) + \frac{c(r+\alpha)}{x}.
\end{aligned} \tag{SA.24}$$

Then, since $V'(\hat{p}) \geq 0$ and $\delta(\hat{p}) = 0$, we obtain $\delta'(\hat{p}_-) > 0$. Using this, one can follow an argument analogous to that in Step 3 to show that $\Delta(p) \leq 0, \forall p \leq \hat{p}$.

Step 5: With $\hat{p} \leq \pi_1$, $\Delta(p) \leq 0$ for $p \leq \hat{p}$.

In this case, $\hat{p} = \hat{p}_M \leq \underline{p}$ by Proposition 8 and Lemma 7, so we have for any $p < \hat{p}$,

$$\sigma(p) - V'(p) > \sigma(\underline{p}) - V'(\hat{p}) = \sigma(\underline{p}) - \frac{\lambda x}{r + \rho_L + \rho_H} = 0,$$

where the inequality follows from the fact that V' is strictly increasing (by Lemma 10) while σ is decreasing. Thus, by (SA.18), we conclude that for any $p < \hat{p}$, $\Delta'(p) > 0$ whenever $\Delta(p) = 0$. Given that $\Delta(\hat{p}) = 0$, this observation implies that $\Delta(p) \leq 0$ for all $p \leq \hat{p}$.

Step 6: With $\hat{p} \le \pi_1$, $\Delta(p) \ge 0$ for $p \ge \hat{p}$.

The result follows immediately from (SA.11) and the fact that $\hat{p} = \hat{p}_M = \frac{c}{\lambda x}$.

D Passive Learning

In this section, we consider a situation in which the detection can occur without enforcement, which we call *passive learning*: for instance, there is a community reporting by the victim of a crime. Let $w \in (0,1)$ denote the probability of such detection (upon a crime being committed). Assuming that the passive learning occurs independently of PM's enforcement, a crime is detected with the probability $y + w - wy$, so the rate at which PM learns $\omega = H$ is $p_t \lambda x(y + w - yw)$. We assume that passive learning, which occurs without any enforcement, has no effect on reducing harm. Thus, by choosing y units of enforcement, PM with posterior belief p_t mitigates the harm by the same rate as before, that is, $p_t \lambda xy$. As a result, PM's flow loss and threshold belief for the myopically optimal policy are unchanged. Therefore, passive learning affects the PM's discounted total loss only through her belief updating.

The belief updating rule (2) changes to

$$\dot{p} = \rho_L(1-p) - \rho_H p - p(1-p)\lambda x(y + w - yw) =: f_w(p, y). \tag{SA.25}$$

With passive learning, learning takes place even when the PM does not enforce. This means that the posterior belief may drift below π_0 even when $y = 0$. Indeed, one can find a unique $\pi_w \in (\pi_1, \pi_0)$ such that $f_w(\pi_w, 0) = 0$.[32]

Recall that, without passive learning, the long-term belief under GP either stays at π_0 or cycles over $[\hat{p}_M, 1]$. In either case, the PM obtains the same payoff under GP as under NP, for information is never collected when enforcement is strictly suboptimal. This is no longer the case when there is passive learning. Suppose $\pi_w < \hat{p}_M$. Then, learning occurs with positive probability even when the posterior falls below $\hat{p}_M$ (where enforcement is strictly suboptimal) until it reaches π_w. In this case, GP prescribes the (myopically) correct action, i.e., no enforcement, unlike NP, so the former performs strictly better:

Proposition 13. *If $\hat{p}_M > \pi_w$, GP achieves a strictly lower long-run loss for PM than NP. Otherwise, GP and NP achieve the same long-run loss.*

[32] This is because $f_w(\pi_0, 0) = f(\pi_0, w) < f(\pi_0, 0) = 0 = f(\pi_1, 1) < f(\pi_1, w) = f_w(\pi_1, 0)$. Note also that the benchmark π_1 remains relevant: Since $f_w(p, 1) = f(p, 1)$, we have $f_w(\pi_1, 1) = f(\pi_1, 1) = 0$.

