To preempt or not to preempt? Clustering *vs.* single entry in an oligopoly*

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Abstract

In this work, we develop a real option model in continuous time in which firms compete perfectly on market entry but compete oligopolistically after having entered. We analytically show that in equilibrium, entry may be clustered and that firms tend to enter in weakly larger clusters as the market becomes more crowded. This equilibrium is unique up to a permutation of players. We also show that with moderate market risk, higher volatility delays entry and leads to larger groups. We explore the following special cases: i) In Cournot competition with a linear demand function, we show that firms always enter individually. ii) In Bertrand competition, a firm enters only for finite market levels. iii) Firm profits follow an inverse exponential function of the number of firms, and we find a closed-form solution in which the cluster size remains constant over time and increases with risk.

JEL: C72, C73, D81, L13

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1 Introduction

Models of real options are commonly used to address questions of the timing of investments or market entry and have been widely used in numerous applications (for example, in natural resources (Paddock et al. (1988)), real estate (Bar-Ilan and Strange (1996)), pharmaceuticals and R&D (Gunther McGrath and Nerkar (2004))). Any real options model needs to make assumptions about the extent of competition, first, over the option itself and, second, in the product market once a firm has entered. The real options literature, starting with McDonald and Siegel (1986) and Dixit and Pindyck (1994), first studied the case of a firm that has a monopoly over the option to enter and faces a perfectly competitive market after entry. For example, this description fits a firm with a license to operate a small mine, such that the firm is a price taker when selling the mined material (Tufano (1996)).

Many natural applications of real options, however, fall outside this description. Depending on the evolution of the market size, a firm may need to decide if and when to enter a market that is served by the limited number of firms that have entered it at some prior point in time. Several papers (Smets (1991), Grenadier (1996), Dutta et al. (1995), Huisman and Kort (1999)) study such situations, assuming that a fixed number of firms can potentially enter a market in which, once entered, they compete with each other imperfectly. These models have a specific feature: as more firms enter, fewer firms are left as potential entrants. This grants more market power over the entry option to later entrants, leading to a monopoly over the entry option for the last firm that has not yet entered. This assumption is convenient because the model can be solved backward from the last potential entrant. However, the model also becomes very complex since the value of the option itself changes as a function of the remaining number of potential entrants. Equilibria are therefore difficult to characterize. Moreover, from a methodological point of view, one cannot explore the limiting case of this model in which the number of potential entrants goes to infinity, as the backward induction argument no longer applies straightforwardly.

The aim of this paper is to study a contestable market for the option to enter, where firms that do enter compete imperfectly with each other in the product market. This setting not only addresses the gap in the real option literature but also reflects real-world situations. In a world without barriers, each time a firm evaluates entry into a market, it considers other potential entrants. Therefore, there is never a definitive "last firm" entering, as each potential entrant assesses the market dynamics on the basis of the presence of other players.[1]

We model an infinite number of potential entrants, who can make costly and irreversible entry decisions in continuous time. Firms that enter receive a profit flow that evolves stochastically, for example, because of demand fluctuations. We model imperfect competition in the product market by assuming that profit flow is a decreasing and convex function of firms that are active in the market, as would be the case, for example, in standard Cournot competition. When firms decide to enter, they need to consider the following trade-off: If they enter, they face the risk of downside losses if market demand decreases in the future. On the other hand, if market demand increases, they can enjoy high profits until further entry potentially increases competition and reduces profits again. Moreover, firms need to be aware that other potential entrants may preempt them at any moment.

[1] Advancements in technology and regulations have decreased barriers to entry—either in terms of know-how or accessing credit—in numerous sectors, leading to a continuous flow of potential competitors who are ready to enter once the market reaches a certain size. Examples in which new technologies reduce the barriers to entry are found in multiple sectors, from e-commerce to transportation services and healthcare. For example, platforms such as Shopify, Etsy and, in a certain way, Amazon favor the creation and development of independent businesses, whereas platforms such as Google Play or Apple allow the easy launch and monetization of new apps. YouTube and, in general, social media platforms have eased the rise of individual video creators.

We prove the existence of an equilibrium that is unique up to a permutation of firm identities and characterize the equilibrium entry decisions of firms. We show that the time between successive entries increases with market size; i.e., as the market grows over time, entry events become less frequent. This is because the downside risk from entry does not change, but the additional profit from being in the market is relatively low, given that there is already a larger number of competing firms. Moreover, we show that entry may be clustered and that the size of clusters weakly increases over time. We analytically characterize the exact size distribution for three special cases. First, for Cournot competition with linear demand, we obtain clusters of size one at each entry time. Second, in Bertrand competition, we have that only one firm enters, and the rest enter only when the market is infinitely large. Third, if profit flow is an exponentially decreasing function of the number of active firms, we obtain clusters with multiple firms and constant cluster sizes.

These clustering results are consistent with empirical studies that have examined the evolution of the number of firms in newly created markets (such as automobiles, computers, and tires). A consistent observation is that firms tend to enter individually, and as the market matures, the rate of firm entry increases, resulting in collective entries, referred to as temporal clustering (Agarwal and Bayus (2002), Klepper and Simons (2000)). Intriguingly, in some cases, potential entrants opt to abstain, even in the presence of lucrative profit margins (see, for example, Agarwal and Gort (1996) and Klepper (2002)).

The findings regarding firm clusters link our paper to Leahy (1993) and Karatzas and Baldursson (1996), who study the special case in which infinitely many entrants compete perfectly after entry. They show that entry may be clustered such that multiple firms enter as soon as the net present value of doing so is zero. However, in those models, the profit flow is independent of the number of active firms, so the size of potential clusters is indeterminate. We endogenize the size of clusters by linking profits to the number of active firms. Furthermore, we show that clustering may occur even when product market competition is imperfect. In other words, the driving force for clustering is the feature that the entry option is contestable. [2]

We show that perfect competition for entry drives the value of the option of waiting down to zero in equilibrium.[3] This finding aligns with real option models that incorporate perfect competition both in the option to invest and in the product market, as discussed by Leahy (1993) and Karatzas and Baldursson (1996). This result contrasts with models in which a finite number of firms have the option to invest. Despite competition on entry reducing the option value, as shown by Smets (1991), Grenadier (1996), and Dutta et al. (1995), the value of the option to invest remains positive and varies with the number of firms that have not yet entered the market (Huisman and Kort (1999)).

Risk emerges as the main determinant that impacts entry strategies, influencing both the timing of entry and the patterns of clustering. We analytically show that for moderate and low levels of risk, increasing risk induces firms to wait longer. With more volatility, extreme demand levels—both higher and lower—occur with higher probability, affecting a firm's payoffs in two ways. First, the increased probability of extreme values does not have a symmetric effect on the firm's payoff: the upside gains increase less than the downside losses do, as the upside potential is limited by future new entries. Second, the expected time during which a firm enjoys a higher profit margin decreases. For these reasons, firms wait longer before entering.

[2] Huisman and Kort (1999) assume that there are at most 3 potential entrants and show that simultaneous entry is possible. This can be viewed as an edge case in which simultaneous entry of 2 firms can occur.

[3] A reduction in option value when the number of potential competitors increases is empirically documented in Pavan et al. (2020). Using an exogenous shock in regulation in the gas industry in Italy, they find that an increase in the number of potential competitors drives earlier entry.

The fact that the greater the uncertainty is, the longer a firm waits to enter has already been reported in traditional real option theory when the firm has a monopoly on the entry option (McDonald and Siegel (1986) and Dixit and Pindyck (1994)). The reasons for this, however, differ. In traditional real option theory, a firm waits longer because the value of the option to invest increases in uncertainty. In our model, the value of this option is constant and equal to zero, due to the perfect competition faced at the entry decision. The lower prospect of high profits induces the firm to wait longer.

This monotonic effect of risk on the timing of entry also differs from that in the oligopolistic case. When a finite number of firms have the option to enter, the last firm that enters has a monopoly on the entry option, and the greater the risk, the longer it waits. The firm that enters before the last one might take advantage of having this extra time and therefore enter earlier (Rossetto and Perotti (2004)). This does not occur in our setting, as the option value is constant.

Risk also affects the cluster size. Higher volatility increases the likelihood of market downturns, but it has a limited effect on the upside potential due to new entries. This reduces the attractiveness of preempting, and clustering, that is, entering along with other competitors for larger market size becomes preferable. Hence, we analytically show that when the risk is moderate or low, the cluster size increases as risk increases.

Our model contributes not only to the literature on real options but also to the literature on clustering. Previous studies explored clustering, usually considering various forms of friction as explanatory factors. These include coordination failures (Levin and Peck (2003)), positive network externalities (Mason and Weeds (2010)), and informational spillovers (Chamley and Gale (1994)). Without assuming entry frictions, Bouis et al. (2009) demonstrated that instances of simultaneous entry can occur when the number of potential entrants is three. To generalize this scenario, Argenziano and Schmidt-Dengler (2014) extended the analysis to include settings with a finite number of potential entrants. However, a key assumption in this model is that investment costs decrease exogenously over time. Our model shows that even without any form of friction and with constant investment costs, simultaneous entry can occur. Given the simplicity of our assumptions, our model serves as a valuable benchmark for both theoretical and empirical investigations aimed at studying market dynamics.

The structure of the paper is as follows: In Section 2, we introduce the model, and the impact of risk is discussed in Section 3. Section 4 examines three specific cases, namely, Cournot with a linear demand function, Bertrand competition and profit margins that exhibit exponential decline. Finally, our conclusions are presented in Section 5. Detailed proofs are provided in the Appendix.

2 The Model

2.1 Setting

We consider a market with an infinite number of identical firms with an infinite time horizon. Each firm can choose if and when to enter the market, which incurs an irreversible cost, K. Once it has entered, the firm competes with the other firms that have already entered the market and receives an instantaneous profit flow of $D_n\theta(t)$. D_n is the profit margin, which is deterministic and varies with the number n of firms in the market; $\theta(t)$ is the market size at instant t.

We assume the following: i) $D_n > D_{n+1}$; ii) $\frac{D_n}{D_{n+1}} \geq \frac{D_{n+1}}{D_{n+2}} \geq 1$; iii) $\lim_{n\to\infty} D_n = 0$ and $\lim_{n\to\infty}(D_n - D_{n+1}) = 0$. These assumptions ensure that as the number of firms active in the market increases, the rate of profit flow decreases in a convex way and that as the number of firms in the market tends toward infinity, the profit flow and the reduction in profit flow due to new entrants tend toward zero. These assumptions capture the idea that a market becomes increasingly competitive as more firms enter.

The market size, $\theta(t)$, is stochastic and obeys a geometric Brownian motion:

$$d\theta\left(t\right)=\mu\theta\left(t\right)dt+\sigma\theta\left(t\right)dz \tag{1}$$

where μ is the drift parameter and σ is the volatility parameter. dt and dz are the time and Wiener process increments, respectively. The geometric Brownian motion assumption is standard in real option theory and implies that the demand and hence the profit flows are always positive. Firms are risk-neutral value maximizers and discount the future at the rate r. To ensure finite valuations, we assume that $\mu < r$.

We assume that the initial $\theta(0)$ is low enough that it is not optimal for any firm to enter immediately. At each point in time, a firm considers whether to enter or not, taking into account the number of firms already in the market, the investment strategy of its competitors and the impact of its (and its competitors') investment decisions on current and future profits. The future entry strategies of its competitors are important because once each firm has entered, it competes with the other firms in the market. Initially, a firm receives a profit flow that depends on the number of firms in the market when it enters. If more firms enter in the future, however, the firm's profit flow will decrease. Competitors' entry strategies affect the number of firms in the market and hence the expected future cash flow from entering.

For more concise notation, we recognize that each firm's strategy can be represented by its choice of entry thresholds (demand levels at which the firm would enter the market given the current market structure, *i.e.*, the number of firms in the market). Furthermore, since all the firms are identical, each firm follows the same strategy, which consists of a set of potential entry thresholds. For this reason, we omit the firm-specific index. We define an entry event as one in which any positive number of firms enter the market, and we index the entry events by $i, i = 1, 2, \ldots,$ and set θ_i as the demand level at which entry event i occurs.[4]

At entry event i, $j(i) \geq 1$ firms enter the market, bringing the overall number of firms in the market to $n(i)$. We call the event of only one firm entering ($j(i) = 1$) a *single* entry, and a *cluster* entry occurs when multiple firms enter simultaneously during entry event i, i.e., when $j(i) > 1$. As is standard in strategic real options, we restrict our attention to symmetric pure strategies where no coordination failures occur.[5]

After entry event i, all firms in the market receive instantaneous profit flows of $D_{n(i)}\theta(t)$ until the market reaches the next entry event threshold, θ_{i+1}. At that point, all active firms' instantaneous profit flows drop to $D_{n(i+1)}\theta(t)$ until market demand reaches θ_{i+2}, *etc.*.[6]

The equilibrium concept adopted here is that of Markov subgame perfection. Hence, an equilibrium is 1) a set of market demand thresholds, $\hat{\theta}_i$, and 2) a set of investment strategies, $\hat{s}(i)$, for all firms in the industry at entry event i, *i.e.*, the number of firms $\hat{j}(i)$ that enter in entry event i. This determines $\hat{n}(i)$, the number of firms active in equilibrium after event i.

