Publicly Persuading Voters: The Single-Crossing Property and the Optimality of Interval Revelation*

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This version: April 15, 2024. Click here for the latest version.

Abstract

We study public Bayesian persuasion in binary-outcome elections using a model that simultaneously allows for heterogeneous voters, a rich set of sender preferences, any super-majority voting rule, and an arbitrary number of senders. We analyze and derive useful properties for a statistical inference problem based on the pivotal voter's choice that any sender who cares about voters' welfare faces. Using these results, we establish a widely held condition under which the public information provided in equilibrium has a simple interval structure that reveals intermediate states but pools extreme states. This condition features a single-crossing property over a sender's and the pivotal voter's indifference curves. We also identify a simple way to pin down the equilibrium outcome under competition from the respective monopoly outcomes. Our results yield insightful implications for welfare-optimal information policies. For example, we show that full disclosure is generically suboptimal for welfare, and derive a novel comparative statics result for how the welfare-optimal information policy varies with the voting rule. Finally, we apply our model to study the welfare effects of media competition and how to optimally provide public information to guide decision making by a committee. More generally, we provide examples where our results hold for small as well as large voting bodies.

JEL Codes: D72, D82, D83

Keywords: single-crossing property, interval revelation, inference from pivotal choice, Bayesian persuasion, information design

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*This paper was previous circulated under title "*Public Persuasion in Elections: Single-Crossing Property and the Optimality of Censorship.*" We are particularly grateful for the insightful comments and suggestions by the editor and three anonymous referees, which have helped us to substantially improve the paper. We also thank (in alphabetical order of surnames) Zeinab Aboutalebi, Ricardo Alonso, Giacomo Calzolari, Piotr Dworczak, Federico Innocenti, Navin Kartik, Nenad Kos, David Levine, Elliot Lipnowski, Laurent Mathevet, Niko Matouschek, Salvatore Modica, Daniel Monte, Mallesh Pai, Alessandro Pavan, David Pearce, Eduardo Perez-Richet, Marzena Rostek, Alex Smolin, Erik Snowberg, Jakub Steiner, Shoshana Vasserman, Asher Wolinsky, Takuro Yamashita, Andriy Zapechelnyuk, Jidong Zhou, and Zhen Zhou for helpful comments and discussions. We also thank audiences at numerous conferences, workshops, and seminars for their useful suggestions and feedback. Junze Sun gratefully acknowledges the grant MIUR PRIN prj-0317_d1, which provided financial support for his research while he was affiliated at the Department of Mathematics and Computer Science at the University of Palermo as a research fellow. All errors are our own.

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1 Introduction

Many important collective decisions are made through voting. Examples range from large-scale elections and referenda to small-scale (often non-political) collective decisions via committee voting in organizations (examples of the latter include an NSF review panel deciding whether to fund a research proposal, or a faculty hiring committee deciding whether to offer a job to a candidate). In all these settings, those with a vote often rely on public information provided by external individuals or organizations (e.g., politicians, mass media outlets, information intermediaries, etc) to evaluate the merits of the candidate options before they vote. These actors, however, may have different preferences with respect to the outcome than those who vote and may try to manipulate information so as to steer the vote in their favor.

This paper studies the strategic provision of public information in elections with binary outcomes, such as whether or not to pass a policy reform or grant funding to a research proposal. We model the environment of interest as a public Bayesian persuasion problem (Kamenica and Gentzkow, 2011) in which one or multiple senders can influence the voting outcome by providing public information about a uni-dimensional payoff-relevant state. In general terms, the research question we address is: *given the senders' objectives, what public information will be provided to voters in equilibrium*? Answering this question is important from both a positive and a normative perspective. From the positive point of view, it uncovers how sender(s)' strategic motives shape voters' information about alternatives and subsequently the voting outcome. From the normative point of view, our analysis allows us to shed light on the optimal way to provide public information to maximize voters' welfare.

To answer these questions in a unified framework, we construct a model with three important features. First of all, we allow for a rich set of sender preferences that range from maximizing self-interest to maximizing a (rank-dependent) weighted average of voters' privately known payoffs. This includes the utilitarian preference as a special case. Second, we allow for any super-majority voting rule, allowing us to study how the equilibrium information provision depends on the voting rule. Third, we allow for an arbitrary number of senders, and thus unify the analyses of monopolistic persuasion by a single sender and competition in persuasion by multiple senders. Our unification of these three features in a single framework enables us to make two substantive contributions that we believe are of independent theoretical interest. One is that we identify and analyze a novel statistical inference problem based on the pivotal voter's choice, which is vital to deriving the optimal information strategy for any sender who cares about voters' welfare. The other is that we provide a simple algorithm to derive the equilibrium outcome under competition in persuasion based solely on each sender's optimal strategy under monopolistic persuasion. Both contributions are explained in more detail below.

