Revealed Preference Analysis of Household Consumption under Risk

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Abstract

We develop a nonparametric approach to analyzing collective household consumption behavior under risk when there is one private commodity in each state of nature. We assume that each household member complies with expected utility theory and is risk averse, and the intrahousehold decision process produces Pareto efficient outcomes. We show how to test data for consistency with this model, recover the individual preferences and the intrahousehold decision process, forecast a household's demand behavior, and quantify the extent of a household's departure from collective rationality. All these questions can be settled by solving a finite number of systems of linear inequalities.

Keywords: Revealed Preference; Collective rationalizability; Expected utility; Pareto efficiency; Household aggregate consumption
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1 Introduction

The standard microeconomic approach to household consumption behavior is to treat a many-person household as a single decision-maker who maximizes a well-behaved utility function subject to a household budget constraint. This "unitary"model has been found to be in conflict with mounting empirical evidence.[1] In an attempt to resolve the conflict, an alternative approach—the collective one—emerges. The collective approach explicitly recognizes that each household member has his or her own preference and the household decision is the outcome of an intrahousehold bargaining process. A nonparametric analysis of the general collective model has been carried out extensively in the literature. In reality, a household often faces risk and has to make decisions under it. The nonparametric implications of the collective model under risk have, however, not been analyzed.

In this paper, we consider a setting with a finite number of states and one good in each state. A household has two members; an extension to household with more than two members is fairly straightforward. A data set consists of finitely many price-quantity pairs. It is collectively rationalizable if there is an expected utility preference for each household member such that each quantity is Pareto efficient among all those affordable at the corresponding price vector. Given a data set, we address three sorts of issues concerning it: When is it collectively rationalizable, and if it is not, how to measure the extent of its departure from collective rationality? How to recover the individual preferences and the intrahousehold decision process underlying it? Can we forecast a household's behavior at new price configurations?

In the unitary model of Varian (1982), each of these issues can be settled by solving a system of linear inequalities. We show that in our collective model under risk, each of them can be settled by solving a finite number of systems of linear inequalities. This is in contrast to the general collective model of Browning and Chiappori (1998), in which the settlement of each of the three issues involves solving an integer programming (Cherchye et al., 2011).

The paper is organized as follows. Section 2 describes the setup and introduces the concept of collective rationalizability. Section 3 presents a characterization of the data set which is collectively rationalizable. In Section 4, we discuss the fore-

[1]See, for instance, Browning et al. (1994), Fortin and Lacroix (197), Browning and Chiappori (1998), Chiappori et al. (2002), and Cherchye and Vermeulen (2008).

cast of a household's behavior at new price configurations, welfare comparison at the individual level of two observed consumption bundles, and recovery of the income sharing rule. Using the result in Section 3, Section 5 provides a characterization of the demand function which is collectively rationalizable. Section 6 gives a measure of the extent of a data set's departure from collective rationality. Finally, Section 7 makes a review of the literature. All proofs of the results in the text are collected in the Appendix.

2 Basic Concepts

We consider a two-member (A and B) household who makes contingent consumption decisions in an uncertain environment. The uncertainty is represented by S possible states of nature, indexed by $s \in \mathbb{S}$, where $\mathbb{S} = \{1, \ldots, S\}$. There is one private commodity in each state, which could be thought of as money. Household consumption is denoted by vector $x \in \mathbb{R}^S_+$ with corresponding price vector $p \in \mathbb{R}^S_{++}$. Each household member has his or her own preference and probabilistic belief about the occurrence of the states of nature. The observables are aggregate household consumption and individual beliefs, with individual consumption unobservable.

Denote the set of strictly positive probability measures on S by

$$\Delta_{++} = \left\{ \pi = (\pi_1, \ldots, \pi_S) \in \mathbb{R}^S_{++} : \sum_{s=1}^{S} \pi_s = 1 \right\}.$$

Suppose that household member m holds belief $\pi^m \in \Delta_{++}$ about the occurrence of the states of nature, $m = A, B$. A data set is a finite number of price-quantity pairs along with individual beliefs, $\mathfrak{D} = \{(p^k, x^k)_{k=1}^K; \pi^A, \pi^B\}$ with $p^k \in \mathbb{R}^S_{++}$, $x^k \in \mathbb{R}^S_+$ for each $k \in \mathbb{K}$, where $\mathbb{K} = \{1, \ldots, K\}$. It is interpreted as that the household aggregately consumes the vector x^k of commodities at the price vector p^k, $k \in \mathbb{K}$. In what follows, we shall use superscripts to refer to observations and subscripts to components of an observation. For instance, x^k_s denotes the sth component of the kth observation x^k.

In the following discussion, we shall compare two settings, namely the unitary setting and the collective setting, and apply results in the former to study the latter. The unitary setting is the one where the two household members share the same preference and the same belief or the household consumption decision is dictated by a single member. In this setting, the data set reduces to $\mathfrak{D}_u = \{(p^k, x^k)_{k=1}^K; \pi\}$, where π is either the common belief of the two household members or the belief of the dictator. The following concept of rationalization is from Kubler et al. (2014).

Definition 2.1 (Unitary rationalizability)**.** The data set $\mathfrak{D}_u$ is unitarily rationalizable by a strictly concave, strictly increasing, and continuous function u on $\mathbb{R}_+$ if for each $k \in \mathbb{K}$, $\sum_{s=1}^S \pi_s u(x_s^k) \geq \sum_{s=1}^S \pi_s u(z_s)$ for all z with $p^k z \leq p^k x^k$.

The collective setting is the one where the two household members interact to generate household level decisions. Following Browning and Chiappori (1998), we assume their interaction is specified by an intrahousehold bargaining process and the outcome of the interaction is Pareto efficient. To define rationalizability in this setting, we introduce the notation of Cherchye et al. (2007). Given a vector $x \in \mathbb{R}_+^S$, we denote by $\hat{x}$ a pair (x^A, x^B) in $\mathbb{R}_+^S \times \mathbb{R}_+^S$ such that $x^A + x^B = x$, and call $\hat{x}$ feasible personalized quantities (FPQ) for x. Note that $\hat{x}$ uniquely pins down x and x has infinitely many FPQ's.

Definition 2.2 (Collective rationalizability)**.** The data set $\mathfrak{D}$ is collectively rationalizable by a pair of strictly concave, strictly increasing, and continuous functions (u^A, u^B) on $\mathbb{R}_+$ if for each $k \in \mathbb{K}$, there exists an FPQ $\hat{x}^k = (x^{Ak}, x^{Bk})$ for x^k such that $\sum_{s=1}^S \pi_s^m u^m(z_s^m) > \sum_{s=1}^S \pi_s^m u^m(x_s^{mk})$ implies $\sum_{s=1}^S \pi_s^\ell u^\ell(z_s^\ell) < \sum_{s=1}^S \pi_s^\ell u^\ell(x_s^{\ell k})$ $(m \neq \ell)$ for any FPQ $\hat{z}$ of z with $p^k z \leq p^k x^k$.

Comparing the above two definitions, we can see that collective rationalizability is a generalization of unitary rationalizability in the sense that a unitarily rationalizable data set must be collectively rationalizable. In the ensuing section, we shall characterize the data set which is collectively rationalizable. A characterization (or a condition) is said to be 'revealed-preference'if it references observable data only; otherwise it is said to be 'non-revealed-preference.'Our objective is to find a revealed-preference characterization of collective rationalizability.

3 Characterization

In this section, we first study the characterization of unitary rationalizability. This is carried out by Kubler et al. (2014) when the observed consumption levels are all distinct. We extend their result by allowing for identical consumption levels at different price vectors. This will considerably simplify our analysis of collective rationalizability, which forms the second part of this section. Based on a non-revealed-preference characterization of collective rationalizability and the extension of Kubler et al.'s result, we finally arrive at a revealed-preference one.

3.1 Unitary Rationalizability

Consider the data set $\mathfrak{D}_u = \{(p^k, x^k)_{k=1}^K; \pi\}$. Let $\rho_s^k = p_s^k/\pi_s$ for $k \in \mathbb{K}$ and $s \in \mathbb{S}$. Define

$$L(i,j) = \max_{(s,s'):x_s^i > x_{s'}^j} \frac{\rho_s^i}{\rho_{s'}^j},\ i,j \in \mathbb{K}, \tag{3.1}$$

where we set $L(i,j) = 0$ if $x_s^i \leq x_{s'}^j$ for all s, s'.