[4] We formally show that the first time the market reaches a specific level θ_i, entry occurs and that entry event $i+1$ will occur only for a $\theta > \theta_i$.

[5] For a discussion of the possibility of coordination failure in the duopoly case, see Thijssen et al. (2012).

[6] With some abuse of notation, the subscript indicates either the number of firms present in the market or the event i.

2.2 Optimal Entry

We first derive some general results related to 1) the value of an inactive firm, that is, a firm that has not yet entered the market; 2) the value of an active firm, that is, a firm that has entered the market; 3) how a firm's optimal entry strategy varies with the number of firms that are already in the market.

The key decision facing each inactive firm is when to enter the market, i.e., at what demand level, θ, to enter. At each point in time, an inactive firm decides whether to enter the market, knowing the entry strategy of its competitors, who are not yet in the market: it can enter individually, preempting its competitors and achieving abnormal profits for a limited period until the competitors optimally enter, or it can enter later, knowing that other firms will enter at the same threshold.

We define $V_i(\theta; \{\hat{\theta}_l, l \geq i+1\})$ as the value of an active firm after event i but before event $i+1$, when the current demand level is θ, and given the future equilibrium entry events l at $\hat{\theta}_l$. Similarly, $W_i(\theta; \{\hat{\theta}_l, l \geq i+1\})$ is the value of an inactive firm after event i but before event $i+1$. Note that each firm's value depends on the strategies (entry thresholds) of its competitors (and its own if it has not yet entered). Therefore, initially, before any firm has entered the market, each firm's value is $W_0(\theta; \{\hat{\theta}_l, l \geq 1\})$ for $\theta \leq \hat{\theta}_1$.

Lemma 1 *The value of an inactive firm is* $W_i = 0$.

Lemma 1 states that the value of an inactive firm is always zero. This result is an implication of perfect competition in the market: even when the number of firms in the market approaches infinity, there are still an infinite number of potential entrants. In the limit, as the market is infinitely large, firms that have not yet entered will enter as soon as the expected value of entering is positive. As the option to enter has zero value, all firms enter as soon as the value of being active net the investment cost, $V_i - K$, becomes positive, irrespective of the market size and the existing number of firms already operating in the market.

The fact that the number of potential entrants remains constant (and infinite) allows us to reconcile two seemingly contradictory results in the real option literature.

On the one hand, some papers have studied situations in which firms compete perfectly both on the option to enter and on the product market. There, as in our case, the entry option has zero value (see Leahy (1993) and Grenadier (2002)).[7]

On the other hand, past studies have looked at the case in which firms compete oligopolistically both at the entry level and in the product market (Smets (1991), Rossetto and Perotti (2004) and Bouis et al. (2009)). In such a case, the number of potential entrants is finite, and at each entry event, the number of potential entrants falls; hence, the competition for the option to enter decreases. In the limit, when only one firm with the option to enter is left, it will enter as in the standard real option model, as discussed in McDonald and Siegel (1986) and Dixit and Pindyck (1994). The value of the option to wait therefore decreases as the number of firms that have not yet entered increases. Note that this approach does not lend itself to the limit case in which n goes to infinity, as the backward induction argument cannot be applied.

As such, our model cannot be considered a special case of the oligopoly case with finite firms. In our model, the number of potential entrants is infinite, firms always face perfect competition on the option to enter, and the number of potential entrants remains infinite. This fundamentally changes the entry dynamics. An inactive firm contemplating entry assesses the value of becoming active versus remaining inactive,

[7] Given the perfect competition in the market, these studies assume that the price, not the market size, is stochastic.

which invariably yields a value of 0 for the latter.

Lemma 2 *Given the optimal entry at event $i+1$, $\hat{\theta}_{i+1}$, the value of an active firm after the i-th entry event when there are $n(i)$ firms in the market is*

$$V_i(\theta) = \frac{D_{n(i)}}{r-\mu}\theta + A_i\theta^{\beta_1} \qquad 0 \le \theta \le \hat{\theta}_{i+1} \tag{2}$$

where

$$A_i = \left(K - \frac{D_{n(i)}}{r-\mu}\hat{\theta}_{i+1}\right)\hat{\theta}_{i+1}^{-\beta_1} \tag{3}$$

$$= \sum_{l=0}^{\infty} \frac{D_{\hat{n}(i+l+1)} - D_{\hat{n}(i+l)}}{r-\mu}\hat{\theta}_{i+l+1}^{-\beta_1+1} \le 0. \tag{4}$$

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{5}$$

The value of an active firm depends only on the present profit flow and future entry events. Past entry events do not play a role. This value can be divided into two components. The first is the present value of profits if $n(i)$ and only $n(i)$ firms remain in the market indefinitely, $\frac{D_{n(i)}}{r-\mu}\theta$. The second component, $A_i\theta^{\beta_1}$, represents the impact of future market entry, which depends on all future entry thresholds, $\{\hat{\theta}_l, l > i\}$, and the number of firms that enter at each future entry event, $\{\hat{j}_l, l > i\}$, as shown in (4). Since the rate of profit flow decreases as more firms enter the market, this term is negative; that is, $A_i \le 0$. The anticipation of future entry reduces the value of an active firm relative to the capitalized value of its current rate of profits. For this reason, A_i increases, that is, becomes less negative, for subsequent entry events (see equation (4)).

We define the incremental value of entering at the i-th entry event, $S_i(\theta)$, for a generic demand level θ as the net present value of the future cash flows when entering at event i, $V_i(\theta)$, minus the investment costs K and the value W_{i-1} of remaining inactive:

$$S_i(\theta) = V_i(\theta) - K - W_{i-1}(\theta) \tag{6}$$

Corollary 1 *The incremental value, $S_i(\theta)$, of entering at the i-th entry event is:*

$$S_i(\theta) = A_i\theta^{\beta_1} + \frac{D_{n(i)}}{r-\mu}\theta - K, \qquad \theta \leq \hat{\theta}_{i+1} \tag{7}$$

where $A_i \leq 0$ is given by (3).

Upon entering, the firm pays the initial investment cost K, foregoes the option of investing later W_{i-1}, and receives the present value of the ongoing stream of profits, accounting for optimal future entry by its competitors, V_i. In practice, since the option of investing later has zero value, the incremental value is equivalent to the net present value (NPV), considering the entry of future competitors.

Since $D_{n(i)} \geq 0$, $A_i \leq 0$ and $\beta_1 > 1$, the net value from entering $S_i(\theta)$ is a concave function of θ. It initially increases from $S_i(0) = -K < 0$, reaches a maximum and then decreases, with $\lim_{\theta\to\infty} S_i(\theta) = -\infty$. It follows that $S_i(\theta)$ crosses the θ axis at most twice. Specifically, $\frac{\partial S_i(\theta)}{\partial\theta} \geq 0$ at the smaller of the two intersections, and $\frac{\partial S_i(\theta)}{\partial\theta} < 0$ at the larger intersection. For a visual representation, refer to Figure 1, where S_i is depicted as a solid line.

Firms enter at the demand level $\hat{\theta}_i$, when the incremental value of entry becomes positive. In other words, they enter when the value of being active, V_i, exceeds the investment costs, K, from below. The optimal entry point, $\hat{\theta}_i$, corresponds to the smallest θ for which $S_i(\theta) \geq 0$. Thus, it must satisfy the condition $\left.\frac{\partial S_i(\theta)}{\partial\theta}\right|_{\theta=\hat{\theta}_i} \geq 0$. As the value of the entry option at any entry event is always zero, the second solution of

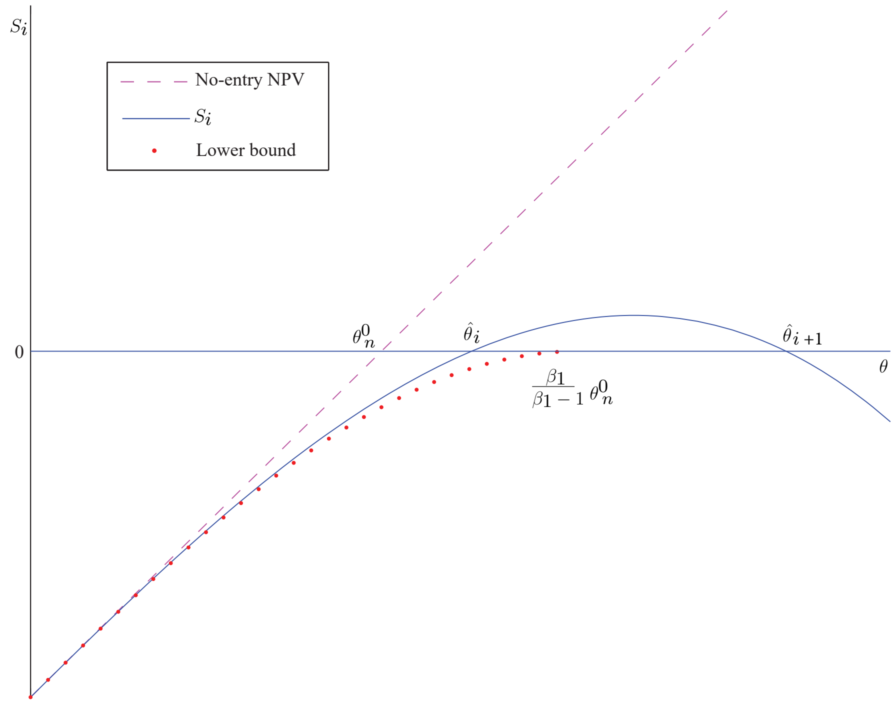

Figure 1: Stylized graph of S_i as a function of θ, with the discounted profit stream assuming no further entry (upper dashed line) and the lower bound for S_i (lower dotted line).

$S_i = 0$ is the optimal entry threshold of event $i+1$, $\hat{\theta}_{i+1}$. From the value-matching condition of optimal entry (see condition (26) of the proof of Lemma 1), $\hat{\theta}_{i+1}$ is also a solution of $S_i(\theta) = 0$. Since the i-th and $i+1$-th entry events are distinct by definition, it follows that $\hat{\theta}_i < \hat{\theta}_{i+1}$ and $\left.\frac{\partial S_i(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}_{i+1}} < 0$, where $\hat{\theta}_{i+1}$ is a summary statistic used to represent the next entry event.[8]

The underlying rationale for this outcome can be described as follows: consider an inactive firm contemplating entry into the market, with the knowledge that at a

[8]We use this condition to determine the optimal entry characteristics for each event i, *viz.*, the number of firms entering and the entry threshold.

specific market size, $\hat{\theta}_{i+1}$, there will be a certain number of firms, $\hat{n}(i+1)$. The firm faces a choice between entering earlier or later. Should it choose to enter, it stands to benefit temporarily from higher profit margins due to limited competition, i.e., $D_{n(i)} > D_{n(i+1)}$. However, it also faces potential losses if the market size decreases, i.e., if θ decreases. On the other hand, if the firm opts to wait, it forgoes the short-term additional profit flow, $D_{n(i)}\theta$, but avoids exposure to downside risks. When market demand is very low, i.e., θ is low, the downside risk outweighs the temporary profit gain. As θ increases, the temporary profits begin to offset the downside risk. Beyond a certain demand threshold, the expected temporary profits start to decrease because of the rising likelihood of a future entry event. Therefore, the firm decides to enter only when the incremental profit margin and the expected time during which it will enjoy these extra profits are substantial enough to outweigh the potential downside losses.

The entry threshold for event i represents the point where the incremental value function for event i increases, going from negative to positive values. This means that the value of becoming active exceeds the value of remaining inactive. However, as the market demand θ approaches the level where other firms are about to enter the market at event $i+1$, the firm's value decreases since the anticipation of future entry reduces the expected profit margin. At this point, the value of the firm starts to decrease in θ.

When there are n firms in the market and no further entry occurs in the future, the value of the firm is $\frac{D_n}{\mu-r}\theta(t)$. We define the "no-entry" zero-NPV threshold, $\theta_n^0 = \frac{K(\mu-r)}{D_n}$, as the entry threshold in such a situation.