To illustrate our model (formally laid out in Section 3), consider an election where voters collectively decide between passing a reform and maintaining the status quo. An ex-ante unknown uni-dimensional state k determines the quality of the reform, which is commonly valued by all. Each voter has her own 'threshold of acceptance' v_i, which we refer to as her *type*, such that she prefers the reform if and only if $k \geq v_i$.[1] We assume that types are voters' private information and they are independently drawn from a commonly known distribution. Voters with higher types receive lower payoffs if the reform is passed and hence need more to be convinced to support the reform.

There is a finite set of senders, who can provide voters with public information about state k. Like voters, each sender prefers the reform to the status quo if and only if k is above some (sender-specific) threshold. A *self-interested* sender has a threshold that is independent of voters' types. A *prosocial* sender instead cares about voters' welfare; his threshold depends on voters' types. To account for all these in a single framework, we allow each sender's utility function to be any convex combination of his own self-interest and some rank-dependent weighted average of all voters' payoffs.[2] Like in Gentzkow and Kamenica (2017*b*), each sender simultaneously chooses an *information policy*, which maps any state k to a (possibly random) public signal. After observing their private types and the public signals sent by all senders, voters simultaneously decide to vote for either the reform or the status quo. The reform is passed if and only if the vote share it receives exceeds a cutoff. For instance, under simple majority rule this cutoff is 50%.

One class of information policies that will prove to be important are the so-called *interval revelation policies* as illustrated in Figure 1. With this information policy, a sender precisely communicates the realized state k if it lies in the *revelation interval* $[a,b]$, but only announces "$k < a$" if the realization is below a and "$k > b$" if the realization is above b. That is, states $k > b$ and $k < a$ are pooled under distinct messages. Under such a policy, voters' posterior expected state equals the realized state k whenever it lies within the revelation interval $[a,b]$, while it equals $\mathbb{E}[k|k > b]$ for $k > b$ and $\mathbb{E}[k|k < a]$ for $k < a$.

Figure 1: Interval Revelation Policy

Our first main result is the following. We establish a widely held sufficient condition under which it is optimal for a sender to focus on interval revelation policies of the kind described in

[1] Throughout this paper, we will refer to voters as feminine and senders as masculine.

[2] More precisely, we allow the weights assigned to voters to depend on their rankings in ex-post payoffs. Aside from allowing for utilitarian preferences, this also captures many relevant non-utilitarian social preferences; e.g., a sender may put more weight on voters with lower ex-post payoffs if he is concerned about the distributional impacts of the reform on low-income voters.

Figure 1, provided that the number of voters exceeds a certain threshold. This is true under both monopolistic and competitive persuasion. Importantly, in arguably realistic and economically relevant cases the required threshold for the number of voters is zero so that our results apply to both large-scale elections and small-scale committee voting.

Our sufficient condition can be interpreted as a *single-crossing property* over the sender's and the pivotal voter's *indifference curves*, which are derived as follows (see Section 4 for formal details). Under any cutoff voting rule the election outcome is essentially determined by a *pivotal voter*, whose realized type is denoted by x. For example, under simple majority rule the median voter is pivotal; her type x is the sample median of voters' realized types. The pivotal voter prefers the reform to the status quo if and only if $k \geq x$. Her indifference curve is therefore the 45-degree line on a two-dimensional plane with on the horizontal axis her realized type x, and on the vertical axis the realized state k. Now we draw a sender's indifference curve on the same plane. This is straightforward for a self-interested sender; his indifference curve is simply a horizontal line, because his threshold to accept the reform is independent of the pivotal voter's type x. Deriving the indifference curve for a prosocial sender is more involved and it calls for solving the statistical inference problem referred to above. The difficulty here is that a prosocial sender's preference over the election outcome depends on voters' private types, which are unobservable to him. The sender must therefore *infer* his threshold to accept the reform by exploiting the *statistical correlations* between the pivotal voter's type and all other voters' types. For example, suppose that the sender is a utilitarian planner and the election outcome is determined by simple majority rule. Then, given the realized type profile of voters, the sender's threshold to accept the reform is given by the average type (denoted by $\bar{v}$) while the pivotal voter's threshold of acceptance is the median type (denoted by v^m). Therefore, conditional on the pivotal voter's type being x, the sender rationally infers his threshold of acceptance to be $\mathbb{E}[\bar{v}|v^m = x]$. Solving this inference problem for all possible pivotal types x gives the utilitarian sender's indifference curve. A similar inference problem applies to other social preferences and voting rules. The wedge between the indifference curves of the pivotal voter and a sender determines their conflict of interests, which is key to shaping the sender's optimal information policy.