Proof. The expected total loss under GP with the initial belief $p_0 = \pi_0$ is given by

$$\begin{aligned} L_{GP}(\pi_0) =& \mathbb{E}\left[\int_0^\infty (p_t\lambda x(1-y_{\text{GP}}(p_t)) + c y_{\text{GP}}(p_t))e^{-rt}dt \middle| p_0 = \pi_0\right] \\ =& \mathbb{E}\left[\int_0^\infty \min\{p_t\lambda x, c\}e^{-rt}dt \middle| p_0 = \pi_0\right] \\ =& \int_0^\infty \mathbb{E}\Big[\min\{p_t\lambda x, c\}\Big| p_0 = \pi_0\Big]e^{-rt}dt \\ \leq& \int_0^\infty \min\Big\{\mathbb{E}[p_t\lambda x | p_0 = \pi_0], c\Big\}e^{-rt}dt \\ =& \int_0^\infty \min\big\{\pi_0\lambda x, c\big\}e^{-rt}dt \\ =& \min\{\pi_0\lambda x, c\}/r = L_{NP}(\pi_0), \end{aligned}$$

where the inequality follows from Jensen's inequality and the equality right after that inequality follows from Claim 5 below.

The weak inequality becomes an equality if $\hat{p}_M \leq \pi_w$. This is because $\min\{p_t\lambda x, c\} = c$ for all $p_t \in [\pi_w, 1]$. However, if $\hat{p}_M > \pi_w$, then both $p_t\lambda x < c$ and $p_t\lambda x > c$ occurs with positive probability. Hence, in this case, the inequality becomes strict.

Claim 5. $\mathbb{E}[p_t|p_0 = \pi_0] = \pi_0$.

Proof. The posterior belief conditional on the initial belief $p_0 = \pi_0$ and any history of enforcement and detection through time t, denoted by h^t, is equal to

$$\Pr[\omega_t = H|p_0 = \pi_0, h^t] = \Pr[\omega_t = H|\omega_0 = H, h^t]\pi_0 + \Pr[\omega_t = H|\omega_0 = L, h^t](1-\pi_0).$$

Thus,

$$\begin{aligned} \mathbb{E}[p_t|p_0 = \pi_0] &= \mathbb{E}[\Pr[\omega_t = H|p_0 = \pi_0, h^t]] \\ &= \mathbb{E}_{h^t}\Big[\Pr[\omega_t = H|\omega_0 = H, h^t]\pi_0 + \Pr[\omega_t = H|\omega_0 = L, h^t](1-\pi_0)\Big] \\ &= \Pr[\omega_t = H|\omega_0 = H]\pi_0 + \Pr[\omega_t = H|\omega_0 = L](1-\pi_0) = \pi_0, \end{aligned}$$

where the last equality follows from the fact that π_0 is the stationary belief that the Markov chain ω_t is at state H. □

□

E Analysis for Section 4

Let $\Phi(p)$ denote the cumulative distribution of p. To identify Φ, we invoke an *balance equation*. As explained, the system spends a significant amount of time at $\hat{p}$, so there is probability mass at $\hat{p}$. Let $m(\hat{p}) := \Phi(\hat{p})$ denote that mass. We expect that Φ admits density at $p > \hat{p}$, denoted by $\phi(p)$.