Corollary 2 *If $\hat{\theta}_i < \hat{\theta}_{i+1}$, then*

$$\theta_n^0 \leq \hat{\theta}_i < \frac{\beta_1}{\beta_1 - 1}\theta_n^0 \tag{8}$$

Figure 1 provides a graphical representation of the corollary above, illustrating the boundaries within which the incremental value curve resides. The dashed line corresponds to the discounted profit flow minus the investment cost when there are $n(i)$ firms in the market, $\frac{D_{n(i)}}{\mu - r}\theta - K$. This is the "no-entry" NPV, which ignores the potential impact of future entries. This line crosses the horizontal axis at θ_n^0, the "no-entry" zero-NPV threshold at which n firms are in the market and no further entry is to occur in the future.

The lower dotted line represents the theoretical lower boundary of $S_i(\theta)$, which touches the axis at a single point, specifically $\frac{\beta_1}{\beta - 1}\theta_{n(i)}^0$. This line represents the incremental value of a firm entering the market under conditions of perfect competition in the product market, as discussed in Leahy (1993) and Grenadier (2002) and reviewed in Dixit and Pindyck (1994). $\frac{\beta_1}{\beta - 1}\theta_n^0$ denotes the entry threshold in such a competitive environment.

The solid line in the graph represents the incremental value of a firm entering at event i, denoted as S_i. It falls within the range defined by the "no-entry" NPV, i.e., the dashed line, and the incremental value under perfect competition, i.e., the dotted line.

The i-th entry threshold, $\hat{\theta}i$, is depicted as the lower point where S_i intersects the horizontal axis (indicating $\frac{\partial S_i}{\partial \theta} > 0$). Conversely, the $i+1$-th entry threshold, $\hat{\theta}_{i+1}$, is the higher point where S_i crosses the axis (and where $\frac{\partial S_i}{\partial \theta} < 0$). It follows that any preemptive threshold $\hat{\theta}_i$ must fall between the "no-entry" threshold, $\theta_{n(i)}^0$, and $\frac{\beta_1}{\beta_1 - 1}\theta_{n(i)}^0$ (where $S_i(\theta)$ touches the axis only once).

We can also derive some features of the expected length of time between entry events.

Lemma 3

$$\frac{\theta_{i+1}}{\theta_i} \leq \frac{\theta_{i+2}}{\theta_{i+1}} \tag{9}$$

This lemma states that as the market expands, the expected time interval between successive entry events also widens. The expected time between entries is $\frac{1}{\mu-\frac{1}{2}\sigma^2}\ln\frac{\theta_{i+1}}{\theta_i}$ when $\mu > \frac{1}{2}\sigma^2$ and ∞ otherwise. With market growth, new entries occur, causing a reduction in profit margins. As firms enter when the NPV becomes positive, the time gap between entry events increases.

The regularity of the expected time between entry events is new in real option theory. In conventional finite oligopolistic real option models, this finding is not demonstrated analytically. The challenge in that case is that the option value differs depending on the number of potential entrants. In our context, where firms engage in perfect competition for the entry option, the payoff remains constant and equals zero.

2.3 Individual vs. Cluster Entry

To complete the characterization of the equilibrium, we need to find the equilibrium number of firms, $\hat{j}(i+1)$, that participate in each entry event $i+1$. The proof proceeds in two steps. We first identify the potential candidates for $\hat{j}(i+1)$; then, among these candidates, we focus on the choices that do not induce any deviations by other firms.

Lemma 2 indicates that $\hat{\theta}_{i+1}$ and $\hat{n}(i+1)$ depend on *future* but not *past* entry events. This implies that at $\hat{\theta}_{i+1}$, event $i+1$ occurs and that after this event, there will be $\hat{n}(i+1)$ firms in the market. We need to find the equilibrium number of firms that enter at entry event $i+1$, $\hat{j}(i+1)$. This corresponds to finding the number of firms that are present in the market after event i, that is, $n(i) = n(i+1) - j(i+1)$.

From corollary 1, we know that the incremental value of a firm entering at event i, S_i, crosses the horizontal axis at θ_i and θ_{i+1}, with $\theta_i < \theta_{i+1}$. Hence, the following definition is obtained:

Definition 1 *j is a **feasible** number of firms to enter at entry event $i+1$ if the optimal entry $\hat{\theta}_{\hat{n}(i+1)-j} < \hat{\theta}_{\hat{n}(i+1)}$.*

Given the number of firms in the market after event $i+1$, $n(i+1)$, each candidate j corresponds to a value of $n(i) = n(i+1) - j$, a value of $D_{n(i)}$ and an incremental value of the firm entering at event i. For this reason, we adapt the notation to include the possibility of having different incremental values of the firm depending on j. We define $S_{\hat{n}(i+1)-j,j}$ as the incremental value of event i when there are $\hat{n}(i+1) - j$ firms in the market and when j firms will enter at event $i+1$. By definition, if j is the equilibrium number of firms entering, i.e., $\hat{j}(i+1)$, then $S_{\hat{n}(i+1)-\hat{j},\hat{j}} = S_i$. Denote $\hat{\theta}_{\hat{n}(i+1)-j,j}$ as the other solution of $S_{\hat{n}(i+1)-j,j}$ in addition to $\hat{\theta}(i+1)$.

Proposition 1 *For each entry event, $i+1$, the number of firms that enter together at $\hat{\theta}_{i+1}$, $\hat{j}(i+1)$, is the smallest feasible j, i.e., the smallest j s.t.*

$$\left.\frac{\partial S_{\hat{n}(i+1)-j,j}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < 0 \tag{10}$$

The proof follows two steps. We first find the feasible number of firms that can enter at entry event $i+1$, $j(i+1)$. We then show that the equilibrium number of firms that enter at event $i+1$ is the smallest feasible j.

The intuition of the proof is as follows. First, note that $S_{\hat{n}(i+1)-j,j}$ has the same characteristics as S_i; that is, it is a concave function of θ and equals 0 at no more than two values of θ, one of which is $\hat{\theta}_{i+1}$ (see Fig. 1). For j to be a feasible number of firms to enter at $\hat{\theta}_{i+1}$, $\hat{\theta}_{i+1}$ must be the largest solution of $S_{\hat{n}(i+1)-j,j} = 0$. It follows that the other solution is smaller than $\hat{\theta}_{i+1}$ and is the candidate optimal entry threshold of event i. A necessary and sufficient condition for $\hat{\theta}_{i+1}$ to be the largest solution is that at $\hat{\theta}_{i+1}$, the value function is decreasing; then, j is feasible if and only if $\left.\frac{\partial S_{\hat{n}(i+1)-j,j}(\theta|\hat{\theta}_{i+1})}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < 0$, and otherwise, j is not feasible.

It follows that an optimal entry threshold $\hat{\theta}(i+1)$ is the crossing point of two successive incremental value functions, $S_{\hat{n}(i+1)-j}(\theta|\hat{\theta}_{i+1})$ and $S_{\hat{n}(i+1)}(\theta|\hat{\theta}_{i+1})$. This proposition indicates that $S_{\hat{n}(i+1)-j}(\theta|\hat{\theta}_{i+1})$ crosses $\hat{\theta}(i+1)$ when it is decreasing while $S_{\hat{n}(i+1)}(\theta|\hat{\theta}_{i+1})$ is increasing.

Figure 2 shows a graphic representation of the different incremental value functions at subsequent entry events given the subsequent entry of the other firms. Each incremental value function crosses the horizontal axis, i.e., the value of the option to remain inactive, for two values of θ, one of which is in common with the next entering firms.

Intuitively, for each demand level, i.e., θ, an inactive firm considers preempting or not, knowing that at the next entry event $i+1$, there will be $n(i+1)$ firms. At each θ, a firm decides to preempt at event i or enter at event $i+1$, trading off the additional profits from preempting against the associated downside risk. If the differential in profit flow, $D_{n(i)} - D_{n(i+1)}$, is large enough, the firm preempts. Conversely, if the differential is not large enough, the firm prefers to wait until event $i+1$. This trade-off determines the minimum number of firms that enter at event $i+1$.

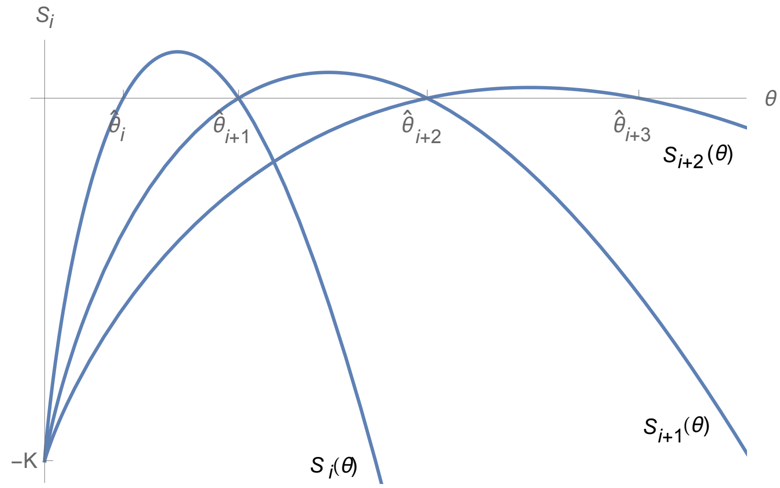

Figure 2: Sequential entry

The final step is to show that among all the feasible j values, the smallest is the equilibrium value. To provide an intuition of this result, consider a scenario in which one fewer firm, the equilibrium number $\hat{j}(i+1)$, attempts to enter. This means that one extra firm preempts, as $n(i) = n(i+1) - \hat{j}(i+1) + 1$. Therefore, the temporary profit flow stemming from event i is lower. If these firms decide to enter the market at the same demand level, the expected future cash flows will not be sufficient to offset the downside risk. These firms do not want to enter earlier to avoid the downside risk, and they will opt to enter at higher market demand levels. These demand levels will be equal to or greater than $\hat{\theta}_{i+1}$. On the other hand, a scenario in which more firms enter than $\hat{j}(i+1)$ also fails to establish equilibrium. In this case, the increased cash flows make it appealing for these firms to enter at lower demand levels. However, this prompts inactive firms to preempt their entry and join the market before event $i+1$, which, in turn, reduces the expected cash flows. Consequently, this scenario cannot be an equilibrium for event i.

Having found the optimal number of firms that enter at each entry event, we summarize our results in the following theorem.

Theorem 1 *For each entry event $i+1$, the number of firms that enter together at $\hat{\theta}_{i+1}$, $\hat{j}(i+1)$, is the smallest j such that*

$$\frac{\beta_1}{\beta_1 - 1}\theta^0_{\hat{n}(i+1)-j} < \hat{\theta}_{i+1} \tag{11}$$

$\hat{\theta}_i$ is then given by the smallest solution to

$$A_i\theta^{\beta_1} + \frac{D_{\hat{n}(i)}}{r-\mu}\theta - K = 0 \tag{12}$$

where $\hat{n}(i) = \hat{n}(i+1) - \hat{j}(i+1)$ and

$$A_i = A_{i+1} + \left(\frac{D_{\hat{n}(i+1)} - D_{\hat{n}(i)}}{r-\mu}\right)\hat{\theta}_{i+1}^{-\beta_1+1} \tag{13}$$

Equation (11) determines the number of firms that will enter at the $i+1$st entry event, $\hat{j}(i+1)$, and thus the number of firms that will be in the market after the preceding entry event i, $\hat{n}(i) = \hat{n}(i+1) - \hat{j}(i+1)$. The number of firms that enter to increase the number of active firms to $\hat{n}(i)$, $\hat{j}(i)$, is determined by the smallest j s.t. $\theta^0_{\hat{n}(i)-j}\frac{\beta_1}{\beta_1-1} < \hat{\theta}_i$. The remainder of Theorem 1 gives the equations required for calculating successive thresholds and value functions.

The next step is to study the regularities in the number of firms entering:

Theorem 2 *$\hat{j}(i) \leq \hat{j}(l)$ for $i < l$.*

Theorem 2 shows that the number of firms entering exhibits a weakly increasing trend. As market demand increases, the number of firms present in the market also increases. Suppose that $j(i)$ enters at event i and that the same number of firms enter at the subsequent event; then, the extra profits from preempting decrease as $D(n)$ is convex in n. Consequently, considering that at event $i+1$, when $\hat{j}(i+1)$ firms enter, bringing the total number of firms in the market to $n(i+1)$, firms contemplating entry at event i encounter a critical trade-off. They must decide whether to enter immediately, incurring a downside risk and securing temporary extra profits. However, if $j(i) = \hat{j}(i+1)$, the differential in temporary profits becomes more substantial, meaning that $D(n(i+1) - 2 \times \hat{j}(i+1)) - D(n(i+1) - \hat{j}(i+1)) > D(n(i+1) - \hat{j}(i+1)) - D(n(i+1))$. This increased profit differential, when it is substantial, leads to condition (11) holding for a smaller j.