With these indifference curves we can define our single-crossing property. Informally speaking, the single-crossing property holds for a sender if his indifference curve crosses the pivotal voter's indifference curve at most once, and if so only from above. We show that the single-crossing property is widely held. It is always satisfied if the sender is self-interested, and satisfied for generic sender preferences and voting rules under a mild assumption on the distribution of voters' types.

In Section 5 we study monopolistic persuasion by a single sender for whom the single-crossing property holds. In this case we show that some interval revelation policy as in Figure 1 is uniquely optimal in sufficiently large elections (Theorem 1). The boundaries of the optimal revelation interval

$[a,b]$ are determined by the tradeoff between the capability of manipulating voters' beliefs in more states on the one hand (providing incentives to pool more states), and the effectiveness of persuasion on the other hand (reducing incentives to pool extra states). We also provide easy-to-check sufficient conditions (in Proposition 2) for Theorem 1 to hold independently of the electorate size, making our results also relevant for small committees. In Section 6 we further characterize and discuss properties of the sender's optimal interval revelation policy and payoff as the electorate size goes to infinity (Theorem 2 and Proposition 3). We also derive comparative statics regarding how the boundaries of the optimal revelation interval vary with the sender's preference and the voting rule (Propositions 4 to 6).

Our results have important normative implications. First, we show that full information disclosure is generally suboptimal even for a social planner who wants to maximize voters' welfare. This is because the pivotal voter decides the election outcome solely based on her own preference, while the planner cares about the (weighted) average payoffs of all voters. Thus, even under full information it is still possible that the preferences of the pivotal voter and the planner disagree, and this creates room for improving welfare by strategically withholding some information. We note that solving for the welfare-optimal information policy is not trivial, because it requires a careful analysis of the preference misalignment between the planner and the pivotal voter. This can be formalized by the aforementioned statistical inference problem based on the pivotal voter's choice. A key technical contribution of our paper is to analyze this inference problem and derive useful properties for it (cf. Proposition 1). Using these properties, we characterize the welfare-optimal information policy for a rich set of social welfare functions in elections with binary outcomes and under any super-majority rule. Second, we establish a novel *sender-preference effect*: ceteris paribus, raising the required vote share to pass the reform makes any prosocial sender more leaning towards the reform. This effect also stems from the inference problem explained above, and it is absent in models where the sender can perfectly observe voters' preferences (as in Alonso and Câmara 2016*a*). The sender-preference effect yields new insights for how the welfare-optimal information policy varies with the voting rule.

In Section 7 we study competitive persuasion where multiple senders simultaneously choose their public information policies. This yields our second substantive contribution by identifying a simple way to pin down the equilibrium outcome under competition from the respective monopoly outcomes. More specifically, we show that if the single-crossing property holds for a sender, then in sufficiently large elections it is *without loss of optimality* for this sender to focus on a subset of interval revelation policies: for any pure strategy profile chosen by other senders, this sender can always find a best response from this subset (Theorem 3). This suggests that interval revelation policies remain optimal for a sender even when voters can get additional public information from other sources that are beyond his control. Moreover, if the single-crossing property holds for all senders, then under a weak regularity condition the information jointly provided by all senders in the minimally informative equilibrium can be replicated by an interval revelation policy, whose revelation interval is simply the convex hull of the optimal revelation intervals for each of the senders under monopolistic persuasion (Theorem 4). In other words, the arguably most plausible (i.e. minimally informative) equilibrium outcome with competing senders coincides with the outcome that would result if all senders use their monopolistic optimal information policies and reveals all states between their individual revelation intervals (if any). Using this theorem, we find that under general conditions senders' competition in persuasion indeed improves information revelation, and derive an intuitive sufficient condition for such competition to ensure full disclosure in all equilibria.