To determine $m(\hat{p})$ and $\phi(p)$ for each $p > \hat{p}$, let us consider any $p \in [\hat{p}, 1)$ and a set $[\hat{p}, p]$. The balance condition for such a set is given by

$$m(\hat{p})\hat{p}\lambda x z(\hat{p})dt + \left[\int_{\hat{p}}^{p} \phi(s)s\lambda x ds\right] dt = [\Phi(p + dp) - \Phi(p)](1 - p\lambda x dt). \tag{SA.26}$$

The LHS represents the outflow of mass out of the set $[\hat{p}, p]$, which occurs when each belief $\tilde{p} \in [\hat{p}, p]$ jumps to 1. The RHS represents the inflow of mass into the set, which occurs when the belief drifts down from right above p. Observe that since the instantaneous rate of belief change is $f(p, 1) < 0$, it follows from (2) that the magnitude of the belief change $dp > 0$ is:

$$dp = -f(p, 1)dt + o(dt).$$

Thus, (SA.26) simplifies to

$$m(\hat{p})\hat{p}\lambda x z(\hat{p})dt + \int_{\hat{p}}^{p} \phi(s)s\lambda x ds = -\phi(p)f(p, 1). \tag{SA.27}$$

With $p = \hat{p}$, (SA.27) becomes

$$m(\hat{p})\hat{p}\lambda x z(\hat{p}) = -\phi(\hat{p})f(\hat{p}, 1), \tag{SA.28}$$

where $\phi(\hat{p})$ is the right limit of ϕ at $\hat{p}$.

With $p > \hat{p}$, we can differentiate both sides of (SA.27) with p to obtain

$$\phi(p)p\lambda x = -\phi'(p)f(p, 1) - \phi(p)f'(p, 1),$$

where $f'(p, 1) = \frac{\partial f(p,1)}{\partial p}$. So we have

$$\phi'(p)f(p, 1) = -\phi(p)\left(p\lambda x + f'(p, 1)\right). \tag{SA.29}$$

This ODE has the following solution:

$$\phi(p) = \phi(\hat{p}) \exp\left(\int_{\hat{p}}^{p} r(s)ds\right), \tag{SA.30}$$

where

$$r(s) = -\frac{s\lambda x + f'(s,1)}{f(s,1)}. \tag{SA.31}$$

Using this, we obtain

$$1 - m(\hat{p}) = \Phi(1) - \Phi(\hat{p}) = \int_{\hat{p}}^{1} \phi(p)dp = \phi(\hat{p}) \int_{\hat{p}}^{1} \exp\left(\int_{\hat{p}}^{p} r(s)ds\right)dp, \tag{SA.32}$$

Lemma 14. *For $\hat{p} \in (\pi_1, \pi_0)$, there exist $\phi(\hat{p}) > 0$ and $m(\hat{p}) \in (0,1)$ that satisfy (SA.28) and (SA.32).*

Proof. Substitute $m(\hat{p}) = -\frac{\phi(\hat{p})f(\hat{p},1)}{\hat{p}\lambda x z(\hat{p})}$ from (SA.28) to express (SA.32) as

$$\phi(\hat{p})\left[\int_{\hat{p}}^{1} \exp\left(\int_{\hat{p}}^{p} r(s)ds\right)dp - \frac{f(\hat{p},1)}{\hat{p}\lambda x z(\hat{p})}\right] = 1. \tag{SA.33}$$

Since the expression in the square brackets is positive, one can find $\phi(\hat{p}) > 0$ satisfying this equation. Then, (SA.32) implies $m(\hat{p}) < 1$ while $m(\hat{p}) = -\frac{\phi(\hat{p})f(\hat{p},1)}{\hat{p}\lambda x z(\hat{p})} > 0$ since $f(\hat{p},1) < 0$. □

Proof of Lemma 1. First, the long-run distribution of posterior belief conditional on $\omega = H$ can be obtained as

$$\Psi(p) := \Pr\{p' \le p | \omega = H\} = \frac{\Pr\{p' \le p \text{ and } \omega = H\}}{\Pr\{\omega = H\}} = \begin{cases} 0 & \text{if } p < \hat{p} \\ \frac{m(\hat{p})\hat{p} + \int_{\hat{p}}^{p} \phi(s)sds}{\pi_0} & \text{if } p \ge \hat{p}. \end{cases}$$