The findings from Theorems 1 and 2 together suggest that in markets that are still limited in size and hence involve a limited number of firms, the optimal strategy for successive entrants is to enter sequentially. However, as the market expands, firms deviate from sequential entry patterns, opting for simultaneous entry in waves, with each wave accommodating an increasing number of entering firms. As the market evolves, the increased presence of firms leads to reduced profit margins and expected firm values, compelling firms to wait and increasingly prefer entering alongside other firms. Hence, as the number of firms in the market increases, so does the number of entering clusters, and entry events become less frequent.

These implications diverge from those of earlier models that examine situations characterized by limited potential entrants and a constrained number of firms in the market (e.g., Smets (1991), Grenadier (1996), and Rossetto and Perotti (2004)). In such settings, individual entry is the norm. In Huisman and Kort (1999), clustered entry can occur. However, under the assumption that the number of potential entrants is limited to three, no regularities in cluster size can be identified.

The assumption of perfect competition in the market for entry allows us to show how the pattern of market entry can evolve as the market expands. This enables us to make comparisons with other studies that consider perfect competition both before and after market entry. Similar to Leahy (1993) and Grenadier (2002), we find that the value of a potential entrant in our context is zero because the option to enter has a zero value. Consequently, firms enter the market as soon as the NPV reaches zero. At the same time, however, in perfect competition, firms are price takers, and thus, each time the market reaches the same price threshold, a new firm enters. In these markets, however, the price is exogenous, so no implications for clustering can be derived.

Therefore, our findings serve as a unifying link between two distinct branches of the real option literature: one focused on imperfect competition in both the option to enter and in the market, and the other focused on perfect competition in both the option to enter and the market.

3 Effect of risk on the entry decision

Risk affects the entry dynamics. To analyze the effect of risk on entry, we first look at what happens when volatility changes marginally so that the number of firms entering is not affected. We then look at the effect of risk on the number of firms entering together.

Proposition 2 *If $j(i)$ is constant and $\beta_1 \geq 2$, $\hat{\theta}_i$ increases as σ increases.*

When volatility changes marginally, the cluster size can be considered constant: $\frac{\partial j(i)}{\partial \sigma} = 0$. The above proposition states that, for a constant number of firms entering simultaneously ($\frac{\partial j(i)}{\partial \sigma} = 0$), the entry threshold increases as the risk increases.

As the risk increases, more extreme market values are reached with a higher probability. In such a case, when deciding whether to enter the market earlier or later, a firm considers the timing of subsequent entry events. If the next entry event will occur at sufficiently higher demand levels, the firm will decide to enter at higher demand levels as well to reduce the risk of extremely low market values. In the above proposition, we show that for moderate risk levels, specifically when $\beta_1 \geq 2$, this behavior arises.[9]

Our result differs from those of standard real option models, in which firms face oligopolistic competition. In those models, the effect of risk on entry cannot be characterized analytically.

We now look at what happens when the change in risk is so large that the cluster size can be affected.

Proposition 3 *If the entry threshold increases as risk increases, the cluster size also increases with risk. If, instead, the entry threshold decreases with risk, the cluster size decreases. That is: if* $\frac{\partial \theta_i}{\partial \sigma} \gtreqless 0$*, then* $\frac{\partial j(i)}{\partial \sigma} \gtreqless 0$*.*

An increase in risk has two effects. On the one hand, it increases the probability of reaching more extreme demand values, and on the other hand, we know from the previous lemma that firms tend to enter later for moderate and low levels of risk. As the risk increases, entering earlier than the competitors does not guarantee higher profits for a long enough period to compensate for the higher probability of lower demand levels, so the incentive to enter earlier decreases. Firms therefore prefer to enter in larger clusters rather than individually.[10]

Combining Propositions 2 and 3, we can conclude that for moderate risk levels, as the risk increases, firms tend to enter later and tend to enter in larger clusters.

[9] In the next section, we show that the same result holds for any risk level in certain specific functional forms of the demand margins, D_n.

[10] Again, these results hold in the general case for moderate and low risk levels. In the special cases analyzed below, we are able to show this result for all risk levels.

Notably, unlike the standard real option model, the valuation of a firm when it enters does not change with risk. The option value is always zero regardless of the level of risk. Therefore, firms opt to enter the market when the level of demand aligns with the point at which the net value of entry equals zero.

4 Special Cases

4.1 Cournot Competition with Linear Demand

A typical special case that is often used in industrial organization is that in which firms compete à la Cournot, facing a linear demand function. We then consider the profit function to be $D(n) = \frac{1}{(n+a)^b}$ with $a \geq 0$ and $b > 1$.[11] As assumed in the model, this profit function is decreasing and convex in the number of firms present in the market.

Lemma 4 *If $D_n = \frac{1}{(n+a)^b}$ with $a \geq 0$ and $b > 1$, firms always enter individually.*

When firms engage in Cournot-style oligopolistic competition, the profits exhibit an inverse relationship with the number of firms participating in the market. When the reduction in profit flow resulting from the entry of additional firms is not as substantial, the firm's upside potential profit flow is not as heavily reduced. Consequently, even if the firm faces the same downside losses, it is more inclined to preempt its competitors by entering the market ahead of them rather than waiting and entering jointly later. As a result, firms in this setting consistently opt to enter the market individually.

[11] This is the textbook example with linear demand functions $a = 1$ and $b = 2$ (Tirole (1988)).

4.2 Bertrand competition

Another textbook case is competition /'a la Betrand. In the equilibrium outcome in this setting, prices are set so that firms do not make profits (Tirole (1988)). It follows that the profit function is $D_1 > 0$ when there is only one firm in the market and $D_n = 0$. Therefore, if $\lim_{\theta\to\infty,n\to\infty} D(n)\theta = K$, one firm can preempt and enjoy monopoly profits until the market is infinitely large. As the market tends to become infinitely large, n firms will enter, and $\lim_{\theta\to\infty} D(n)\theta = K$.

4.3 Exponential profit margins

We now consider the special case in which $D_n = e^{-n}$. The interesting feature of this functional form of profit margins is that we are able to find a closed-form solution and derive more general conclusions for the comparative statics.[12]

Lemma 5 *If $D_n = e^{-n}$, then the number of firms jointly entering at $\hat{\theta}_i$ is constant at each entry event i and is given by:*

$$\hat{j}(i) = \left\lceil \ln \frac{\beta_1}{\beta_1 - 1} \right\rceil \tag{14}$$

The threshold for each entry event is

$$\hat{\theta}_i = e^{\hat{n}(i)} = e^{\hat{j}i} \tag{15}$$

The value of a firm after the i-th entry event is given by

$$V_i = \frac{D_{\hat{n}(i)}}{r-\mu}\theta - K + A_i\theta^{\beta_1} \tag{16}$$

[12]Note that qualitatively identical results are obtained for more generic functional forms of the type $D_n = ae^{-bn} + c$.

where

$$A_i = \frac{(1 - e^j)}{(1 - e^{\beta_1 n})} \frac{e^{\beta_1 (n-j)}}{(r - \mu)} \tag{17}$$

Corollary 3 *When demand is more volatile, the number of firms entering jointly at each entry event, $\hat{j}(i)$, increases. Moreover, entry is delayed ($\hat{\theta}_i$ increases).*

When profit margins are exponentially decreasing in the number of firms, we obtain $\frac{D_i - D_{i+j}}{D_{i+j} - D_{i+2j}} = \exp(j)$. This simplifies the analysis and facilitates the derivation of an explicit solution.

Because profit margin ratios are constant, we can investigate each entry event in isolation without needing to account for subsequent entries. Additionally, we can infer that the number of firms entering during each event remains constant, as captured in equation (14).

This special case also yields an explicit expression for the entry thresholds. As equation (15) shows, each entry threshold is an exponential function of the number of active firms after the entry event occurs. Consequently, the more firms are active in the market upon entry, the later firms choose to enter.

In line with the results of the previous section, the number of firms entering at each event increases with the risk. In other words, greater risk is associated with larger clusters of firms entering the market. The intuition of this conclusion relies on the fact that higher market volatility exposes entering firms to greater downside risk while simultaneously capping their upside potential due to future competitors' entry. Consequently, firms tend to prefer entering with other firms to avoid downside risk rather than entering early. Following a similar line of reasoning, firms tend to delay their entry as profits become more volatile; this is a strategic response to mitigate the effects of downside risk.

5 Conclusion

We model investment decisions under uncertainty in situations where firms face an infinite number of potential entrants to a market and thus compete perfectly on the entry decision. Moreover, product market competition is imperfect since, in general, only a finite number of firms are in the market. This setting differs from those of existing models, which consider either a fixed and finite number of potential entrants (Bouis et al. (2009) and Argenziano and Schmidt-Dengler (2014)) or an infinite number of firms both outside and inside the market (Leahy (1993) and Grenadier (2002))).

As in Leahy (1993) and Grenadier (2002)), owing to perfect competition in the market for entry, the value of the option to wait is zero. However, oligopolistic competition in the product market means that the profit flow varies with the number of firms in the market and thus creates an asymmetry in the level of profit flow depending on whether demand decreases or increases, since increased demand will induce further entry and lead to reduced profit flows in the future. Thus, when the number of firms in the market is sufficiently small, there is an incentive for firms to preempt, and thus, in contrast to (Leahy (1993) and Grenadier (2002)), the entry thresholds vary depending on the number of firms in the market.

The preemption motive is similar to that of models with a finite number of potential entrants (Bouis et al. (2009) and Argenziano and Schmidt-Dengler (2014)). However, unlike these authors, we find that if the number of firms in the market is sufficiently large, the benefits of preempting alone are no longer sufficient, and firms prefer to wait and enter together with other firms. For convex decreasing demand functions, the minimum number of firms (weakly) increases with the number of firms in the market.

Most economic models of competing firms either consider a restricted number of firms competing oligopolistically or consider the limit of an infinite number of firms and perfect competition. Our results provide a link between these two extremes.

Owing to the special cases considered, this model offers a tractable solution to a very generic competition setting in continuous time and can be applied in various settings. In addition, this model can be extended to the case of exit. Thus far, it is not clear whether clustering can also occur at the exit level. The study of entry and exit at the same time can explain sector dynamics.

References

Agarwal, Rajshree and B. L. Bayus. "The Market Evolution and Sales Takeoff of Product Innovations." *Management Science 48*.8, 1024–1041 (2002).

Agarwal, Rajshree and M. Gort. "The Evolution of Markets and Entry, Exit and Survival of Firms." *The Review of Economics and Statistics 78*.3, 489–498 (1996).

Argenziano, Rossella and P. Schmidt-Dengler. "Clustering in N-Player Preemption Games." *Journal of the European Economic Association 12*.2, 368–396 (2014, 04).

Bar-Ilan, Avner and W. C. Strange. "Investment Lags." *The American Economic Review 86*.3, 610–622 (1996).

Bouis, Romain, K. J. Huisman, and P. M. Kort. "Investment in oligopoly under uncertainty: The accordion effect." *International Journal of Industrial Organization 27*.2, 320 – 331 (2009).

Chamley, Christophe and D. Gale. "Information Revelation and Strategic Delay in a Model of Investment." *Econometrica 62*.5, 1065–1085 (1994, September).

Dixit, Avinash K. and R. P. Pindyck. *Investment under Uncertainty.* Princeton, New Jersey: Princeton University Press (1994).

Dutta, Prajit K., S. Lach, and A. Rustichini. "Better Late than Early: Vertical Differentiation in the Adoption of a New Technology." *Journal of Economics & Management Strategy 4*.4, 563–589 (1995).

Grenadier, Steven R. "The Strategic Exercise of Options: Development cascades and Overbuilding in Real Estate Markets." *Journal of Finance 51*.5, 1653–1679 (1996, December).

Grenadier, Steven R. "Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms." *Review Financial Studies 15*.3, 691–721 (2002).

Gunther McGrath, Rita and A. Nerkar. "Real options reasoning and a new look at the R&D investment strategies of pharmaceutical firms." *Strategic Management Journal 25*.1, 1–21 (2004).

Huisman, Kuno J. and P. M. Kort. "Effects of Strategic Interactions on the Option Value of Waiting." Technical Report 9992, Discussion Paper CentER, (1999).