Finally, in Section 8 we present three applications of our results. The first application concerns the electoral impact of media competition. Conventional wisdom has it that more information is better, and media competition is thus best for welfare if it induces full revelation. As explained above, however, this seemingly appealing wisdom is incorrect because full revelation is typically suboptimal for welfare. In fact, we show that media competition can harm welfare by inducing too much information revelation, rather than too little. In large elections, the harm of such excess revelation is negligible only when there is no ex-ante conflict of interests between the average and pivotal voters, and it can be substantial otherwise. This yields an important lesson for institution design: to ensure that elections produce a welfare-optimal outcome, simply regulating the media market to induce more information revelation is not enough; the voting rule must also be carefully designed to align the interests of the average and pivotal voters.

Our second and third applications study how to optimally provide public information to guide decision making by a small committee. More specifically, we consider a science foundation that has received a large number of research proposals that are heterogeneous in quality. Each proposal is independently evaluated by a scientific committee, which uses some super-majority rule to decide whether it should be eligible for funding. While all committee members see higher quality as a merit, their thresholds of approval may differ. An intermediary can provide public information about the qualities of all proposals. We show that under certain conditions our main insights carry over: the socially optimal information policy is in general not full disclosure, but instead has an interval revelation structure. Moreover, despite the small electorate size, the boundaries of the optimal revelation interval satisfy many properties (including the comparative statics) that we derive for large elections. Importantly, the optimal information policy can be credibly enforced in practice even when the intermediary may have a substantial conflict of interests with the committee and can manipulate signals ex-post. Finally, we show that our approach can also be used to characterize the welfare-optimal information policy in a Condorcet jury environment with boundedly rational voters.

2 Related literature

This paper speaks to several strands of the literature. First of all, our paper relates to the literature on Bayesian persuasion and information design in elections.[3] Aside from some notable exceptions discussed below, this literature predominantly focuses on persuasion by a monopoly sender whose goal is to sway the election outcome in favor of his preferred alternative (Wang, 2013; Alonso and Câmara, 2016*a*,*b*; Bardhi and Guo, 2018; Chan et al., 2019; Ginzburg, 2019; Kerman, Herings and Karos, 2020; Heese and Lauermann, 2021; Sun, Schram and Sloof, 2021; Gitmez and Molavi, 2022; Gradwohl, Heller and Hillman, 2022).[4] We contribute to this literature by studying an environment that simultaneously allows for heterogeneous voters, a rich set of sender preferences, any super-majority voting rule, and an arbitrary number of senders.

Two studies closely related to ours are Alonso and Câmara (2016*b*) and Kolotilin, Mylovanov and Zapechelnyuk (2022). Their models can be interpreted as a monopoly sender persuading a privately informed representative voter. In Alonso and Câmara (2016*b*), the sender is an incumbent party leader who aims at maximizing the re-election probability. Kolotilin, Mylovanov and Zapechelnyuk (2022) allow for a wider set of sender preferences, ranging from maximizing the winning probability to maximizing the payoff of the representative voter. Both papers show that when the distribution of the representative voter's type has a strictly log-concave density, the sender's optimal information policy takes the form of *upper censorship*, which is a specific interval revelation policy that reveals states below a cutoff and pools states above it. Our paper generalizes their results to an environment that allows for multiple senders and voters. Looking at a setup with multiple voters instead of a single representative voter enables us to account for a broader class of relevant social preferences and to study the impact of voting rules on the optimal information policy. We show that the optimality of interval revelation can be ensured under more general conditions when there are multiple voters, and it extends also to the case with multiple senders.

Finding the welfare-optimal information policies that maximize some weighted average of voters' payoffs is an important normative question. Yet aside from our work, Van der Straeten and Yamashita (2023) and Ferguson (2020) are the only papers we know that consider this question. Both papers study persuasion problems for a utilitarian planner in models different from ours, and show that full disclosure can be suboptimal in some circumstances. We contribute by analyzing and

[3] Of course, strategic information transmission in elections has been extensively studied under various other communication protocols, such as cheap talk (Schnakenberg, 2015, 2017; Kartik and Van Weelden, 2019) and verifiable disclosure (Liu, 2019). One important feature that separates our paper from these is that we can also address the normative question regarding the optimal information policy for a social planner.