Definition 3.1 (Kubler et al. 2014)**.** The data set $\mathfrak{D}_u$ satisfies the strong axiom of revealed expected utility (SAREU) if for any $t > 1$ and all $i_1, i_2, \ldots, i_t \in \mathbb{K}$, we have

$$L(i_1,i_2) \cdot L(i_2,i_3) \cdot \cdots \cdot L(i_{t-1},i_t) \cdot L(i_t,i_1) < 1. \tag{3.2}$$

Lemma 3.1. The following three conditions are equivalent:

(i) The data set $\mathfrak{D}_u$ is unitarily rationalizable.
(ii) There exist $\lambda^k \in \mathbb{R}_{++}$, $k \in \mathbb{K}$, such that for all $i,j \in \mathbb{K}$ and $s, s'\mathbb{S}$

$$x_s^i > x_{s'}^j \Rightarrow \lambda^i \rho_s^i < \lambda^j \rho_{s'}^j. \tag{3.3}$$

(iii) The data set $\mathfrak{D}_u$ satisfies SAREU.

Theorem 1 of Kubler et al. (2014) establishes the validity of the lemma in the case where $x_s^i \neq x_{s'}^j$ for all $s, s' \in \mathbb{S}$ and $i, j = 1, \ldots, K$. We show the latter restriction can be dispensed with. The key point is that since the rationalizing function u is not required to be differentiable, it can have more than one supergradient at a point. In the proof of Lemma 3.1 we construct a rationalizing function u whose superdifferential is the set of numbers between $\lambda^i \rho_s^i$ and $\lambda^j \rho_{s'}^j$ when $x_s^i = x_{s'}^j$.[2] Echenique et al. (2023) provide a characterization of unitary rationalizability by a concave (rather than strictly concave) function. This characterization is based on Theorem 1 of Echenique and Saito (2015) whose proof is more involved than that of Theorem 1 of Kubler et al. (2014).

[2]For the definitions of supergradient and superdifferential, one is referred to Aliprantis and Border (2006, Section 7.4, p. 264).

3.2 Collective Rationalizability

We proceed to characterize collective rationalizability. As an immediate consequence of Lemma 3.1, we have the following result.

Lemma 3.2. The data set $\mathfrak{D}$ is collectively rationalizable if and only if there exists an FPQ $\hat{x}^k = (x^{Ak}, x^{Bk})$ for x^k, $k \in \mathbb{K}$, such that both $\{(p^k, x^{Ak})_{k=1}^K; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^K; \pi^B\}$ satisfy SAREU.

One way for checking whether there exists $\hat{x}^k$ such that $\{(p^k, x^{mk})_{k=1}^K; \pi^m\}$ satisfies SAREU is to verify the solvability of the system of inequalities corresponding to (3.2). This is not easy to implement, however, because the inequalities are nonlinear in $\hat{x}^k$. On the other hand, referring to the definition of $L(i, j)$, we note that whether $\{(p^k, x^{mk})_{k=1}^K; \pi^m\}$ satisfies SAREU depends not upon the absolute magnitudes of x_s^{mk}, $s \in \mathbb{S}$ and $k \in \mathbb{K}$, but upon their relative magnitudes. Since the data set $\mathfrak{D}$ is finite, the relative magnitudes of x_s^{mk} have only a finite number of possibilities, which provides another way for verifying the if-condition in Lemma 3.2.

Specifically, let $\mathscr{I} = \{(s, k) : s \in \mathbb{S}, k \in \mathbb{K}\}$. Given a weak order $\succcurlyeq$ on $\mathscr{I}$, let $\succ$ and $\sim$ denote its asymmetric and symmetric parts, respectively. We generalize the operator L by defining it relative to $\succcurlyeq$ as

$$L(i, j; \succcurlyeq) = \max_{(s,s'):(s,i)\succ(s',j)} \frac{\rho_s^i}{\rho_{s'}^j}, i, j \in \mathbb{K}, \tag{3.4}$$

where we set $L(i, j) = 0$ if $(s', j) \succcurlyeq (s, i)$ for all s, s'.

Definition 3.2. The data set $\mathfrak{D}_u$ satisfies SAREU relative to $\succcurlyeq$ if for any $t > 1$ and all $i_1, i_2, \ldots, i_t \in \mathbb{K}$, we have

$$L(i_1, i_2; \succcurlyeq) \cdot L(i_2, i_3; \succcurlyeq) \cdot \cdots \cdot L(i_{t-1}, i_t; \succcurlyeq) \cdot L(i_t, i_1; \succcurlyeq) < 1. \tag{3.5}$$

By analogy with Cherchye et al. (2007), we assign each household member m a weak order $\succcurlyeq^m$ on $\mathscr{I}$, $m = A, B$. By Lemma 3.2, the collective rationalizability of $\mathfrak{D}$ implies the existence of an FPQ $\hat{x}^k = (x^{Ak}, x^{Bk})$ such that both $\{(p^k, x^{Ak})_{k=1}^K; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^K; \pi^B\}$ satisfy SAREU. Define

$$(s, i) \succcurlyeq^m (s', j) \Leftrightarrow x_s^{mi} \geq x_{s'}^{mj}, m = A, B.$$

Then the data set $\{(p^k, x^k)_{k=1}^K; \pi^m\}$ satisfies SAREU relative to $\succcurlyeq^m$, $m = A, B$. Consequently, a necessary condition for $\mathfrak{D}$ to be collectively rationalizable is that the data set $\{(p^k, x^k)_{k=1}^K; \pi^m\}$ satisfies SAREU relative to some weak order on $\mathscr{I}$,

$m = A, B$. Since it references observable data only, this condition is of a revealed-preference nature. As is easy to see, however, it cannot be sufficient without any restrictions imposed on the weak orders involved.

To seek such restrictions, assume given a pair of weak orders $(\succcurlyeq^A, \succcurlyeq^B)$ on $\mathscr{I}$. It is said to be consistent with $\mathfrak{D}$ if the following system of linear inequalities has a solution in $z = (z_s^{km})_{s\in\mathbb{S};k\in\mathbb{K};m=A,B} \in \mathbb{R}^{2SK}$:

$$\begin{cases} z_s^{mi} - z_{s'}^{mj} > 0 & \text{if } (s,i) \succ^m (s',j) \\ z_s^{mi} - z_{s'}^{mj} = 0 & \text{if } (s,i) \sim^m (s',j) \\ z_s^{Ak} + z_s^{Bk} = x_s^k, & \\ z_s^{mk} \geq 0. & \end{cases} \tag{3.6}$$

Technically, the solvability of (3.6) can be determined by the Fourier-Motzkin elimination method (Stoer and Witzgall, 1970, Section 1.2). This method is also able to find all solutions of a system of linear inequalities, which will be useful in the recovery analysis of the next section. From the construction of (3.6), we immediately have the following result.

Lemma 3.3. $(\succcurlyeq^A, \succcurlyeq^B)$ is consistent with $\mathfrak{D}$ if and only if there exists an FPQ (x^{Ak}, x^{Bk}) for each x^k, $k \in \mathbb{K}$, such that $(s,i) \succcurlyeq^m (s',j) \Leftrightarrow x_s^{mi} \geq x_{s'}^{mj}$ for all $s, s' \in \mathbb{S}$, all $i, j \in \mathbb{K}$, and $m = A, B$.

Combining Lemmas 3.2 and 3.3, we arrive at another characterization of collective rationalizability. To avoid cumbersome notation let $\mathfrak{D}_m = \{(p^k, x^k)_{k=1}^K; \pi^m\}$, $m = A, B$. We say $\mathfrak{D}$ satisfies the collective axiom of revealed expected utility (CAREU) if there is a pair of weak orders $(\succcurlyeq^A, \succcurlyeq^B)$ on $\mathscr{I}$ consistent with $\mathfrak{D}$ and such that $\mathfrak{D}_m$ satisfies SAREU relative to $\succcurlyeq^m$ for $m = A, B$. To indicate the weak orders explicitly, we sometimes also say $\mathfrak{D}$ satisfies CAREU with respect to $(\succcurlyeq^A, \succcurlyeq^B)$.

Theorem 3.1. The data set $\mathfrak{D}$ is collectively rationalizable if and only if it satisfies CAREU.