The expected enforcement conditional on $\omega = H$ is thus

$$\begin{aligned} \mathbb{E}_p[y_{\hat{p}}(p)|\omega = H, x] = \Psi(\hat{p})z(\hat{p}) + 1 - \Psi(\hat{p}) &= 1 - \Psi(\hat{p})(1 - z(\hat{p})) \\ &= 1 - \frac{1}{\pi_0} m(\hat{p})\hat{p}(1 - z(\hat{p})), \end{aligned} \tag{SA.34}$$

where the last equality holds since $\Psi(\hat{p}) = \frac{m(\hat{p})\hat{p}}{\pi_0}$. We first show that this expression is decreasing in $\hat{p}$. To do so, it suffices to prove that $m(\hat{p})$ is increasing since $\hat{p}(1 - z(\hat{p}))$ is increasing.

To this end, let

$$\eta(\hat{p}) := \int_{\hat{p}}^{1} \exp\left(\int_{\hat{p}}^{p} r(s)ds\right) dp.$$

Using this and (SA.33), we obtain

$$\phi(\hat{p}) = \frac{1}{\eta(\hat{p}) - \frac{f(\hat{p},1)}{\hat{p}\lambda x z(\hat{p})}},$$

which can be plugged into (SA.32) to yield

$$1 - m(\hat{p}) = \frac{\eta(\hat{p})}{\eta(\hat{p}) - \frac{f(\hat{p},1)}{\hat{p}\lambda x z(\hat{p})}} = \frac{1}{1 + \left(\frac{1}{\hat{p}\lambda x z(\hat{p})}\right)\left(\frac{-f(\hat{p},1)}{\eta(\hat{p})}\right)},$$

which is continuous in $\hat{p} \in (\pi_1, \pi_0)$.

Observe first that $0 = f(\hat{p}, z(\hat{p})) = h(\hat{p}) - \lambda x \hat{p}(1 - \hat{p}) z(\hat{p})$ and thus

$$\hat{p}\lambda x z(\hat{p}) = \frac{h(\hat{p})}{1 - \hat{p}} = \rho_L - \frac{\hat{p}}{1 - \hat{p}}\rho_H, \tag{SA.35}$$

which is decreasing in $\hat{p}$. So, it suffices to show that $\left(\frac{-f(\hat{p},1)}{\eta(\hat{p})}\right)$ is increasing in $\hat{p}$. For this, observe

$$\begin{aligned}
\frac{d}{d\hat{p}}\left(\frac{-f(\hat{p},1)}{\eta(\hat{p})}\right) &= \frac{-f'(\hat{p},1)\eta(\hat{p}) + f(\hat{p},1)\eta'(\hat{p})}{\eta(\hat{p})^2} \\
&= \frac{-f'(\hat{p},1)\eta(\hat{p}) + f(\hat{p},1)(-r(\hat{p})\eta(\hat{p}) - 1)}{\eta(\hat{p})^2} \\
&= \frac{\hat{p}\lambda x \eta(\hat{p}) - f(\hat{p},1)}{\eta(\hat{p})^2} > 0,
\end{aligned}$$

where the second equality follows from the fact that $\eta'(\hat{p}) = -r(\hat{p})\eta(\hat{p}) - 1$ (by the definition of η), the third equality from (SA.31), and the inequality from $f(\hat{p}, 1) < 0$.

As $\hat{p} \to \pi_1$, we have $z(\hat{p}) \to 1$. Then, by (SA.28), we must have $m(\hat{p}) \to 0$ since $f(\hat{p}, 1) \to 0$, which, by (SA.34), implies $\mathbb{E}_p[y_{\hat{p}}(p)|\omega = H] \to 1$ as $\hat{p} \to \pi_1$.

As $\hat{p} \to \pi_0$, we have $z(\hat{p}) \to 0$ by definition of $z(\hat{p})$. Then, by (SA.28), we have $\phi(\hat{p}) \to 0$, which implies $m(\hat{p}) \to 1$ by (SA.32). Thus, by (SA.34), $\mathbb{E}_p[y_{\hat{p}}(p)|\omega = H] \to 0$ as $\hat{p} \to \pi_0$.