Karatzas, Ioannis and F. o. Baldursson. "Irreversible Investment and Industry Equilibrium." *Finance and Stochastics* 1, 69–89 (1996, 02).

Klepper, Steven. "Firm Survival and the Evolution of Oligopoly." *The RAND Journal of Economics 33*.1, 37–61 (2002).

Klepper, Steven and K. L. Simons. "The Making of an Oligopoly: Firm Survival and Technological Change in the Evolution of the U.S. Tire Industry." *Journal of Political Economy 108*.4, 728–760 (2000, August).

Leahy, John V. “Investment in Competitive Equilibrium: The Optimality of Myopic Behavior.” *The Quarterly Journal of Economics 108*.4, 1105–1133 (1993, Nov.).

Levin, Dan and J. Peck. “ To Grab for the Market or to Bide One’s Time: A Dynamic Model of Entry.” *RAND Journal of Economics 34*.3, 536–556 (2003, Autumn).

Mason, Robin and H. Weeds. “Investment, uncertainty and pre-emption.” *International Journal of Industrial Organization 28*.3, 278–287 (2010).

McDonald, Rpbert and D. Siegel. “The Value of Waiting to Invest.” *Quarterly Journal of Economics 101*.4, 707–728 (1986, November).

Paddock, James L., D. R. Siegel, and J. Smith. “Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases.” *The Quarterly Journal of Economics 103*.3, 479–508 (1988).

Pavan, Giulia, A. Pozzi, and G. Rovigatti. “Strategic entry and potential competition: Evidence from compressed gas fuel retail.” *International Journal of Industrial Organization* 69, 102566 (2020).

Rossetto, Silvia and E. Perotti. *Venture capital contracting and the valuation of high tech firms*, Chapter Internet Portals as Portfolio of Entry Options, pp. 281–296. Oxford University Press (2004).

Smets, Frank. “Exporting versus Foreign Direct Investement: The Effect of Uncertainty, Irreversibilities and Strategic Interactions.” (1991). unpublished manuscript.

Thijssen, Jacco J.J., K. J. Huisman, and P. M. Kort. “Symmetric equilibrium strategies in game theoretic real option models.” *Journal of Mathematical Economics 48*.4, 219–225 (2012).

Tirole, Jean. *The Theory of Industrial Organization.* MIT Press (1988).

Tufano, Peter. "Who Manages Risk? An Empirical Examination of Risk Management Practices in the Gold Mining Industry." *The Journal of Finance 51*.4, 1097–1137 (1996).

A Appendices

Proof of Lemma 1:

Following the standard steps of Dixit and Pindyck (1994), we apply Ito's lemma and Bellman's principle of optimality to find the value of an active firm, $V_i(\theta)$, and the value of an inactive firm, $W_i(\theta)$, after the i-th entry event. It is found that they must satisfy the following differential equations:

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 W_i}{\partial\theta^2} + \mu\theta\frac{\partial W_i}{\partial\theta} - rW_i = 0 \tag{18}$$

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V_i}{\partial\theta^2} + \mu\theta\frac{\partial V_i}{\partial\theta} - rV_i + D_{n(i)}\theta = 0. \tag{19}$$

These equations have the following general solutions:

$$W_i = a_i\theta^{\beta_1} + b_i\theta^{\beta_2} \qquad V_i = A_i\theta^{\beta_1} + B_i\theta^{\beta_2} + \frac{D_{n(i)}}{r-\mu}\theta \tag{20}$$

where β_1 and β_2 are the positive and negative solutions of the characteristic equation:

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{21}$$

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0 \tag{22}$$

and the constants a_i, b_i, A_i, B_i need to be found together with the optimal entry thresholds, $\hat{\theta}_i$, as the solutions to the following boundary conditions:

$$W_i(0) = 0 \qquad 0 \leq i \tag{23}$$

$$V_i(0) = 0 \qquad 1 \leq i \tag{24}$$

$$W_{i-1}(\hat{\theta}_i) = V_i(\hat{\theta}_i) - K \qquad 1 \leq i \tag{25}$$

$$V_{i-1}(\hat{\theta}_i) = V_i(\hat{\theta}_i) \qquad 2 \leq i \tag{26}$$

$$W_{i-1}(\hat{\theta}_i) = W_i(\hat{\theta}_i) \qquad 1 \leq i \tag{27}$$

$$\lim_{n,\theta \to \infty} W_i(\theta) = 0 \tag{28}$$

$$\lim_{n,\theta \to \infty} V_i(\theta) = K \tag{29}$$

These conditions are needed to define the firm value when the firm is (in)active. Conditions (23) and (24) are obtained by observing that if θ ever equals zero, it always remains at zero, and hence, the firm value is zero. Equation (25) is a value-matching condition for the investment decision at the i-th entry event: at $\hat{\theta}_i$, the value obtained on entry, $V_i(\theta) - K$, equals the value of the option to invest later, which is foregone: $W_{i-1}(\theta)$. Equations (26) and (27) capture the standard value-matching condition; that is, at the i-th entry event, the value of remaining (in)active does not jump. Conditions (28) and (29) set the boundary conditions when the market is infinitely large. As the market demand tends toward infinity, the market tends toward perfect competition: the number of firms in the market is so large that the value of an active firm tends toward the investment costs, $V_i \to K$, and the value of an inactive firm, which equals the value of the option to choose the timing of future investment, reduces to zero.

From condition (23), we see that $b_i = 0$ for all i. Hence, condition (27) yields

$a_i = a_{i-1}$ for all i. Finally, condition (28) yields $\lim_{i\to\infty} a_i = 0$, and hence, $a_i = 0$ for all i. ■

Proof of Lemma 2

From condition (24), we see that $B_i = 0$ for all i and hence obtain equation (2). Condition (29) implies that $\lim_{i\to\infty} A_i = 0$. Rearranging condition (26) yields:

$$A_i = A_{i+1} + \hat{\theta}_{i+1}^{-\beta_1+1} \frac{D_{n(i+1)} - D_{n(i)}}{r-\mu} \tag{30}$$

Rearranging $V_i(\hat{\theta}_{i+1}) = K$ via (2) yields (3), whereas iterating (30) yields (4).

Entry events are distinct, so $n(i) < n(i+1)$; hence, $D_{n(i)} > D_{n(i+1)}$, which implies that $A_i < A_{i+1}$. This, combined with $lim_{i\to\infty} A_i = 0$, yields $A_i \leq 0$ for all i. ■

Proof of Lemma 3

We first present a set of preliminary lemmas.

Lemma 6

$$\frac{A_{i+1}}{A_i} < \frac{D_{n(i+1)}}{D_{n(i)}} \tag{31}$$

Proof Define the θ value at which $S_i(\theta)$ is maximized as $\theta_{\max(i)}$. As $\left.\frac{\partial S_i(\theta)}{\partial\theta}\right|_{\hat{\theta}_{i+1}} < 0$ and $\left.\frac{\partial S_{i+1}(\theta)}{\partial\theta}\right|_{\hat{\theta}_{i+1}} > 0$, $\theta_{\max(i)} < \theta_{\max(i+1)}$.

$$\theta_{\max(i)} = \left(-\frac{D_{n(i)}}{\beta_1 A_i (r-\mu)}\right)^{\frac{1}{\beta_1-1}} < \left(-\frac{D_{n(i+1)}}{\beta_1 A_{i+1} (r-\mu)}\right)^{\frac{1}{\beta_1-1}} = \theta_{\max(i+1)} \tag{32}$$

Rearranging, we obtain (31). ■

Lemma 7

$$\frac{\theta_{\max(i)}}{\theta_{\max(i+1)}} < \frac{\hat{\theta}_i}{\hat{\theta}_{i+1}} \tag{33}$$

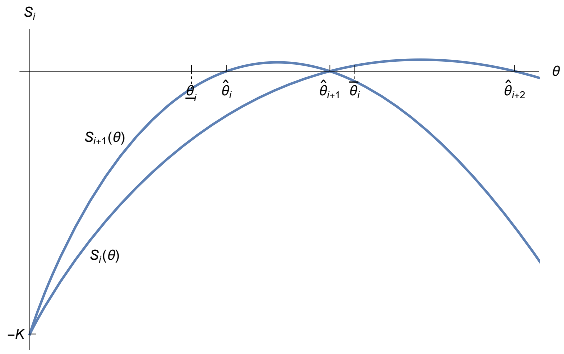

Figure 3: Graphic representation of the relationships among $\underline{\theta}_i$, $\hat{\theta}_i$, $\hat{\theta}_{i+1}$, $\bar{\theta}_{i+1}$ and $\hat{\theta}_{i+2}$.

Proof The proof condition (33) is equivalent to proving

$$\frac{A_{i+1}}{A_i}\frac{D_{n(i)}}{D_{n(i+1)}} < \left(\frac{\hat{\theta}_i}{\hat{\theta}_{i+1}}\right)^{\beta_1-1} \tag{34}$$

$$\hat{\theta}_i > \left(\frac{A_{i+1}}{A_i}\frac{D_{n(i)}}{D_{n(i+1)}}\right)^{\frac{1}{\beta_1-1}}\hat{\theta}_{i+1} \equiv \underline{\theta}_i \tag{35}$$

As $S_i(\theta)$ is concave, a necessary and sufficient condition for proving the lemma is that at $\theta = \underline{\theta}_i$, $S_i(\theta) < 0$ and $\underline{\theta}_i < \hat{\theta}_{i+1}$ (see Fig. 3 for a graphic representation).

From the boundary conditions of Lemma 1, we know that $S_i(\hat{\theta}_{i+1}) = 0$ and thus that $S_i(\underline{\theta}_i) < 0$ is equivalent to $S_i(\underline{\theta}_i) < S_i(\hat{\theta}_{i+1})$; that is,

$$A_i\underline{\theta}_i^{\beta_1} + \frac{D_{n(i)}}{r-\mu}\underline{\theta}_i - K < A_i\hat{\theta}_{i+1}^{\beta_1} + \frac{D_{n(i)}}{r-\mu}\hat{\theta}_{i+1} - K$$

$$A_i\underline{\theta}_i^{\beta_1-1}\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} + \frac{D_{n(i)}}{r-\mu}\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} < A_i\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i)}}{r-\mu}$$

Substituting equation (35), the above relationship is equivalent to:

$$A_i\frac{A_{i+1}}{A_i}\hat{\theta}_{i+1}^{\beta_1-1}\frac{D_{n(i)}}{D_{n(i+1)}}\hat{\theta}_{i+1}^{\beta_1-1}\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} + \frac{D_{n(i)}}{r-\mu}\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} < A_i\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i)}}{r-\mu}$$

$$
\begin{aligned}
\left(A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1}\frac{D_{n(i)}}{D_{n(i+1)}}+\frac{D_{n(i)}}{r-\mu}\right)\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} &< A_i\hat{\theta}_{i+1}^{\beta_1-1}+\frac{D_{n(i)}}{r-\mu} \\
\left(A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1}+\frac{D_{n(i+1)}}{r-\mu}\right)\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} &< \frac{D_{n(i+1)}}{D_{n(i)}}A_i\hat{\theta}_{i+1}^{\beta_1-1}+\frac{D_{n(i+1)}}{r-\mu} \\
\frac{K}{\hat{\theta}_{i+1}}\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} &< \frac{D_{n(i+1)}}{D_{n(i)}}\frac{K}{\hat{\theta}_{i+1}} \\
\frac{\underline{\theta}_i}{\hat{\theta}_{i+1}} &< \frac{D_{n(i+1)}}{D_{n(i)}} \\
\left(\frac{A_{i+1}}{A_i}\frac{D_{n(i)}}{D_{n(i+1)}}\right)^{\frac{1}{\beta_1-1}} &< \frac{D_{n(i+1)}}{D_{n(i)}} \\
\frac{A_{i+1}}{A_i}\frac{D_{n(i)}}{D_{n(i+1)}} &< \left(\frac{D_{n(i+1)}}{D_{n(i)}}\right)^{\beta_1-1} \\
\frac{A_{i+1}}{A_i} &< \left(\frac{D_{n(i+1)}}{D_{n(i)}}\right)^{\beta_1} \qquad (36)
\end{aligned}
$$

As $\beta_1 > 1$, from lemma 6, we know that this is always the case. As $\frac{A_{i+1}}{A_i}\frac{D_{n(i)}}{D_{n(i+1)}} < 1$, it follows that $\underline{\theta}_i < \hat{\theta}_i < \hat{\theta}_{i+1}$; hence, condition (33) holds. ■

Lemma 8

$$\frac{\hat{\theta}_{i+1}}{\hat{\theta}_{i+2}} < \frac{\theta_{\max(i)}}{\theta_{\max(i+1)}} \qquad (37)$$

Proof Proving condition (37) is equivalent to proving the following:

$$\left(\frac{\hat{\theta}_{i+1}}{\hat{\theta}_{i+2}}\right)^{\beta_1-1} < \frac{A_{i+1}}{A_i}\frac{D_{n(i)}}{D_{n(i+1)}} \qquad (38)$$

$$\hat{\theta}_{i+1} < \left(\frac{A_{i+1}}{A_i}\frac{D_{n(i)}}{D_{n(i+1)}}\right)^{\frac{1}{\beta_1-1}}\hat{\theta}_{i+2} \equiv \bar{\theta}_{i+1} \qquad (39)$$

As $S_i(\theta)$ is concave, a sufficient condition for obtaining $\hat{\theta}_{i+1} < \bar{\theta}_{i+1}$ is that $S_i(\bar{\theta}_{i+1}) < 0$ and $\left.\frac{dS_i(\theta)}{d\theta}\right|_{\theta=\bar{\theta}_{i+1}} < 0$ (see Fig. 3 for a graphic representation).