Theorem 3.1 indicates that checking collective rationalizability can be carried out by solving a finite number of systems of linear inequalities, one for each pair of weak orders on $\mathscr{I}$. The key fact is that given any consistent pair of weak orders on $\mathscr{I}$, if $(x^{Ak}, x^{Bk})_{k=1}^K$ and $(y^{Ak}, y^{Bk})_{k=1}^K$ are both solutions to the corresponding system (3.6), then $\{(p^k, x^{mk}); \pi^m\}$ is unitarily rationalizable if and only if so is $\{(p^k, y^{mk}); \pi^m\}$ for $m = A, B$.

To illustrate the usefulness of Theorem 3.1, let us consider two examples. Example 3.1 gives a data set which is not collectively rationalizable and Example 3.2 provides a data set which is collectively rationalizable but not unitarily rationalizable. In all the examples of this and the ensuing sections, we take $S = 2$, $\pi^A = [1/2, 1/2]$, $\pi^B = [1/3, 2/3]$, $\pi = \pi^A$, $\mathfrak{D}_u = \{(p^k, x^k)_{k=1}^K; \pi\}$, and $\mathfrak{D} = \{(p^k, x^k)_{k=1}^K; \pi^A, \pi^B\}$, where the values of (p^k, x^k), $k = 1, \ldots, K$, will be specified in the examples.
Example 3.1. Take $K = 1$ and let

$$p^1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, x^1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then $\rho_1^{A1} = 10, \rho_2^{A1} = 8$ and $\rho_1^{B1} = 15, \rho_2^{B1} = 6$. Since $x_1^1 > x_2^1$, it follows that $(1,1) \succ^A (2,1)$ or $(1,1) \succ^B (2,1)$ for any pair of weak orders $(\succcurlyeq^A, \succcurlyeq^B)$ on $\mathscr{I}$ that is consistent with $\mathfrak{D}$. Therefore, if $\mathfrak{D}$ is collectively rationalizable, we must have $\rho_1^{A1} < \rho_2^{A1}$ or $\rho_1^{B1} < \rho_2^{B1}$, a contradiction. This means the data set $\mathfrak{D}$ is not collectively rationalizable.
Remark. Example 3.1 indicates that in the collective household consumption model of this paper, it is sufficient to have two states and one observation for rejecting the collective rationalizability of observed household behavior. This is in contrast to the general collective household consumption model of Browning and Chiappori (1998), in which at least three goods and three observations are needed for rejecting the collective rationalizability of observed household behavior (see Cherchye et al., 2007, Proposition 3).
Example 3.2. Take $K = 2$ and let

$$p^1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, x^1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}; p^2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, x^2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

For the data set $\mathfrak{D}_u$, we have $L(1,2) = L(2,1) = 1$, and therefore $\mathfrak{D}_u$ is not unitarily rationalizable. On the other hand, take

$$(1,2) \sim^A (2,2) \succ^A (2,1) \succ^A (1,1),$$
$$(2,1) \succ^B (1,1) \succ^B (1,2) \sim^B (2,2).$$

This pair of weak orders is consistent with $\mathfrak{D}$ because

$$x^{A1} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, x^{B1} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, x^{A2} = \begin{bmatrix} 1.6 \\ 1.6 \end{bmatrix}, x^{B2} = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix}$$

constitute FPQ's for x^1 and x^2 such that $(s,i) \succcurlyeq^m (s',j) \Leftrightarrow x_s^{mi} \geq x_{s'}^{mj}$ for $i, j = 1, 2$,

$s = 1, 2$, and $m = A, B$. Moreover, direct calculation shows

$$L(1,1;\succcurlyeq^A) = \frac{4}{5}, L(1,1;\succcurlyeq^B) = \frac{2}{5}$$
$$L(2,2;\succcurlyeq^m) = 0, m = A, B,$$
$$L(1,2;\succcurlyeq^A) = L(2,1;\succcurlyeq^B) = 0.$$

It follows that $\mathfrak{D}$ satisfies CAREU, and therefore is collectively rationalizable.

We make some remarks about Theorem 3.1.

(i) In the case of certainty, Cherchye et al. (2007) present a necessary condition for collective rationalizability. This condition is of a revealed-preference nature, but is not sufficient in general. Cherchye et al. (2011) provide a non-revealed-preference characterization of collective rationalizability in terms of an integer programming. The solution of this programming entails a large amount of computational burden. In contrast, the verification of CAREU can be implemented by solving a finite number of linear programming problems. It should be pointed out, however, that Cherchye et al. (2007, 2011) work in a framework with both private and public commodities, while it is unclear how to generalize Theorem 3.1 to this more general setup.

(ii) In the unitary setting and when the household members' common belief is fixed but unobservable, Echenique and Saito (2015) characterize the data sets that are unitarily rationalizable by subjective expected utility preferences and by state-dependent utility functions. Following the argument of this section, their characterization can be readily extended to the current collective setting. The extension, in particular, yields a characterization of collective rationalizability by additively separable utility functions, although such a characterization is not available for general preferences (Cherchye et al., 2007).

(iii) The household members' beliefs are assumed to be fixed in this paper. Kubler et al. (2014) characterize unitary rationalizability for the data sets in which the household's beliefs are allowed to vary. With the aid of this characterization, Theorem 3.1 can be extended to this more general setup.

(iv) Information on "assignable quantities"for the goods can be easily incorporated. This information describe how much a household member consumes of a good, and is increasingly available in budget surveys (see, e.g., Browning and Bonke, 2006). Note that z_s^{mk} in (3.6) represents the consumption by individual m of good s in observation k. The information on "assignable quantities"takes the form of a lower bound on z_s^{mk}, i.e.

$$z_s^{mk} \geq \underline{x}_s^{mk}, \tag{3.7}$$

wherein $\underline{x}_s^{mk} \in \mathbb{R}_+$ is observable. Such information can be taken into account by replacing the last inequality in (3.6) with (3.7).

4 Recovery Analysis

Suppose that we have observed the data set $\mathfrak{D} = \{(p^k, x^k)_{k=1}^K; \pi^A, \pi^B\}$ which is collectively rationalizable, and that the government intends to make a tax reform, after which the prices of the commodities will be given by p^0. Can we forecast the household's demand behavior under this new price configuration? Does the reform improve the welfare of the household? What is the effect of the tax reform on the income allocation between the household members? This section is devoted to a study of these questions.

4.1 Extrapolation

Any consumption bundle that is consistent with the structural decision model underlying the data set $\mathfrak{D}$ is a possible chosen bundle of the household at p^0. Formally, let $\bar{\mathbb{K}} = \{0\} \cup \mathbb{K}$ and

$$S(p^0) = \left\{x^0 \in \mathbb{R}_+^S : (p^k, x^k), k \in \bar{\mathbb{K}}, \text{ satisfies CAREU}\right\}.$$

Then any bundle in $S(p^0)$ is a plausible demand of the household at p^0. Next we discuss the computation of $S(p^0)$.

For any $x^0 \in \mathbb{R}_+^S$, let $\hat{\mathfrak{D}}(x^0) = \{(p^k, x^k)_{k=0}^K; \pi^A, \pi^B\}$ and $\hat{\mathscr{I}} = \{(s,k) : s \in \mathbb{S}, k \in \bar{\mathbb{K}}\}$. Given a pair of weak orders $(\succcurlyeq^A, \succcurlyeq^B)$ on $\hat{\mathscr{I}}$, x^0 is a match with $(\succcurlyeq^A, \succcurlyeq^B)$ if the latter is consistent with $\hat{\mathfrak{D}}(x^0)$. Let $\mathfrak{M}(\succcurlyeq^A, \succcurlyeq^B)$ be the set of matches with $(\succcurlyeq^A, \succcurlyeq^B$). Referring to (3.6), the set $\mathfrak{M}(\succcurlyeq^A, \succcurlyeq^B)$ can be obtained by solving a system of linear inequalities. Let Ψ_0 be the set of pairs of weak orders $(\succcurlyeq^A, \succcurlyeq^B)$ on $\hat{\mathscr{I}}$ for which $\mathfrak{M}(\succcurlyeq^A, \succcurlyeq^B) \neq \emptyset$ and Ψ_0^* the set of pairs of weak orders $(\succcurlyeq^A, \succcurlyeq^B) \in \Psi_0$ such that $\hat{\mathfrak{D}}_m(x^0)$ satisfies SAREU relative to $\succcurlyeq^m$, where $x_0 \in \mathfrak{M}(\succcurlyeq^A, \succcurlyeq^B)$ and $\hat{\mathfrak{D}}_m(x^0) = \{(p^k, x^k)_{k=0}^K; \pi^m\}$, $m = A, B$. Then we have

$$S(p^0) = \bigcup_{(\succcurlyeq^A, \succcurlyeq^B) \in \Psi_0^*} \mathfrak{M}(\succcurlyeq^A, \succcurlyeq^B).$$