[4] Some of these papers (e.g., Heese and Lauermann (2021)) allow the sender's preferred alternative to be state-dependent. They do not, however, allow for the sender's utility to depend on voters' private types. Moreover, all these papers except Alonso and Câmara (2016*a*,*b*) and Ginzburg (2019) study targeted persuasion in which the sender can privately communicate to voters (Bergemann and Morris, 2016; Taneva, 2019; Mathevet, Perego and Taneva, 2020). Our paper instead focuses on public persuasion whereby a sender must reveal the same information to all voters.

deriving useful properties for an inference problem based on the pivotal voter's choice, which is key to understanding the preference misalignment between a prosocial sender and the pivotal voter. Building on these results, we characterize the welfare-optimal information policies for a large class of social welfare functions in elections with binary outcomes and super-majority rules. Moreover, the inference problem also gives rise to a novel sender-preference effect on how a social planner's incentive to persuade voters depends on the voting rule. This effect yields new comparative static results on how voting rules may affect the amount of information revelation.

Alonso and Câmara (2016*a*) study public persuasion in elections by a single sender and also consider how the voting rule affects the optimal information policy. A crucial difference between our paper and theirs is that we allow voters to have private types, while in their model the sender knows all voters' types. This implies that the inference problem based on the pivotal voter's choice is absent in the framework they study. For this reason, the structure of the optimal public information policies in our paper is qualitatively different from theirs, and our sender-preference effect is also absent in theirs.

Second, our results also relate to papers studying competition in Bayesian persuasion with multiple senders (Gentzkow and Kamenica, 2017*a*,*b*; Cui and Ravindran, 2020; Au and Kawai, 2020, 2021; Li and Norman, 2021; Mylovanov and Zapechelnyuk, 2021; Innocenti, 2021; Sun, 2022*a*). Most papers in this field study conditions under which senders' competition can improve information revelation or induce full disclosure. In general, however, there is no clear relationship between the structures of equilibrium outcomes under competition and each sender's optimal information policy under monopoly persuasion. We contribute to this literature by showing that, in the context of publicly persuading voters, there is a simple way to derive the minimally informative equilibrium outcome under competition solely from each sender's monopoly optimal information policy. Using this result, we show that sender competition indeed improves information revelation and obtain an intuitive sufficient condition for full disclosure to hold in all equilibria.

Third, methodologically, our paper builds on a recent literature that exploits the duality approach to study linear persuasion problems, where senders' utilities depend only on the posterior expected state (Kolotilin, 2018; Dworczak and Martini, 2019; Dworczak and Kolotilin, 2019; Dizdar and Kováč, 2020; Kolotilin, Mylovanov and Zapechelnyuk, 2022; Sun, 2022*a*,*b*).[5] In particular, Dworczak and Martini (2019) show that finding an equilibrium outcome under competition can be converted to solving the monopolistic persuasion problems of each sender with modified utility functions. This allows us to treat monopolistic and competitive persuasion in a unified framework. Kolotilin, Mylovanov and Zapechelnyuk (2022) show that upper (resp. lower) censorship policies

[5] Several papers study linear persuasion problems using other methods. For instance, Gentzkow and Kamenica (2016) and Kolotilin et al. (2017) characterize the sets of implementable outcomes under public and private signals, respectively, using the Blackwell theorem. More recently, Kleiner, Moldovanu and Strack (2021) and Arieli et al. (2022) apply theories of extreme points and majorization to study general structures of solutions to linear persuasion problems.

are optimal if the sender's utility function is S-shaped (resp. inverse S-shaped). Sun (2022*b*) extends this observation to competitive persuasion in an environment a la Gentzkow and Kamenica (2017*b*). Sun (2022*a*) derives sufficient and necessary conditions for senders' competition to induce full disclosure in all equilibria in linear persuasion games. We fruitfully exploit these earlier findings to derive our main results. Along the way, we also contribute to this literature by identifying a novel *increasing-slope property* for a sender's indirect utility function and show that it ensures that any solution to his monopolistic persuasion problem must precisely communicate whether the realized state is above or below a cutoff (cf. Lemma 4 in Section 5.2).

Finally, our applications in Section 8 produce insights that contribute to the literature on the welfare implications of media competition in large elections and of information intermediaries. The related literature for these applications is discussed in Section 8.

3 Framework

We consider an election[6] in which $n+1$ voters collectively decide between two options, which for ease of reference we label *Reform* and *Status quo*. The outcome is determined by a cutoff rule with threshold $q \in [0,1]$; the reform is adopted if and only if it obtains strictly more than nq votes. For instance, $q = 0.5$ corresponds to simple majority rule. For ease of exposition, we assume that nq is an integer (unless explicitly mentioned otherwise).