1) Proof that $S_i(\bar{\theta}_{i+1}) < 0$.

From the boundary conditions of lemma 1, we know that $S_{i+1}(\hat{\theta}_{i+2}) = 0$. Thus, showing that $S_i(\bar{\theta}_{i+1}) < 0$ is equivalent to showing that $S_i(\overline{\theta}_{i+1}) < S_{i+1}(\hat{\theta}_{i+2})$; that is,

$$
\begin{aligned}
A_i\bar{\theta}_{i+1}^{\beta_1} + D_{n(i)}\bar{\theta}_{i+1} &< A_{i+1}\hat{\theta}_{i+2}^{\beta_1} + D_{n(i+1)}\hat{\theta}_{i+2} \\
D_{n(i)}\frac{A_{i+1}}{D_{n(i+1)}}\hat{\theta}_{i+2}^{\beta_1-1}\bar{\theta}_{i+1} + D_{n(i)}\bar{\theta}_{i+1} &< A_{i+1}\hat{\theta}_{i+2}^{\beta_1} + D_{n(i+1)}\hat{\theta}_{i+2} \\
D_{n(i)}\frac{A_{i+1}}{D_{n(i+1)}}\hat{\theta}_{i+2}^{\beta_1-1}\frac{\bar{\theta}_{i+1}}{\hat{\theta}_{i+2}} + D_{n(i)}\frac{\bar{\theta}_{i+1}}{\hat{\theta}_{i+2}} &< A_{i+1}\hat{\theta}_{i+2}^{\beta_1-1} + D_{n(i+1)} \\
\left(A_{i+1}\hat{\theta}_{i+2}^{\beta_1-1} + D_{n(i+1)}\right)\frac{\bar{\theta}_{i+1}}{\hat{\theta}_{i+2}} &< \left(A_{i+1}\hat{\theta}_{i+2}^{\beta_1-1} + D_{n(i+1)}\right)\frac{D_{n(i+1)}}{D_{n(i)}} \\
\frac{\bar{\theta}_{i+1}}{\hat{\theta}_{i+2}} &< \frac{D_{n(i+1)}}{D_{n(i)}} \\
\left(\frac{A_{i+1}}{D_{n(i+1)}}\frac{D_{n(i)}}{A_i}\right)^{\frac{1}{\beta_1-1}} &< \frac{D_{n(i+1)}}{D_{n(i)}}
\end{aligned}
$$

From equation (36), we know that this is always the case.

2) We prove that $\left.\frac{dS_i(\theta)}{d\theta}\right|_{\theta=\bar{\theta}_{i+1}} \leq 0$:

$$
\begin{aligned}
\left.\frac{dS_i(\theta)}{d\theta}\right|_{\theta=\bar{\theta}_{i+1}} &= \beta_1 A_i\bar{\theta}_{i+1}^{\beta_1-1} + D_{n(i)} \\
&= \beta_1 A_{i+1}\frac{D_{n(i)}}{+D_{n(i+1)}}\left(\bar{\theta}_{i+1}\right)^{\beta_1-1} + D_{n(i)}
\end{aligned}
$$

This is negative, as $A_{i+1}\bar{\theta}_{i+1}^{\beta_1-1} + D_{n(i+1)} < 0$.

■

From Lemmas 7 and 8, we can conclude the following:

$$\frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} < \frac{\theta_{\max(i+1)}}{\theta_{\max(i)}} < \frac{\hat{\theta}_{i+2}}{\hat{\theta}_{i+1}} \tag{40}$$

■

Proof of Proposition 1

To prove this proposition, we proceed in two steps. First, we find the feasible j values, and then we find the equilibrium j.

Lemma 9 *Given $\hat{\theta}_{i+1}$ and $\hat{n}(i+1)$, a sufficient condition for j firms to enter at event $i+1$ is that*

$$\hat{\theta}_{n(i+1)-j,j} < \hat{\theta}_{i+1} \quad \Leftrightarrow \quad \left.\frac{\partial S_{\hat{n}(i+1)-j,j}(\theta|\hat{\theta}_{i+1})}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < 0 \tag{41}$$

Proof of Lemma 9

When j firms enter at entry event $i+1$ with threshold $\hat{\theta}_{i+1}$, $n = \hat{n}(i+1) - j$ firms are present after event i. From corollary 1, the incremental value function for a firm entering at i when j firms enter at the next entry event, $S_{n,j}$, is as follows:

$$S_{\hat{n}(i+1)-j,j}(\theta|\hat{\theta}_{i+1}) = \frac{D_{\hat{n}(i+1)-j}}{r-\mu}\theta + A_{\hat{n}(i+1)-j}(\hat{\theta}_{i+1})\theta^{\beta_1} - K \tag{42}$$

where

$$A_{\hat{n}(i+1)-j}(\hat{\theta}_{i+1}) = \left(K - \frac{D_{\hat{n}(i+1)-j}}{r-\mu}\hat{\theta}_{i+1}\right)\hat{\theta}_{i+1}^{-\beta_1} \tag{43}$$

$$= A_{i+1} + \left(\frac{D_{\hat{n}(i+1)} - D_{\hat{n}(i+1)-j}}{r-\mu}\right)\hat{\theta}_{i+1}^{1-\beta_1} \tag{44}$$

Note that $S_{\hat{n}(i+1)-j,j}(\theta|\hat{\theta}_{i+1})$ has the same characteristics as $S_i(\theta)$ discussed above; it is a concave function with $S_{\hat{n}(i+1)-j,j}(0) < 0$, which crosses the axis at at most two values of θ, one of which is $\hat{\theta}_{i+1}$. If it crosses twice, then $\frac{\partial S_{\hat{n}(i+1)-j,j}}{\partial \theta} \geq 0$ at the lower value of θ, and $\frac{\partial S_{\hat{n}(i+1)-j,j}}{\partial \theta} < 0$ at the higher. ■

Lemma 10 *If $j' > j$, then*

$$\left.\frac{\partial S_{\hat{n}(i+1)-j',j'}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < \left.\frac{\partial S_{\hat{n}(i+1)-j,j}}{\partial \theta}\right|_{\hat{\theta}_{i+1}}.$$

Proof

$$\left.\frac{\partial S_{\hat{n}(i+1)-j',j'}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} = \frac{D_{\hat{n}(i+1)-j'}}{r-\mu} + \beta_1 A_{\hat{n}(i+1)-j'}\hat{\theta}_{i+1}^{\beta_1 - 1}$$

Inserting equation (43) and rearranging yields:

$$\begin{aligned}
\left.\frac{\partial S_{\hat{n}(i+1)-j',j'}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} &= \frac{D_{\hat{n}(i+1)-j'}}{r-\mu}(1-\beta_1) + \beta_1 \frac{K}{\hat{\theta}_{i+1}} \\
\left.\frac{\partial S_{\hat{n}(i+1)-j',j'}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} &< \frac{D_{\hat{n}(i+1)-j}}{r-\mu}(1-\beta_1) + \beta_1 \frac{K}{\hat{\theta}_{i+1}} = \left.\frac{\partial S_{\hat{n}(i+1)-j,j}}{\partial \theta}\right|_{\hat{\theta}_{i+1}}
\end{aligned}$$

as $1-\beta_1 < 0$ and $D_{\hat{n}(i+1)-j'} > D_{\hat{n}(i+1)-j}$. ■

Given the definition of $\hat{j}(i+1)$ as the smallest j s.t. $\left.\frac{\partial S_{n(i+1)-j,j}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < 0$, if $j' < \hat{j}(i+1)$, then $\left.\frac{\partial S_{n(i+1)-j',j'}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} \geq 0$, so $\hat{\theta}_{\bar{n}(i+1)-j',j'} = \hat{\theta}_{i+1}$; i.e., j' is not feasible. If $j' > \hat{j}(i+1)$, then $\left.\frac{\partial S_{n(i+1)-j',j'}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < \left.\frac{\partial S_{n(i+1)-\hat{j}(i+1),\hat{j}(i+1)}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < 0$, so it is worthwhile to enter at $\hat{\theta}_{n(i+1)-j',j'}$ if the next entry is at $\hat{\theta}_{i+1}$; *i.e.*, j' is feasible. However, if the next entry event will be at $\hat{\theta}_{i+1}$, then it is also worthwhile for other firms to enter at the higher threshold $\hat{\theta}_{\hat{n}(i+1)-\hat{j}(i+1),\hat{j}(i+1)}$, which is between $\hat{\theta}_{n(i+1)-j',j'}$ and $\hat{\theta}_{i+1}$. Therefore, entry at $\bar{\theta}_{n(i+1)-j',j'}$ is not credible. ■

Proof of Theorem 1

From Proposition 1, $\hat{j}(i+1)$ is the smallest j for which

$$\left.\frac{\partial S_{\hat{n}(i+1)-j,j}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} = \frac{D_{\hat{n}(i+1)-j}}{r-\mu}(1-\beta_1) + \beta_1 \frac{K}{\hat{\theta}_{i+1}} < 0 \tag{45}$$

Rearranging and substituting $\theta^0_{\hat{n}(i+1)-j} = \frac{K(\mu-r)}{D_{\hat{n}(i+1)-j}}$ yields (11). ■

Proof of Theorem 2

$\hat{j}(i+1)$ is defined as the smallest j such that

$$\left.\frac{\partial S_{\hat{n}(i+1)-j}}{\partial \theta}\right|_{\hat{\theta}_{i+1}} < 0 \tag{46}$$

Using (44), this is equivalent to

$$(1-\beta_1)D_{\hat{n}(i+1)-j} + \beta_1 D_{\hat{n}(i+1)} + \beta_1(r-\mu)A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1} < 0 \tag{47}$$

or

$$\frac{D_{\hat{n}(i+1)-j}}{D_{\hat{n}(i+1)}} > \left(\frac{\beta_1}{\beta_1-1}\right)\left(1 + \frac{(r-\mu)A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1}}{D_{\hat{n}(i+1)}}\right) \tag{48}$$

$$D_{\hat{n}(i+1)-j} > \left(\frac{\beta_1}{\beta_1-1}\right)\frac{K}{\hat{\theta}_{i+1}} \tag{49}$$

We know that $\frac{D_{\hat{n}(i+1)-j}}{D_{\hat{n}(i+1)}}$ decreases (and is convex) as n increases. Moreover, the right-hand side increases as $\hat{\theta}_{i+1}$ decreases. Hence, the condition becomes less binding as i decreases. ■

Proof of Proposition 2

Note that σ affects the model through β_1 and $\frac{\mathrm{d}\beta_1}{\mathrm{d}\sigma} < 0$.

To prove this proposition, we first prove some intermediate results.