Example 4.1. Given

$$p^1 = \begin{bmatrix}10\\4\end{bmatrix}, x^1 = \begin{bmatrix}5\\6\end{bmatrix}, p^0 = \begin{bmatrix}4\\10\end{bmatrix},$$

let us compute $S(p^0)$. First, direct calculation shows

$$\rho^{A1} = \begin{bmatrix}20\\8\end{bmatrix}, \rho^{B1} = \begin{bmatrix}30\\6\end{bmatrix}, \rho^{A0} = \begin{bmatrix}8\\20\end{bmatrix}, \rho^{B0} = \begin{bmatrix}12\\15\end{bmatrix}.$$

So for any $(\succcurlyeq^A, \succcurlyeq^B) \in \Psi^*$, we must have $(2,1) \succcurlyeq^m (1,1)$ and $(1,0) \succcurlyeq^m (2,0)$ for $m = A, B$. To pin down $\succcurlyeq^m$, there are several possibilities to consider. For instance, suppose $(2,0) \succ^m (2,1)$, $m = A, B$. Then we have $L(0,1;\succcurlyeq^m) = 5/2$ and $L(1,0;\succcurlyeq^m) = 0$, so that $\hat{\mathfrak{D}}_m(x^0)$ satisfies SAREU relative to $\succcurlyeq^m$. It is straightforward to show that $\mathfrak{M}(\succcurlyeq^A, \succcurlyeq^B) = \{x^0 \in \mathbb{R}^2_+ : x^0_1 \geq x^0_2 > 6\}$. After considering all other possibilities, we will obtain $S(p^0) = \{x^0 \in \mathbb{R}^2_+ : x^0_1 \geq x^0_2\}$.

4.2 Comparison of Consumption Bundles

Suppose that the consumption bundle of the household demanded after the tax reform is given by $x^0 \in S(p^0)$. Then the set of data becomes $\hat{\mathfrak{D}} = \hat{\mathfrak{D}}(x^0)$. Sometimes it is important to assess the impact of the reform on individual, as opposed to household, welfare (see, e.g., Duflo, 2003; Cherchye et al., 2011; Lise and Seitz, 2011). Given this, it is interesting to ask whether the reform is Pareto-improving for the household compared with the consumption bundle before the reform, say (p^1, x^1)?

To answer this question, let us first make a review of the unitary case. In this case, the concept of Pareto-improvement reduces to that of preference. Consider the data set $\hat{\mathfrak{D}}_u = \{(p^k, x^k)_{k=0}^K; \pi\}$ which is unitarily rationalizable. It is well justified to conclude that the reform is a desirable one if x^0 is revealed preferred to x^1 (Varian, 1982).The concept of revealed preference refers to the quantities x^k. In the collective setup, however, individual consumption quantities are unobservable. For this reason, we take into account the set $\mathfrak{F}$ of all possible FPQ's that are consistent with a collective rationalization of $\hat{\mathfrak{D}}$. Specifically, let Ψ_1 be the set of pairs of weak orders on $\hat{\mathscr{I}}$ with respect to which $\hat{\mathfrak{D}}$ satisfies CAREU. For any $(\succcurlyeq^A, \succcurlyeq^B) \in \Psi_1$, let $\mathfrak{F}(\succcurlyeq^A, \succcurlyeq^B)$ be the set of solutions to the corresponding system (3.6). Then define $\mathfrak{F} = \cup_{(\succcurlyeq^A, \succcurlyeq^B) \in \Psi_1} \mathfrak{F}(\succcurlyeq^A, \succcurlyeq^B)$.

Given any $(\hat{x}^k) = (x^{Ak}, x^{Bk})_{k=0}^K$, we say x^0 is revealed preferred to x^1 by individual m with respect to $(\hat{x}^k)$, written $x^0 \succcurlyeq^m x^1(\hat{x}^k)$, if x^{m0} is revealed preferred to x^{m1} for the data set $(p^k, x^{mk})_{k=0}^K$. And we say x^0 is revealed preferred to x^1 by individual m, written $x^0 \succcurlyeq^m x^1$, if $x^0 \succcurlyeq^m x^1(\hat{x}^k)$ for every $(\hat{x}^k) \in \mathfrak{F}$. Then the reform is Pareto-improving if $x^0 \succcurlyeq^m x^1$ for $m = A, B$.

Example 4.2. Let

$$p^1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, x^1 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}; p^0 = \begin{bmatrix} 2 \\ 1.9 \end{bmatrix}, x^0 = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

It is straightforward to verify that $\mathfrak{D}_u = \{(p^k, x^k)_{k=0}^1; \pi\}$ is unitarily rationalizable, so $x^0 \in S(p^0)$. In the unitary setting, where the household is taken as a single decision-maker, as $x^0 \gg x^1$, we have x^0 revealed preferred to x^1. This means the tax reform improves the welfare of the household as a whole. But, in the collective setting, we are obliged to see whether it improves the welfare of each household member. The answer is, not necessarily. To see this, take

$$x^{A1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, x^{B1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; x^{A0} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, x^{B0} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

Then $\hat{x}^k = (x^{Ak}, x^{Bk})_{k=0}^1 \in \mathfrak{F}$. Since $x^1 \succcurlyeq^A x^0(\hat{x}^k)$, the tax reform cannot be Pareto-improving. On the other hand, suppose we have information on assignable quantities: $\underline{x}_2^{m0} = 5.5$, $m = A, B$. Let $\mathfrak{F}'$ be the set of points in $\mathfrak{F}$ which are consistent with this information. Then we must have $x_1^{m0} \geq x_1^{m1}$ for any $(\hat{x}^0, \hat{x}^1) \in \mathfrak{F}'$ and for $m = A, B$. To see this, assume by way of contradiction that $x_1^{A0} < x_1^{A1}$ for some $(\hat{x}^0, \hat{x}^1) \in \mathfrak{F}'$. Suppose $(\hat{x}^0, \hat{x}^1) \in \mathfrak{F}(\succcurlyeq^A, \succcurlyeq^B)$. Then we have

$$L(0,1; \succcurlyeq^A) \cdot L(1,0; \succcurlyeq^A) = \frac{1}{2} \cdot \frac{5}{2} > 1,$$

a contradiction. Since $x_2^{m0} > x_2^{m1}$ for any $(\hat{x}^0, \hat{x}^1) \in \mathfrak{F}'$ and for $m = A, B$, it follows that the reform is Pareto-improving.

We conclude this subsection with two remarks in relation to Varian (1982) and Cherchye et al. (2011). Varian (1982) considers the unitary setting and studies, among other things, the preference relations of two consumption bundles x^0 and x' which have not been previously observed. This question (i.e. comparing two consumption bundles without specifying their associated price vectors) is not interesting in the collective setting, because in the latter case either bundle could be chosen by the household. To see this, note that the household will, in principle, choose a bundle which has a Pareto efficient FPQ. Suppose that $\mathfrak{D}$ is collectively rationalizable by (u^A, u^B). Let $\partial u^m(x)$ denote the superdifferential of

u^m at x, $m = A, B$. Take the price vectors p^0 and p' such that $\rho^{A0} \in \partial u^A(x^0)$ and $\rho^{B'} \in \partial u^B(x')$. Then $\hat{x}^0 = (x^0, 0)$ is an FPQ for x^0 which is Pareto efficient among all FPQ $\hat{z}$ for z with $p^0 z \leq p^0 x^0$. Similarly, $\hat{x}' = (0, x')$ is an FPQ for x' which is Pareto efficient among all FPQ $\hat{z}$ for z with $p'z \leq p'x'$. But neither of $\hat{x}^0$ and $\hat{x}'$ Pareto dominates the other, and therefore either of x^0 and x' could be a chosen bundle for the household.