Thus, there exists $\bar{p}(x) \in (\pi_1, \pi_0)$ that satisfies (9). The continuity of $\bar{p}(x)$ in x follows from the continuity of $z(\cdot)$ and $m(\cdot)$ in x.

Given that $\mathbb{E}_p[y_{\hat{p}}(p)|\omega = H, x]$ is decreasing from 1 to 0 as $\hat{p}$ increases from π_1 to π_0, it is clear that $\bar{p}(x)$ is decreasing from π_0 to π_1 as $\bar{y}$ increases from 0 to 1. □

Lemma 15. *As $\hat{p}$ becomes lower, $\Phi(p)$ falls for all $p \geq \hat{p}$. Hence, Φ exhibits a mean-preserving spread as $\hat{p}$ becomes lower.*

Proof. For $p \geq \hat{p}$, we have $\Phi(p) = m(\hat{p}) + \phi(\hat{p}) \int_{\hat{p}}^{p} \exp\left(\int_{\hat{p}}^{t} r(s)ds\right) dt$ and thus

$$\begin{aligned}\frac{\partial \Phi(p)}{\partial \hat{p}} &= m'(\hat{p}) + \phi'(\hat{p}) \int_{\hat{p}}^{p} \exp\left(\int_{\hat{p}}^{t} r(s)ds\right) dt - \phi(\hat{p}) \left[1 + r(\hat{p}) \int_{\hat{p}}^{p} \exp\left(\int_{\hat{p}}^{t} r(s)ds\right) dt\right] \\ &= m'(\hat{p}) - \phi(\hat{p}) + [\phi'(\hat{p}) - \phi(\hat{p})r(\hat{p})] \int_{\hat{p}}^{p} \exp\left(\int_{\hat{p}}^{t} r(s)ds\right) dt = m'(\hat{p}) - \phi(\hat{p}) > 0,\end{aligned}$$

where the second equality follows from the definition of ϕ. To prove the inequality, observe that

$$m(\hat{p}) = -\frac{\phi(\hat{p}) f(\hat{p}, 1)}{\hat{p}\lambda x z(\hat{p})} = -\frac{\phi(\hat{p}) f(\hat{p}, 1)(1 - \hat{p})}{h(\hat{p})} \quad \text{or} \quad m(\hat{p})h(\hat{p}) = -\phi(\hat{p}) f(\hat{p}, 1)(1 - \hat{p}),$$

where the second equality follows from (SA.35). By differentiating both sides, we obtain

$$\begin{aligned}m'(\hat{p})h(\hat{p}) + m(\hat{p})h'(\hat{p}) &= -\phi'(\hat{p}) f(\hat{p}, 1)(1 - \hat{p}) - \phi(\hat{p})(f'(\hat{p}, 1)(1 - \hat{p}) - f(\hat{p})) \\ &= \phi(\hat{p})(\lambda x\hat{p} + f'(\hat{p}, 1))(1 - \hat{p}) - \phi(\hat{p})[f'(\hat{p}, 1)(1 - \hat{p}) - f(\hat{p})] \\ &= \phi(\hat{p})(\lambda x\hat{p}(1 - \hat{p}) + f(\hat{p})) = \phi(\hat{p})h(\hat{p}),\end{aligned}$$

where the second equality follows from (SA.29). Rearranging the resulting equation yields

$$m'(\hat{p}) - \phi(\hat{p}) = -\frac{m(\hat{p})h'(\hat{p})}{h(\hat{p})} = \frac{m(\hat{p})(\rho_L + \rho_H)}{(1 - \hat{p})\rho_L - \hat{p}\rho_H} > 0,$$

where the inequality holds since $\hat{p} < \pi_0 = \frac{\rho_L}{\rho_L + \rho_H}$. □