Lemma 11 *When* $\dfrac{\mathrm{d}j(i)}{\mathrm{d}\beta_1} = 0$,

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}\beta_1} \leq 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}A_i}{\mathrm{d}\beta} + A_i \ln \theta_i \geq 0 \tag{50}$$

Proof

From the boundary conditions of lemma 1, we know that:

$$A_i \hat{\theta}_i^{\beta_1} + D_i \hat{\theta}_1 = K \tag{51}$$

It follows that

$$\begin{aligned} \frac{\mathrm{d}A_i}{\mathrm{d}\beta_1}\hat{\theta}_i^{\beta_1} + \beta_1 A_i \hat{\theta}_i^{\beta_1 - 1}\frac{\mathrm{d}\hat{\theta}_i}{\mathrm{d}\beta_1} + A_i \hat{\theta}_i^{\beta_1} \ln \hat{\theta}_i + D_i \frac{\mathrm{d}\hat{\theta}_i}{\mathrm{d}\beta_1} = 0 \\ \frac{\mathrm{d}\hat{\theta}_i}{\mathrm{d}\beta_1} = -\frac{\dfrac{\mathrm{d}A_i}{\mathrm{d}\beta_1} + A_i \ln \hat{\theta}_i}{\beta_1 A_i \hat{\theta}_i^{\beta_1 - 1} + D_i}\hat{\theta}_i^{\beta_1} \end{aligned} \tag{52}$$

The denominator is the slope of S_i at $\hat{\theta}_i$ and hence positive. It follows that:

$$\frac{\mathrm{d}\hat{\theta}_i}{\mathrm{d}\beta_1} \leq 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}A_i}{\mathrm{d}d\beta_1} + A_i \ln \hat{\theta}_i \geq 0 \tag{53}$$

■

Lemma 12 *For real numbers* x_1 *and* x_2 *such that* $1 < x_1 < x_2$, *we have*

$$\frac{\ln x_1}{\ln x_2} > \frac{x_1 - 1}{x_2 - 1} \tag{54}$$

Proof

$$\lim_{x_2 -> x_1} \frac{\ln x_1}{\ln x_2} - \frac{x_1 - 1}{x_2 - 1} = 0 \tag{55}$$

$$\frac{\mathrm{d}\frac{\ln x_1}{\ln x_2}}{\mathrm{d}x_2} = -\frac{1}{x_2}\frac{\ln x_1}{\ln x_2} \tag{56}$$

$$\frac{\mathrm{d}\frac{x_1 - 1}{x_2 - 1}}{\mathrm{d}x_2} = -\frac{1}{x_2 - 1}\frac{x_1 - 1}{x_2 - 1} \tag{57}$$

Given the result of (55),

$$\frac{\mathrm{d}\frac{\ln x_1}{\ln x_2}}{\mathrm{d}x_2} > \frac{\mathrm{d}\frac{x_1 - 1}{x_2 - 1}}{\mathrm{d}x_2} \quad \forall x_2 > x_1 \iff -\frac{1}{x_2} > \frac{1}{x_2 - 1} \tag{58}$$

This is always the case, as $x_2 > 1$. ■

Lemma 13 *The following holds:*

$$\frac{\hat{\theta}_{i+1} - \hat{\theta}_i}{\hat{\theta}_i} = \frac{\hat{\theta}_{i+1}^{\beta_1 - 1} - \hat{\theta}_i^{\beta_1 - 1}}{\beta_1 \hat{\theta}_{max(i)}^{\beta_1 - 1} - \hat{\theta}_{i+1}^{\beta_1 - 1}} \tag{59}$$

Proof From condition (26), we know:

$$A_i \hat{\theta}_{i+1}^{\beta_1} + D_i \hat{\theta}_{i+1} = A_i \hat{\theta}_i^{\beta_1} + D_i \hat{\theta}_i$$

$$\frac{\hat{\theta}_{i+1}}{\hat{\theta}_i}\left(\frac{A_i}{D_i}\hat{\theta}_{i+1}^{\beta_1 - 1} + 1\right) = \frac{A_i}{D_i}\hat{\theta}_i^{\beta_1 - 1} + 1$$

$$\frac{\hat{\theta}_{i+1}}{\hat{\theta}_i}\left(-\frac{\hat{\theta}_{i+1}^{\beta_1 - 1}}{\hat{\theta}_{\max(i)}} + \beta_1\right) = -\frac{\hat{\theta}_i^{\beta_1 - 1}}{\hat{\theta}_{\max(i)}} + \beta_1$$

$$\frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} = \frac{\beta_1 \hat{\theta}_{\max(i)} - \hat{\theta}_i^{\beta_1 - 1}}{\beta_1 \hat{\theta}_{\max(i)} - \hat{\theta}_{i+1}^{\beta_1 - 1}}$$

$$\frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} - 1 = \frac{\beta_1 \hat{\theta}_{\max(i)} - \hat{\theta}_i^{\beta_1 - 1}}{\beta_1 \hat{\theta}_{\max(i)} - \hat{\theta}_{i+1}^{\beta_1 - 1}} - 1$$

$$(60)$$

Equation (59) follows. ■

Lemma 14 *Keeping $j(i)$ constant, if $\beta \geq 2$ and $\frac{\mathrm{d}\hat{\theta}_{(i+1)}}{\mathrm{d}\beta_1} \leq 0$, then $\frac{\mathrm{d}\hat{\theta}_i}{\mathrm{d}\beta_1} < 0$.*

Proof From lemma 11, we know that $\frac{\mathrm{d}\hat{\theta}_i}{\mathrm{d}\beta_1} < 0$ iff $\frac{\mathrm{d}A_i}{\mathrm{d}\beta_1} + A_i \ln \hat{\theta}_i \geq 0$.

From equation (3), we can obtain:

$$\begin{aligned}
\frac{\mathrm{d}A_i}{\mathrm{d}\beta_1} = & \frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1} + (D_i - D_{i+1})(\beta_1 - 1)\,\hat{\theta}_{i+1}^{-\beta_1} \frac{\mathrm{d}\hat{\theta}_{i+1}}{\mathrm{d}\beta_1} + (D_i - D_{i+1})\,\hat{\theta}_{i+1}^{-(\beta_1-1)} \ln \hat{\theta}_{i+1} + A_i \ln \hat{\theta}_i \\
= & \frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1} + (D_i - D_{i+1})(\beta_1 - 1)\,\hat{\theta}_{i+1}^{-\beta_1} \frac{\mathrm{d}\hat{\theta}_{i+1}}{\mathrm{d}\beta_1} + (D_i - D_{i+1})\,\hat{\theta}_{i+1}^{-(\beta_1-1)} \ln \hat{\theta}_{i+1} + A_i \ln \hat{\theta}_i \\
& + A_{i+1} \ln \hat{\theta}_{i+1} - A_{i+1} \ln \hat{\theta}_{i+1} \\
= & \frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1} + A_{i+1} \ln \hat{\theta}_{i+1} - A_i \ln \hat{\theta}_{i+1} + A_i \ln \hat{\theta}_i + (D_i - D_{i+1})(\beta_1 - 1)\,\hat{\theta}_{i+1}^{-\beta_1} \frac{\mathrm{d}\hat{\theta}_{i+1}}{\mathrm{d}\beta_1} \\
= & \frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1} + A_{i+1} \ln \hat{\theta}_{i+1} - A_i \ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} + (D_i - D_{i+1})(\beta_1 - 1)\,\hat{\theta}_{i+1}^{-\beta_1} \frac{\mathrm{d}\hat{\theta}_{i+1}}{\mathrm{d}\beta_1}
\end{aligned}$$

Using equation (52), we substitute $\frac{\mathrm{d}\hat{\theta}_{i+1}}{\mathrm{d}\beta_1}$ and obtain:

$$\begin{aligned}
= & - A_i \ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} + \frac{\frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1} + A_{i+1} \ln \hat{\theta}_{i+1}}{\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta_1 - 1} + D_{i+1}} \hat{\theta}_{i+1}^{\beta_1} \left(\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta_1 - 1} + D_{i+1} - \beta_1 D_i + D_i + \beta_1 D_{i+1} - D_{i+1} \right) \\
= & - A_i \ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} + \frac{\frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1} + A_{i+1} \ln \hat{\theta}_{i+1}}{\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta_1 - 1} + D_{i+1}} \hat{\theta}_{i+1}^{\beta_1} \left(\beta_1 \frac{K}{\hat{\theta}_{i+1}} - D_i (\beta_1 - 1) \right) \\
= & - A_i \ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} + \frac{\frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1} + A_{i+1} \ln \hat{\theta}_{i+1}}{\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta_1 - 1} + D_{i+1}} \hat{\theta}_{i+1}^{\beta_1} \left(\beta_1 A_i \hat{\theta}_{i+1} + D_i \right)
\end{aligned}$$

$$
\begin{aligned}
= &- A_i \ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} - A_{i+1} \ln \frac{\hat{\theta}_{i+2}}{\hat{\theta}_{i+1}} \frac{\beta_1 A_i \hat{\theta}_{i+1}^{\beta-1} + D_i}{\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta-1} + D_{i+1}} \\
&+ \left(\frac{\mathrm{d}A_{i+2}}{\mathrm{d}\beta_1} + A_{i+2} \ln \hat{\theta}_{i+2} \right) \frac{\beta_1 A_{i+1} \hat{\theta}_{i+2}^{\beta_1-1} + D_{i+1}}{\beta_1 A_{i+2} \hat{\theta}_{i+2}^{\beta_1-1} + D_{i+2}} \frac{\beta_1 A_i \hat{\theta}_{i+1}^{\beta_1-1} + D_i}{\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta_1-1} + D_{i+1}}
\end{aligned}
$$

If $\frac{\mathrm{d}\hat{\theta}_{i+2}}{\mathrm{d}\beta_1} < 0$, the last part is always positive. A sufficient condition for $\frac{\mathrm{d}\hat{\theta}_i}{\mathrm{d}\beta_1} < 0$ is therefore:

$$
\begin{aligned}
-A_i \ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} - A_{i+1} \ln \frac{\hat{\theta}_{i+2}}{\hat{\theta}_{i+1}} \frac{\beta_1 A_i \hat{\theta}_{i+1}^{\beta-1} + D_i}{\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta-1} + D_{i+1}} &\geq 0 \\
\ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} &\geq -\frac{A_{i+1}}{A_i} \frac{\beta_1 A_i \hat{\theta}_{i+1}^{\beta-1} + D_i}{\beta_1 A_{i+1} \hat{\theta}_{i+1}^{\beta-1} + D_{i+1}} \ln \frac{\hat{\theta}_{i+2}}{\hat{\theta}_{i+1}} \\
\ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} \Bigg/ \ln \frac{\hat{\theta}_{i+2}}{\hat{\theta}_{i+1}} &\geq \frac{\hat{\theta}_{i+1}^{\beta-1} - \theta_{\max(i)}^{\beta-1}}{\theta_{\max(i+1)}^{\beta-1} - \hat{\theta}_{i+1}^{\beta-1}}
\end{aligned} \tag{61}
$$

From lemma 3, we know that $1 < \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} < \frac{\hat{\theta}_{i+2}}{\hat{\theta}_{i+1}}$. From lemma 12, we know that:

$$
\ln \frac{\hat{\theta}_{i+1}}{\hat{\theta}_i} \Bigg/ \ln \frac{\hat{\theta}_{i+2}}{\hat{\theta}_{i+1}} > \frac{\hat{\theta}_{i+1} - \hat{\theta}_i}{\hat{\theta}_i} \Bigg/ \frac{\hat{\theta}_{i+2} - \hat{\theta}_{i+1}}{\hat{\theta}_{i+1}} \tag{62}
$$

Hence, a sufficient condition for equation (61) is as follows:

$$
\frac{\hat{\theta}_{i+1} - \hat{\theta}_i}{\hat{\theta}_i} \geq \frac{\hat{\theta}_{i+1}^{\beta-1} - \theta_{\max(i)}^{\beta-1}}{\theta_{\max(i+1)}^{\beta-1} - \hat{\theta}_{i+1}^{\beta-1}} \frac{\hat{\theta}_{i+2} - \hat{\theta}_{i+1}}{\hat{\theta}_{i+1}} \tag{63}
$$

Applying Lemma 13, the above relationship is equivalent to:

$$
\frac{\hat{\theta}_{i+1}^{\beta_1-1} - \hat{\theta}_i^{\beta_1-1}}{\beta_1 \hat{\theta}_{max(i)}^{\beta_1-1} - \hat{\theta}_{i+1}^{\beta_1-1}} \geq \frac{\hat{\theta}_{i+1}^{\beta-1} - \theta_{\max(i)}^{\beta-1}}{\theta_{\max(i+1)}^{\beta-1} - \hat{\theta}_{i+1}^{\beta-1}} \frac{\hat{\theta}_{i+2}^{\beta_1-1} - \hat{\theta}_{i+1}^{\beta_1-1}}{\beta_1 \hat{\theta}_{max(i+1)}^{\beta_1-1} - \hat{\theta}_{i+2}^{\beta_1-1}}
$$

$$\frac{\hat{\theta}_{i+1}^{\beta_1-1}-\hat{\theta}_{i}^{\beta_1-1}}{\hat{\theta}_{i+1}^{\beta-1}-\theta_{\max(i)}^{\beta-1}} \geq \frac{\hat{\theta}_{i+2}^{\beta_1-1}-\hat{\theta}_{i+1}^{\beta_1-1}}{\theta_{\max(i+1)}^{\beta-1}-\hat{\theta}_{i+1}^{\beta-1}} \frac{\beta_1\hat{\theta}_{max(i)}^{\beta_1-1}-\hat{\theta}_{i+1}^{\beta_1-1}}{\beta_1\hat{\theta}_{max(i+1)}^{\beta_1-1}-\hat{\theta}_{i+2}^{\beta_1-1}} \tag{64}$$