The second remark is concerned with Cherchye et al. (2011), who compare two observed consumption bundles by inquiring whether the hypothesis that household member m prefers x^0 to x^1 should be rejected. If there exists an $(\hat{x}^k) \in \mathfrak{F}$ such that $x^0 \succcurlyeq^m x^1(\hat{x}^k)$, the hypothesis should not be rejected. But failing to reject a hypothesis is different from accepting it as true. This subsection provides a condition under which the hypothesis is justified to be accepted as true.

4.3 Recovery of the Income Sharing Rule

We now turn to the last question: the recovery, or set identification, of the income sharing rule. This rule describes how a household's total income is shared across its members. It can be taken as a measure of the wealth or poverty of the individual members. Identifying the rule is, therefore, of particular importance if one cares about the economic well-being of some special members of a household, such as the elderly (Cherchye et al., 2012) or the children (Dunbar et al., 2013). Consider the data set $\mathfrak{D} = \{(p^k, x^k)_{k=1}^K; \pi^A, \pi^B\}$. At each observation k and for an FPQ (x^{Ak}, x^{Bk}) of x^k, the income share received by individual m is $p^k x^{mk}$, $m = A, B$. Then the whole set of possible income sharing rules that are consistent with a collective rationalization of $\mathfrak{D}$ is given by

$$\bar{W} = \left\{(w^A, w^B) \in \mathbb{R}^2 : w^A = p^k x^{Ak}, w^B = p^k x^{Bk}, k \in \mathbb{K}, (x^{Ak}, x^{Bk}) \in \mathfrak{F}\right\}, \qquad (4.1)$$

where $\mathfrak{F} = \cup_{(\succcurlyeq^A, \succcurlyeq^B) \in \Psi} \mathfrak{F}(\succcurlyeq^A, \succcurlyeq^B)$ and Ψ is the set of pairs of weak orders on $\mathscr{I}$ with respect to which $\mathfrak{D}$ satisfies CAREU.

The above identification technique differs from that of Cherchye et al. (2015) in several respects. These authors assume the whole household demand function is known, and study the identification of bounds on the sharing rule for general collective consumption models under certainty with both private and public goods marketed. They exploit the weak axiom of revealed preference (WARP) and formulate the problem as a nonlinear programming. The bounds they identify on the sharing rule can be made tighter by strengthening WARP to the strong axiom of revealed preference (SARP), but doing so will make the nonlinear programming much more difficult to solve, if not completely intractable. In contrast, this paper assumes given a finite number of observations on the household demand function, and studies a collective consumption model under uncertainty with private goods only. The method for identifying the sharing rule described above exploits SARP and boils down to the solution of a number of linear programs. With knowledge of the data set $\mathfrak{D}$ only, the bounds obtained on the sharing rule is the tightest possible.

5 Collective Rationalizability of Demand Functions

The previous sections examine the collective rationalizability of finitely many observations on an aggregate demand functions. Now if we are given the whole function, a natural question is to characterize this function as a Pareto-efficient outcome of some intrahousehold decision process (Browning and Chiappori, 1998; Chiappori and Ekeland, 2006). In this section, we examine to what extent one can exploit the results of the preceding sections to bear on that characterization. In the unitary setting, the same question is raised and studied by Mas-Colell (1978) in the context of competitive markets under certainty and extended to various other contexts, for instance, by Forges and Minelli (2009) and Kübler and Polemarchakis (2017).

Specifically, let $h : \mathbb{P} \to \mathbb{R}^S_{++}$ be a function which is continuous and homogeneous of degree zero on a compact subset $\mathbb{P}$ of $\mathbb{R}^S_{++}$. It is interpreted as a household's aggregate demand function. Because of the homogeneity of h, it is without loss of generality to normalize the prices by restricting its domain to Δ_{++}, and henceforth we shall assume $\mathbb{P} \subset \Delta_{++}$. The function h satisfies CAREU if so does the data set $\{(p, h(p) : p \in \mathcal{P}\}$ for any finite subset $\mathcal{P} \subset \mathbb{P}$. An FPQ for h is a pair of functions (h^A, h^B) from Δ_{++} to $\mathbb{R}^S_{++}$ such that $h = h^A + h^B$. We say h is collectively rationalizable by a pair of functions (u^A, u^B) on $\mathbb{R}_+$ if there exists an FPQ (h^A, h^B) for h such that for every $p \in \mathbb{P}$, $\sum_{s=1}^S \pi^m_s u^m(z^m_s) > \sum_{s=1}^S \pi^m_s u^m(h^m_s(p))$ implies $\sum_{s=1}^S \pi^\ell_s u^\ell(z^\ell_s) < \sum_{s=1}^S \pi^\ell_s u^\ell(h^\ell_s(p))$ $(m \neq \ell)$ for any FPQ $\hat{z}$ of z with $pz \leq ph(p)$, where $h_s(p)$ denotes the sth component of $h(p)$.

Proposition 5.1. If h satisfies CAREU, then it is collectively rationalizable.

We now make a discussion of this proposition and its proof. Let $\mathcal{P}^K$ be an increasing sequence of finite sets of observed prices with $\cup_K \mathcal{P}^K$ dense in $\mathbb{P}$. Since h satisfies CAREU, the data set $\mathfrak{D}^K = \{(p, h(p) : p \in \mathcal{P}^K\}$ is collectively rationalizable by a pair of strictly concave, strictly increasing, and continuous functions, say, (u^{AK}, u^{BK}). These functions can be taken in such a way that the sequence $\{u^{mK}\}_{K=1}^{\infty}$ is equicontinuous, and hence has an accumulation point, say u^m, $m = A, B$. We show that h is collectively rationalizable by (u^A, u^B).

At this moment we cannot determine what restrictions should be imposed on h such that it uniquely identifies the underlying pair of rationalizing expected utility preferences. This problem is referred to as the identifiability problem in the literature. It has been studied in the collective setting under certainty by Chiappori and Ekeland (2009) and in the unitary setting under uncertainty by Kübler and Polemarchakis (2017). We reserve for future study this problem in the collective setting under uncertainty.

6 Approximate Collective Rationalizability

The data set in Example 3.1 is not collectively rationalizable. In reality, it is very unlikely that a household's behavior is exactly collectively rationalizable, and, for most purposes, "nearly collectively rationalizable"is just as good as "collectively rationalizable"(Varian, 1990). In this section, we develop a way for quantifying the term "nearly collectively rationalizable", i.e. a measure of deviations from collective rationalizability. In the unitary setting, various such measures have been constructed. In particular, Echenique et al. (2023) propose a measure of how far a data set is from being unitarily rationalizable by an expected utility preference. In the following we extend their measure to the collective setting.

Within the expected utility framework, several reasons account for the failure of a data set to be rationalizable (either unitarily or collectively), as, for instance, miss-perceived prices, random taste, and incorrect beliefs. These three reasons are equivalent in terms of their ability to reconcile a data set with expected utility theory (Echenique et al., 2023, Theorem 1). So, without loss of generality, we focus on miss-perceived prices. Recall that $\mathfrak{D}_u = \{(p^k, x^k)_{k=1}^K; \pi\}$ and $\mathfrak{D} = \{(p^k, x^k)_{k=1}^K; \pi^A, \pi^B\}$. The following notion of perturbed unitary rationalizability is from Definition 4 of Echenique et al. (2023).

Definition 6.1. (i) Given $e \in \mathbb{R}_+$, the data set $\mathfrak{D}_u$ is e-perturbed unitarily rationalizable if there exists a set of price vectors $\tilde{p}^k$, $k \in \mathbb{K}$, such that $\{(\tilde{p}^k, x^k)_{k=1}^K; \pi\}$ is unitarily rationalizable and

$$\frac{1}{1+e} \leq \frac{\tilde{p}_s^k/\tilde{p}_{s'}^k}{p_s^k/p_{s'}^k} \leq 1+e, \forall s, s' \text{ and } \forall k. \tag{6.1}$$

(ii) The data set $\mathfrak{D}$ is e-perturbed collectively rationalizable if there exists a set of price vectors $\tilde{p}^k$, $k \in \mathbb{K}$, which satisfies (6.1) and such that $\{(\tilde{p}^k, x^k)_{k=1}^K; \pi^A, \pi^B\}$ is collectively rationalizable.

We are interested in the minimum e, written e_*, which makes $\mathfrak{D}$ e-perturbed collectively rationalizable. It can be taken as a measure of how far the data set $\mathfrak{D}$ is from being collectively rationalizable. When $e_* = 0$, $\mathfrak{D}$ is collectively rationalizable. To compute e_*, note that by the remark after Theorem 3.1, given $(\succcurlyeq^A, \succcurlyeq^B) \in \Psi$, for any two $(x^{Ak}, x^{Bk})_{k=1}^K, (y^{Ak}, y^{Bk})_{k=1}^K \in \mathfrak{F}(\succcurlyeq^A, \succcurlyeq^B)$, $\{(p^k, x^{mk}); \pi^m\}$ is e-perturbed unitarily rationalizable if and only if so is $\{(p^k, y^{mk}); \pi^m\}$ for $m = A, B$. Therefore, it makes sense to define a function $e_*^m : \Psi \to \mathbb{R}_+$ such that $e_*^m(\succcurlyeq^A, \succcurlyeq^B)$ is the minimum e which makes $\{(p^k, x^{mk}); \pi^m\}$ e-perturbed unitarily rationalizable.

Proposition 6.1. $e_* = \min_{(\succcurlyeq^A, \succcurlyeq^B) \in \Psi} \max_{m \in \{A,B\}} e_*^m(\succcurlyeq^A, \succcurlyeq^B)$.

An an illustration, let us compute the e_* for the data set in Example 3.1.

Example 6.1 (Example 3.1-continued)**.** Recall the data set in Example 3.1. Since $\mathscr{I}$ has only two elements, we use $\succ^m$ to indicate $(1,1) \succ^m (2,1)$, $\prec^m$ to indicate $(1,1) \prec^m (2,1)$, and $\sim^m$ to indicate $(1,1) \sim^m (2,1)$, $m = A, B$. It is straightforward to verify that

$$\Psi = \{(\succ^A, \succ^B), (\succ^A, \sim^B), (\succ^A, \prec^B), (\sim^A, \succ^B), (\prec^A, \succ^B)\}.$$

Take $(\succ^A, \succ^B)$ for instance. For any $(x^A, x^B) \in \mathfrak{F}(\succ^A, \succ^B)$ and any $\tilde{p} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{R}_{++}^2$, it can be verified that $(\tilde{p}, x^A)$ is unitarily rationalizable if and only if $\tilde{p}_1 < \tilde{p}_2$. Referring to (6.1), we get $e_*^A(\succ^A, \succ^B) = 1/4$. Similarly, $(\tilde{p}, x^B)$ is unitarily rationalizable if and only if $\tilde{p}_1 < \tilde{p}_2/2$, so that $e_*^B(\succ^A, \succ^B) = 3/2$. Repeating the above procedure for all other pairs of weak orders in Ψ, we find $e_* = e_*^A(\succ^A, \sim^B) = 1/4$.

7 Literature Review

This paper is related to two strands of literature. The first is about the collective consumption model. Based on earlier works of Manser and Brown (1980) and McElroy and Horney (1981), Chiappori (1988, 1992) makes the first formal investigation of a collective consumption model under the sole assumption that household decisions result in Pareto efficient outcomes. Chiappori's model is then studied both parametrically and non-parametrically. The parametric approach imposes some (nonverifiable) functional structure on the household decision process. Using this approach, Browning and Chiappori (1998) and Chiappori and Ekeland (2006) provide, and empirically test, a set of necessary and sufficient restrictions that fully characterize a general collective consumption model; Chiappori and Ekeland (2009) discuss the recovery of the underlying structural model (i.e. individual preferences and the intrahousehold decision process) from the household's aggregate behavior.

The nonparametric approach to the collective model is similar to that to the unitary model (see, for instance, Afriat, 1967; Varian, 1982). It does not impose any functional structure on the household decision process. Adopting this approach, Cherchye et al. (2007, 2010, 2011) study the characterization and empirical test of the collective consumption model of Browning and Chiappori (1998).

The second relevant strand of literature is concerned with the nonparametric test of expected utility maximization in the unitary setting. In the setting which allows for the consumption of multiple goods in each state, Varian (1983) and Green and Srivastava (1986) characterize the data sets which are consistent with objective expected utility theory in terms of the solvability of a system of Afriat inequalities. Confining to the setting with one good in each state, Kubler et al. (2014) give a revealed preference characterization. In the same setting but with probabilities unobservable, Echenique and Saito (2015) provide a revealed preference characterization of the data sets which are consistent with subjective expected utility theory. Chambers et al. (2016) make an attempt to extend Kubler et al. (2014) to the setting with multiple goods in each state. While the tests in all papers cited above hold valid only for risk-averse expected utility preference, Polisson et al. (2020) develop a nonparametric test of expected utility maximization without the risk aversion requirement.

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Appendix A Proofs

A.1 Proof of Lemma 3.1

We prove the lemma by showing (i)⇔(ii) and (ii)⇔(iii). The equivalence (i)⇔(ii) can be proved in almost the same way as Lemma 2 of Kubler and Schmedders (2010) and Lemma 7 of Echenique and Saito (2015). For the sake of completeness, we present the details here. For (i)⇒(ii), suppose $\mathfrak{D}_u$ is unitarily rationalizable by u. Let $\partial u(x)$ denote the superdifferential of u at x. By Theorem 28.3 of Rockafellar (1970), there are numbers $\lambda^k \in \mathbb{R}_+$, $k \in \mathbb{K}$, such that $\lambda^k \rho_s^k \in \partial u(x_s^k)$ if $x_s^k > 0$ and $\lambda^k \rho_s^k \geq \underline{w}$ for some $\underline{w} \in \partial u(x_s^k)$ if $x_s^k = 0$. The strict monotonicity of u implies $\lambda^k > 0$, $k \in \mathbb{K}$. And the strict concavity of u implies $w_1 > w_2$ if $w_1 \in \partial u(z_1)$, $w_2 \in \partial u(z_2)$, and $z_1 < z_2$. From this, statement (ii) follows immediately.

For (ii)⇒(i), suppose that $\{x_s^k : s = 1, \ldots, S, k \in \mathbb{K}\} = \{x_1, \ldots, x_L\}$ with $x_1 < x_2 < \cdots < x_L$. For each $n = 1, \ldots, L$, let $I(n) = \{(s,k) : x_s^k = x_n\}$ and let

$$\theta_n^0 = \min_{(s,k)\in I(n)} \lambda^k \rho_s^k, \quad \theta_n^1 = \max_{(s,k)\in I(n)} \lambda^k \rho_s^k.$$

If $x_1 > 0$, set $x_0 = 0$, $\theta_0^0 = \theta_0^1 = 1 + \theta_1^1$. Define a function $g : \mathbb{R}_+ \to \mathbb{R}_{++}$ such that

$$g(x) = \begin{cases} \theta_n^0 + \frac{\theta_n^0 - \theta_{n+1}^1}{x_n - x_{n+1}}(x - x_n), & \text{if } x \in [x_n, x_{n+1}), n = 0, \ldots, L-1 \\ \theta_L^0 \frac{\pi/2 - \arctan x}{\pi/2 - \arctan x_L}, & \text{if} x \geq x_L, \end{cases}$$

where π is the circumference ratio and arctan indicates the inverse tangent function. Otherwise if $x_1 = 0$, define g such that

$$g(x) = \begin{cases} \theta_n^0 + \frac{\theta_n^0 - \theta_{n+1}^1}{x_n - x_{n+1}}(x - x_n), & \text{if } x \in [x_n, x_{n+1}), n = 1, \ldots, L-1 \\ \theta_L^0 \frac{\pi/2 - \arctan x}{\pi/2 - \arctan x_L}, & \text{if} x \geq x_L, \end{cases}$$

It is easily seen that g is strictly decreasing, strictly positive, and piecewise continuous, so that it is integrable. Define a function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$u(x) = \int_0^x g(t)dt,$$

so that u is strictly concave, strictly increasing, and continuous. By the construction of u, we have $\lambda^k \rho_s^k \in \partial u(x_s^k)$ for all s and k, and hence the data set $\mathfrak{D}_u$ is unitarily rationalizable by u.