If $\beta = 2$, S_i, $\theta_{\max(i+1)}^{\beta-1} - \hat{\theta}_{i+1}^{\beta-1} = \hat{\theta}_{i+2}^{\beta-1} - \theta_{\max(i+1)}^{\beta-1}$. If $\beta > 2$, S_i is skewed to the right, and $\theta_{\max(i+1)}^{\beta-1} - \hat{\theta}_{i+1}^{\beta-1} > \hat{\theta}_{i+2}^{\beta-1} - \theta_{\max(i+1)}^{\beta-1}$. Hence, the left-hand side of equation 64 is greater than 2, the first part of the right-hand side is smaller than $\frac{1}{2}$, and the second part is smaller than 1. Hence, the above relationship is always true for $\beta \geq 2$. ■

As $\lim_{i\to\infty} \dfrac{\mathrm{d}\hat{\theta}_{(i+1)}}{\mathrm{d}\beta_1} = 0$, from Lemma 14, it follows that $\dfrac{\mathrm{d}\hat{\theta}_{(i)}}{\mathrm{d}\beta_1} < 0$ when $\beta_1 \geq 2$. ■

Proof of Proposition 3

The number of firms that enter at $i+1$ is the smallest j such that:

$$\left.\frac{\partial S_{n(i+1)-j,j}(\theta|\hat{\theta}_{i+1})}{\partial\theta}\right|_{\hat{\theta}_{i+1}} = \beta_1 A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1} - \beta_1\frac{D_{n(i,j)}-D_{n(i+1)}}{r-\mu} + \frac{D_{n(i,j)}}{r-\mu} < 0 \tag{65}$$

We also know that if $j(i+1)$ increases, $j(i)$ cannot decrease; i.e., it can increase or remain constant. Hence, it is sufficient to show that $j(i)$ remains constant or increases while keeping $j(i+1)$ constant, that is, keeping $D(i+1)$ constant.

j weakly increases as σ increases, and if j is constant, then $\frac{\partial\frac{\partial S_{n(i+1),j}(\theta)}{\partial\theta}}{\partial\beta_1} < 0$; that is:

$$\frac{\partial\frac{\partial S_{n(i+1),j}(\theta)}{\partial\theta}}{\partial\beta_1} = \beta_1\frac{\mathrm{d}A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1}}{\mathrm{d}\beta_1} + A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1} - \frac{D_{n(i+1-j)}-D_{n(i+1)}}{r-\mu} < 0 \tag{66}$$

A sufficient condition for satisfying condition (66) is that:

$$\beta_1\frac{\mathrm{d}A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1}}{\mathrm{d}\beta_1} + A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1} \leq 0 \tag{67}$$

Lemma 15 *If*

$$\beta_1\frac{\mathrm{d}A_{i+1}\hat{\theta}_{(i+2)}^{\beta_1-1}}{\mathrm{d}\beta_1} + A_{i+1}\hat{\theta}_{(i+2)}^{\beta_1-1} \leq 0, \tag{68}$$

$$\beta_1 \frac{\mathrm{d}A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1}}{\mathrm{d}\beta_1} + A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1} \leq 0. \tag{69}$$

Proof

$$\frac{\mathrm{d}A_{i+1}\theta^{\beta_1-1}}{\mathrm{d}\beta_1} = \frac{\mathrm{d}A_{i+1}}{\mathrm{d}\beta_1}\theta^{\beta_1-1} + A_{i+1}\theta^{\beta_1-1}\ln\theta \tag{70}$$

As θ increases, the above derivative decreases. It goes to $+\infty$ for $\theta \to 0$ and goes to $-\infty$ as $\theta \to \infty$. ■

Finally, note that $i \to \infty$, $A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1} \to \infty$ for $\frac{\mathrm{d}j(i)}{\mathrm{d}\beta} \geq 0$.

■

Proof of Lemma 4

From condition (49), we know that the optimal number of firms entering at $i+1$ is the smallest j that satisfies the following condition:

$$\frac{D_{\hat{n}(i+1)-j}}{D_{\hat{n}(i+1)}} > \frac{\beta_1}{\beta_1 - 1}\frac{K}{D_{\hat{n}(i+1)}\hat{\theta}_{i+1}} \tag{71}$$

$$\lim_{n\to\infty} \frac{K}{D_{\hat{n}(i+1)}\hat{\theta}_{i+1}} = 1 \tag{72}$$

$$\lim_{n\to\infty} \frac{(n+a-1)^b}{(n+a)^b} > \frac{\beta_1}{\beta_1 - 1} \tag{73}$$

This is always the case. From Theorem 2, we know that the cluster size is weakly larger for later entry events. Hence, in this case, firms always enter individually.

■

Proof of Lemma 5

From corollary 1, we know that $S_i(\hat{\theta}_i) = S_i(\hat{\theta}_{i+1}) = 0$.

Lemma 16 *Given $j(i+1)$ and $\hat{\theta}_{i+1}$,*

$$\hat{\theta}_i = \frac{D_{n(i)}}{D_{n(i+1)}}\hat{\theta}_{i+1} \tag{74}$$

$$= e^{-j(i+1)}\hat{\theta}_{i+1} \tag{75}$$

$$\frac{D_{n(i+1)}}{r-\mu}\hat{\theta}_{i+1} = \frac{D_{n(i)}}{r-\mu}\hat{\theta}_i \tag{76}$$

$$A_i\theta_i^{\beta_1 - 1} = A_{i+1}\theta_{i+1}^{\beta_1 - 1} \tag{77}$$

Proof To prove this lemma, we check whether $\hat{\theta}_i = e^{-\hat{j}(i+1)}\hat{\theta}_{i+1}$ is the solution of our system. Given the functional form of D_n, we have:

$$D_{n(i+1)} = be^{-n-j(i+1)} = D_{n(i)}e^{-j(i+1)} \tag{78}$$

$$\frac{D_{n(i+1)}}{r-\mu}\hat{\theta}_{i+1} = \frac{D_{n(i)}}{r-\mu}e^{-j(i+1)}\hat{\theta}_{i+1} \tag{79}$$

$$= \frac{D_{n(i)}}{r-\mu}e^{-j(i+1)}\hat{\theta}_i e^{j(i+1)} \tag{80}$$

$$= \frac{D_{n(i)}}{r-\mu}\hat{\theta}_i \tag{81}$$

From equation (25), we know that:

$$A_i\hat{\theta}_i^{\beta_1} + \frac{D_{n(i)}}{r-\mu}\hat{\theta}_i = A_{i+1}\hat{\theta}_{i+1}^{\beta_1} + \frac{D_{n(i+1)}}{r-\mu}\hat{\theta}_{i+1} \tag{82}$$

$$A_i\hat{\theta}_i^{\beta_1} = A_{i+1}\hat{\theta}_{i+1}^{\beta_1} \tag{83}$$

$$A_{i+1} = A_i\left(\frac{\hat{\theta}_i}{\hat{\theta}_{i+1}}\right)^{\beta_1} = A_i e^{-j(i+1)\beta_1} \tag{84}$$

The final step is to check that for $\hat{\theta}_i = e^{-j(i+1)}\hat{\theta}_{i+1}$, condition (26) holds. Thus far, we know that

$$A_i\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i)}}{r-\mu} = A_{i+1}\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i+1)}}{r-\mu} \tag{85}$$

$$A_i\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i)}}{r-\mu} = A_i\frac{\hat{\theta}_i^{\beta_1}}{\hat{\theta}_{i+1}^{\beta_1}}\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i+1)}}{r-\mu} \tag{86}$$

$$A_i\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i)}}{r-\mu} = A_i\frac{\hat{\theta}_i^{\beta_1}}{\hat{\theta}_{i+1}^{\beta_1}}\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i+1)}}{r-\mu} \tag{87}$$

$$A_i\hat{\theta}_{i+1}^{\beta_1} + \frac{D_{n(i)}}{r-\mu}\hat{\theta}_{i+1} = A_i\hat{\theta}_i^{\beta_1} + \frac{D_{n(i)}}{r-\mu}e^{-j(i+1)}\hat{\theta}_{i+1} \tag{88}$$

$$A_i\hat{\theta}_i^{\beta_1} + \frac{D_{n(i)}}{r-\mu}\hat{\theta}_i = A_i\hat{\theta}_i^{\beta_1} + \frac{D_{n(i)}}{r-\mu}e^{-j(i+1)}\hat{\theta}_{i+1} \tag{89}$$

$$\frac{D_{n(i)}}{r-\mu}\hat{\theta}_i = \frac{D_{n(i)}}{r-\mu}e^{-j(i+1)}\hat{\theta}_{i+1} \tag{90}$$

$$\hat{\theta}_i = e^{-j(i+1)}\hat{\theta}_{i+1} \tag{91}$$

■

Lemma 17 *When $D_{n(i)} = e^{-n(i)}$, $j(i)$ is constant for every i and is the smallest integer such that:*

$$e^j > \frac{\beta_1}{\beta_1 - 1} \tag{92}$$

Proof From Lemma 2, we know that $j(i) \leq j(i+1)$. Therefore, to prove the lemma, it is sufficient to show that if $j(i) < j(i+1)$, then condition (10) holds.

Given equations (84) and (79), condition (10) can be written as:

$$\beta_1 A_i\hat{\theta}_{i+1}^{\beta_1-1} + \frac{D_{n(i)}}{r-\mu} \tag{93}$$

$$e^{j(i+1)}\left(\beta_1 A_{i+1}\hat{\theta}_{i+2}^{\beta_1-1} + \frac{D_{n(i+1)}}{r-\mu}\right) < 0 \tag{94}$$

The second part of equation (94) is the condition (10) of event $(i+2)$. Therefore, the above condition is negative for the same condition, and hence, j is the same as in the $i+2$ case.

To compute how many firms enter at each event, we check what happens as the number of firms tends toward infinity. We know that condition (10) can be written as:

$$D_{n(i)} > \frac{\beta_1}{\beta_1 - 1} \left(D_{n(i+1)} + A_{i+1} \theta_{i+1}^{\beta_1 - 1} \right) \tag{95}$$

$$\lim_{n->\infty} A_1 \theta_{i+1}^{\beta_1 - 1} = 0 \tag{96}$$

$$\lim_{n->\infty} D_{n(i+1)} e^{j(i+1)} > \frac{\beta_1}{\beta_1 - 1} \left(D_{n(i+1)} \right) \tag{97}$$

Hence, j is the smallest integer such that:

$$e^j > \frac{\beta_1}{\beta_1 - 1} \tag{98}$$

■

It follows that $\hat{\theta}_i = e^{n(i)}$, and from equation (4), we have:

$$A_i = \sum_{n(i)=0}^{\infty} \frac{-e^{-n(i)+e^{-n(i)-j}}}{r - \mu} \left(e^{n(i)+j} \right)^{-\beta_1 + 1} \tag{99}$$

$$= \sum_{n(i)=0}^{\infty} \frac{-e^{-n(i)} \left(1 - e^{-j}\right)}{r - \mu} e^{(n(i)+j)(-\beta_1 + 1)} \tag{100}$$

$$= \sum_{n(i)=0}^{\infty} -\frac{\left(1 - e^{-j}\right)}{r - \mu} e^{-\beta_1 (n(i)+j)} \tag{101}$$

$$= -\frac{1 - e^{-j}}{r - \mu} \sum_{n(i)=0}^{\infty} e^{-\beta_1 (n(i)+j)} \tag{102}$$

$$= -\frac{1 - e^{-j}}{r - \mu} e^{-\beta_1 j} \sum_{n(i)=0}^{\infty} e^{-\beta_1 n(i)} \tag{103}$$

When $\beta_1 \geq 1$ and $n \geq 1$, $e^{-\beta_1 n(i)} \leq 1$; hence, the sum of the geometric series converges, and we can write:

$$A_i = -\frac{1 - e^{-j}}{1 - e^{-\beta_1 n(i)}} \frac{e^{-\beta_1 j}}{r - \mu} \tag{104}$$

$$= \frac{(1 - e^{j})}{(1 - e^{\beta_1 n})} \frac{e^{\beta_1 (n-j)}}{(r - \mu)} \tag{105}$$

■