We now turn to the proof of the equivalence (ii)⇔(iii). The implication (ii)⇒(iii) can be proved in the same way as in the argument of Theorem 1 of Kubler et al. (2014). Therefore, it remains to show (iii)⇒(ii). Given that SAREU holds and that there are a finite number of observations, there must exist an $\epsilon > 0$ such that for any $t > 1$ and any sequence $i_1, \dots, i_t \in \{1, \dots, K\}$, we have

$$L(i_1, i_2) \cdot L(i_2, i_3) \cdot \cdots \cdot L(i_{t-1}, i_t) \cdot L(i_t, i_1) < \frac{1}{(1+\epsilon)^t}. \tag{A.1}$$

Define $\mathbb{K}_1 = \{k \in \mathbb{K} : x^k_s = x_L, \forall s \in \mathbb{S}\}$. When $\mathbb{K}_1 = \emptyset$, the argument is the same as in the proof of Theorem 1 of Kubler et al. (2014) and so is omitted here. Let us focus on the case where $\mathbb{K}_1 \neq \emptyset$. Set $\lambda^k = 1$ for all $k \in \mathbb{K}_1$. For any $j \notin \mathbb{K}_1$, define

$$\lambda^j = \max_k \max_{(i_1, i_2, \dots, i_{k-1}, j) \in \mathfrak{I}_k} L(i_1, i_2) \cdot \cdots \cdot L(i_{k-1}, j)(1+\epsilon)^k, \tag{A.2}$$

where $\mathfrak{I}_k = \{(i_1, \dots, i_k) : i_1 \in \mathbb{K}_1, i_m \in \mathbb{K}, m = 2, \dots, k\}$. By (A.1), λ^j is well-defined. And as $j \notin \mathbb{K}_1$, we have $L(i, j) > 0$ for any $i \in \mathbb{K}_1$, hence $\lambda^j > 0$. Now take any (s, i) and (s', j) with $x^i_s > x^j_{s'}$. This means $j \notin \mathbb{K}_1$. If $i \in \mathbb{K}_1$, then $\lambda^i = 1$ and $\lambda^j > L(i, j)$ (by (A.2)), and hence $\lambda^i \rho^i_s < \lambda^j \rho^j_{s'}$. If $i \notin \mathbb{K}_1$, there exists a sequence $(i_1, i_2, \dots, i_{k-1}) \in \mathfrak{I}_k$ for some k such that $\lambda^i = L(i_1, i_2) \cdot \cdots \cdot L(i_{k-1}, i)(1+\epsilon)^k$. Again by (A.2), we have

$$\lambda^j > L(i_1, i_2) \cdot \cdots \cdot L(i_{k-1}, i)(1+\epsilon)^k \cdot L(i, j) = \lambda^i L(i, j),$$

and hence $\lambda^i \rho^i_s < \lambda^j \rho^j_{s'}$.

A.2 Proof of Lemma 3.2

Let us start with the necessity part of Lemma 3.2. Suppose that the data set $\mathfrak{D}$ is collectively rationalizable. Then there exist two strictly concave, strictly increasing, and continuous functions u^m on $\mathbb{R}_+$, $m = A, B$, an FPQ $\hat{x}^k = (x^{Ak}, x^{Bk})$, and $\mu_k \in \mathbb{R}_+$, $k \in \mathbb{K}$, such that $\hat{x}^k$ is the solution to the following program

$$\begin{gathered}\max_{(z^A, z^B)} \sum_{s=1}^{S} \pi^A_s u^A(z^A_s) + \mu_k \sum_{s=1}^{S} \pi^B_s u^B(z^B_s), \\ \text{s.t. } p^k(z^A + z^B) \leq p^k x^k.\end{gathered}$$

It follows that x^{mk} must solve the problem

$$\max_{z^m} \sum_{s=1}^{S} \pi^m_s u^m(z^m_s) \quad \text{s.t. } p^k z^m \leq p^k x^{mk}, m = A, B.$$

This implies the data set $\{(p^k, x^{mk})_{k=1}^K; \pi^m\}$ is unitarily rationalizable, and hence, by Lemma 3.2, satisfies SAREU.

For the sufficiency part, suppose that there exists an FPQ $\hat{x}^k = (x^{Ak}, x^{Bk})$ for x^k, $k \in \mathbb{K}$, such that both $\{(p^k, x^{Ak})_{k=1}^K; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^K; \pi^B\}$ satisfy SAREU. By Lemma 3.2, the data set $\mathfrak{D}_m = \{(p^k, x^{mk})_{k=1}^K; \pi^m\}$ is unitarily rationalizable for $m = A, B$. That is, there exists a strictly concave, strictly increasing, and continuous function u^m on $\mathbb{R}_+$ such that $\sum_{s=1}^S \pi_s^m u^m(x_s^{mk}) > \sum_{s=1}^S \pi_s^m u^m(z_s)$ for every $z \in \mathbb{R}_+^S$ with $p^k \cdot z \leq p^k \cdot x^{mk}$ and $z \neq x^{mk}$. We claim that the data set $\mathfrak{D}$ is collectively rationalizable by the pair of functions (u^A, u^B). To see this, it suffices to show $\hat{x}^k$ is Pareto efficient in the budget set $\{z \in \mathbb{R}_+^S : p^k z \leq p^k x^k\}$, $k \in \mathbb{K}$. Suppose otherwise; then for some $k \in \mathbb{K}$, there exist a $z \in \mathbb{R}_+^S$ with $p^k z \leq p^k x^k$ and an FPQ $\hat{z} = (z^A, z^B)$ for z such that $\sum_{s=1}^S \pi_s^m u^m(z_s^m) \geq \sum_{s=1}^S \pi_s^m u^m(x_s^{mk})$ for $m = A, B$, with at least one inequality being strictly. This implies $p^k z^m \geq p^k x^{mk}$ for $m = A, B$, with at least one inequality being strictly. Therefore, $p^k z > p^k x^k$, a contradiction. This completes the proof of the sufficiency part.

A.3 Proof of Theorem 3.1

Let us begin with the necessity part. Suppose that the data set $\mathfrak{D}$ is collectively rationalizable. By Lemma 3.2, there exists an FPQ $(\hat{x}^k) = (x^{Ak}, x^{Bk})$ for each x^k, $k \in \mathbb{K}$, such that both $\{(p^k, x^{Ak})_{k=1}^K; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^K; \pi^B\}$ satisfy SAREU. Define the weak order $\succcurlyeq^m$ on $\mathscr{I}$ such that for all $s, s' \in \mathbb{S}$ and all $i, j \in \mathbb{K}$

$$(s, i) \succcurlyeq^m (s', j) \Leftrightarrow x_s^{mi} \geq x_{s'}^{mj}, m = A, B.$$

Hence $\mathfrak{D}_m$ satisfies SAREU relative to $\succcurlyeq^m$ for $m = A, B$. It is easily seen the vector $(x^{Ak}, x^{Bk})_{k=1}^K$ is a solution to the system (3.6) corresponding to $(\succcurlyeq^A, \succcurlyeq^B)$. This means $(\succcurlyeq^A, \succcurlyeq^B)$ is consistent with $\mathfrak{D}$ and therefore $\mathfrak{D}$ satisfies CAREU.

We now turn to the sufficiency part. Suppose that $\mathfrak{D}$ satisfies CAREU. Then there is a pair $(\succcurlyeq^A, \succcurlyeq^B)$ of weak orders on $\mathscr{I}$ consistent with $\mathfrak{D}$ and $\mathfrak{D}_m$ satisfies SAREU relative to $\succcurlyeq^m$ for $m = A, B$. Then, by Lemma 3.3, there exists an FPQ (x^{Ak}, x^{Bk}) for each x^k such that $(s, i) \succcurlyeq^m (s', j) \Leftrightarrow x_s^{mi} \geq x_{s'}^{mj}$ for $m = A, B$ and $i, j \in \mathbb{K}$. It follows that the two sets, $\{(s, s') : (s, i) \succ^m (s', j)\}$ and $\{(s, s') : x_s^{mi} > x_{s'}^{mj}\}$, must coincide for all $i, j \in \mathbb{K}$. Consequently, the two data sets, $\{(p^k, x^{Ak})_{k=1}^K; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^K; \pi^B\}$, both satisfy SAREU, and hence, by Lemma 3.2, $\mathfrak{D}$ is collectively rationalizable.

A.4 Proof of Proposition 5.1

Let $\mathcal{P}^K$ be an increasing sequence of finite sets of observed prices with $\cup_K \mathcal{P}^K$ dense in $\mathbb{P}$ and let $\mathfrak{D}^K = \{(p, h(p)) : p \in \mathcal{P}^K\}$. Since h satisfies CAREU, $\mathfrak{D}^K$ is collectively rationalizable by a pair of strictly concave, strictly increasing, and continuous functions (u^{AK}, u^{BK}). We can normalize u^{mK} such that $u^{mK}(0) = 0$ and

$$\min_{\varpi \in \partial u^{mK}(0)} \varpi \le 1. \tag{A.3}$$

Let $X = \max\{h_s(p) : s = 1, \dots, S, p \in \mathbb{P}\}$, where $h_s(p)$ denotes the sth component of $h(p)$. It suffices to consider u^{mK} on $[0, X]$. By our normalization of u^{mK}, we have

$$u^{mK}(x) \le X \text{ and } u^{mK}(x) - u^{mK}(y) \le |x - y|, \forall x, y \in [0, X].$$

This means the sequence $\{u^{mK}\}$ is uniformly bounded and equicontinuous, and hence, by the Arzela-Ascoli Theorem, has a convergent subsequence, which without loss of generality we assume is the sequence itself. Suppose $\{u^{mK}\} \to u^m$, $m = A, B$. In the following we show h is collectively rationalizable by (u^A, u^B).

Take any $p \in \mathbb{P}$. Since $\cup_K \mathcal{P}^K$ is dense in $\mathbb{P}$, there exists a sequence $\{p^k\} \subset \cup_K \mathcal{P}^K$ with $p^k \to p$. Let $\gamma(k)$ be the smallest integer such that $p^k \in \mathcal{P}^{\gamma(k)}$. For notational convenience let $x^k = h(p^k)$, $u^{mk} = u^{m\gamma(k)}$, and $U^{mk}(x) = \sum_s \pi_s^m u^{mk}(x_s)$, $m = A, B$. Since $\mathfrak{D}^{\gamma(k)}$ is collectively rationalizable by (u^{Ak}, u^{Bk}), there exist $(\alpha^{Ak}, \alpha^{Bk}) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $\alpha^{Ak} + \alpha^{Bk} = 1$ and an FPQ (x^{Ak}, x^{Bk}) for x^k such that (x^{Ak}, x^{Bk}) solves the problem

$$\begin{cases} \max\limits_{(z^A, z^B) \in \mathbb{R}_+^S \times \mathbb{R}_+^S} & \alpha^{Ak} U^{Ak}(z^A) + \alpha^{Bk} U^{Bk}(z^B) \\ s.t. & p^k(z^A + z^B) \le p^k x^k. \end{cases} \tag{A.4}$$

Since the sequences $\{x^{mk}\}$ and $\{\alpha^{mk}\}$ are bounded, they have convergent subsequences, which without loss of generality we assume are the sequences themselves. Suppose $x^{mk} \to x^m$ and $\alpha^{mk} \to \alpha^m$. Because h is continuous and $x^{Ak} + x^{Bk} = x^k$, we have $x^A + x^B = h(p)$, and therefore (x^A, x^B) is an FPQ for $h(p)$. To prove h is collectively rationalizable by (u^A, u^B), it suffices to show (x^A, x^B) is a solution to the problem

$$\begin{cases} \max\limits_{(z^A, z^B) \in \mathbb{R}_+^S \times \mathbb{R}_+^S} & \alpha^A U^A(z^A) + \alpha^B U^B(z^B) \\ s.t. & p(z^A + z^B) \le p h(p), \end{cases} \tag{A.5}$$

where $U^m(x) = \sum_s \pi_s^m u^m(x_s)$, $m = A, B$. Take $q = (\alpha^A, \alpha^B, p)$ as parameters in (A.5).

Let $V : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^S_+ \to \mathbb{R}$ be the value function of Program (A.5) and

$$(\hat{x}^A, \hat{x}^B) : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^S_+ \rightrightarrows \mathbb{R}^S_+ \times \mathbb{R}^S_+$$

the "argmax"correspondence, i.e. the correspondence such that each element in $(\hat{x}^A(q), \hat{x}^B(q))$ is a solution to (A.5). It follows from the Berge Maximum Theorem (Aliprantis and Border, 2006, Theorem 17.31, p. 570) that V is continuous and $(\hat{x}^A, \hat{x}^B)$ is upper hemicontinuous.

Let $q^k = (\alpha^{Ak}, \alpha^{Bk}, p^k)$ and take $(\bar{x}^A(q^k), \bar{x}^B(q^k)) \in (\hat{x}^A(q^k), \hat{x}^B(q^k))$, so that $(\bar{x}^A(q^k), \bar{x}^B(q^k))$ has a subsequence which converges to $(\bar{x}^A(q), \bar{x}^B(q)) \in (\hat{x}^A(q), \hat{x}^B(q))$ (Hildenbrand, 1974, Theorem 1, p. 24). Since $(\bar{x}^A(q^k), \bar{x}^B(q^k))$ satisfies the constraint of (A.4), we have

$$\alpha^{Ak} U^{Ak}(x^{Ak}) + \alpha^{Bk} U^{Bk}(x^{Bk}) \geq \alpha^{Ak} U^{Ak}(\bar{x}^A(q^k)) + \alpha^{Bk} U^{Bk}(\bar{x}^A(q^k)). \tag{A.6}$$

Note that

$$\begin{aligned} |U^{mk}(x^{mk}) - U^m(x^m)| &\leq |U^{mk}(x^{mk}) - U^{mk}(x^m)| + |U^{mk}(x^m) - U^m(x^m)| \\ &\leq |x^{mk} - x^m| + |U^{mk}(x^m) - U^m(x^m)| \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

where the second inequality is a consequence of (A.3). Letting $k \to \infty$ in (A.6), we get $\alpha^A U^A(x^A) + \alpha^B U^B(x^B) \geq \alpha^A U^A(\bar{x}^A(q)) + \alpha^B U^B(\bar{x}^A(q))$. On the other hand, as an FPQ for $h(p)$, (x^A, x^B) satisfies the constraint of (A.5). This proves (x^A, x^B) is a solution to (A.5).

A.5 Proof of Proposition 6.1

Note that perturbed collective (resp. unitary) rationalizability just means another data set with perturbed prices is collectively (resp. unitarily) rationalizable. As an immediate consequence of Lemma 3.2, we have the following result.

Lemma A.1. The data set $\mathfrak{D}$ is e-perturbed collectively rationalizable if and only if there exists an FPQ $\hat{x}^k = (x^{Ak}, x^{Bk})$ for x^k, $k \in \mathbb{K}$, such that both $\{(p^k, x^{Ak})_{k=1}^K; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^K; \pi^B\}$ are e-perturbed unitarily rationalizable.

Furthermore, it follows directly from Definition 6.1 that if a data set is e-perturbed collectively (resp. unitarily) rationalizable, it is also e'-perturbed collectively (resp. unitarily) rationalizable for any $e' > e$. For notational convenience, let $e_0 = \min_{(\succcurlyeq^A, \succcurlyeq^B) \in \Psi} \max_{m \in \{A,B\}} e_*^m(\succcurlyeq^A, \succcurlyeq^B)$ and suppose $e_0 = \max_{m \in \{A,B\}} e_*^m(\succcurlyeq^{*A}, \succcurlyeq^{*B})$ for some

$(\succcurlyeq^{*A}, \succcurlyeq^{*B}) \in \Psi$. For any $(x^{Ak}, x^{Bk})_{k=1}^{K} \in \mathcal{F}(\succcurlyeq^{*A}, \succcurlyeq^{*B})$, we then have $\{(p^k, x^{Ak})_{k=1}^{K}; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^{K}; \pi^B\}$ are e_0-perturbed unitarily rationalizable. Hence $\mathfrak{D}$ is e_0-perturbed collectively rationalizable and, therefore, $e_* \leq e_0$.

On the other hand, for any $e < e_0$, it follows from the definition of e_0 that at least one of the data sets $\{(p^k, x^{Ak})_{k=1}^{K}; \pi^A\}$ and $\{(p^k, x^{Bk})_{k=1}^{K}; \pi^B\}$ is not e-perturbed unitarily rationalizable, and hence by Lemma A.1, $\mathfrak{D}$ is not e-perturbed collectively rationalizable. This implies $e_* > e$ and thus $e_* = e_0$.

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