An ex-ante unknown state k is drawn from a common prior F that admits a positive and continuous density f on $[-1,1]$. Without loss of generality, we normalize all players' payoffs to zero under the status quo. If the reform is adopted, each voter i's payoff equals $k - v_i$, where v_i is her private type. Because voter i prefers the reform if and only if $k \geq v_i$, her private type represents her threshold of acceptance of the reform. We assume that each v_i is independently drawn from a commonly known distribution G, which admits a positive and twice continuously differentiable density g on $[\underline{v}, \overline{v}]$ with $\underline{v} < -1$ and $\overline{v} > 1$.[7] For any profile of type realizations $v = (v_1, \cdots, v_{n+1})$, we let $v^{(1)} \leq v^{(2)} \leq \cdots \leq v^{(n+1)}$ be its ascending permutation. Since $k - v_i$ decreases in v_i, voters with lower type realizations receive higher ex-post payoffs if the reform is adopted.

Consider first a monopoly sender; in Section 7 we extend our model to allow for multiple senders competing in persuading voters. The sender's payoff under the reform is given by

$$u(k,v) = \rho \sum_{j=1}^{n+1} w_j \cdot \left(k - v^{(j)}\right) + (1-\rho)(k-\chi) \tag{1}$$

[6] For semantic reasons, we use the term 'election' when presenting our model. The reader should bear in mind that voting may take place in anything from a small committee to a national election.

[7] All our analyses and results remain unchanged if either or both of $\underline{v}$ and $\overline{v}$ become unbounded, so long as both $g(\cdot)$ and $g'(\cdot)$ are uniformly continuous on their domains.

where $\rho \in [0,1]$, $\chi \in \mathbb{R}$ and $(w_1, \cdots, w_{n+1})$ is a non-negative vector of weights that sum up to 1. Parameter ρ captures the extent to which the sender cares about 'voter welfare' relative to his 'self-interest'. If $\rho = 0$ then the sender prefers reform to be adopted if and only if $k \geq \chi$. In this case, the sender is *self-interested* in the sense that his preference over alternatives is independent of voters' interests.[8] Conversely, if $\rho = 1$ then $u(k,v)$ is a weighted average of voters' realized payoffs when the reform is adopted. For each $j = 1, \cdots, n+1$, w_j is the *rank-dependent welfare weight* the sender assigns to the voter whose payoff under reform is ranked the j-th highest under the realized type profile v. The vector $(w_1, \cdots, w_{n+1})$ is generated by a *weighting function* $w(\cdot)$ that is non-decreasing, absolutely continuous on $[0,1]$ and satisfies $w(0) = 0$ and $w(1) = 1$. Hence, $w(\cdot)$ is the cumulative distribution function (cdf) of a random variable on $[0,1]$.[9] For any integer $n \geq 0$ and $j \in \{1, \cdots, n+1\}$, element w_j is uniquely generated by

$$w_j = w\left(\frac{j}{n+1}\right) - w\left(\frac{j-1}{n+1}\right) . \qquad (2)$$

This setup captures a wide class of social welfare functions in a unified way. For instance, the utilitarian welfare function can be obtained by letting $\rho = 1$ and $w(y) = y$ for all $y \in [0,1]$. With this $w(\cdot)$, it follows from (2) that $w_j = \frac{1}{n+1}$ for each j so that the welfare weights are equal across voters. If $w(\cdot)$ is not the cdf of a uniform distribution on $[0,1]$, then it represents the preference of a non-utilitarian social planner who may care about the welfare of some voters more than others depending on the rankings of their ex-post payoffs. We will discuss some examples in Section 5.

The sender can affect voters' information about k by designing an *information policy*. Following the convention of the Bayesian persuasion literature, we define an information policy $\pi : [-1,1] \mapsto \Delta(S)$ as a measurable mapping from each state k to a distribution of signals, where S is a sufficiently large signal space. Let Π denote the set of all feasible information policies.

The timing of the game is as follows. First, prior to observing state k, the sender chooses an information policy $\pi \in \Pi$. Second, state k is realized and a public signal is drawn according to π. Observing the realized public signal, voters simultaneously decide to vote for either the reform or the status quo. The reform is adopted if and only if its vote tally strictly exceeds nq. All players' payoffs then realize. Throughout, we focus on equilibria in weakly undominated strategies.[10]

[8] This captures state-independent motives (which are most extensively explored in the literature) as limiting cases. For instance, the preference of a sender whose aim is to maximize the winning probability of reform (resp. status quo) independent of state realizations can be captured by letting $\chi \to -\infty$ (resp. $\chi \to \infty$).

[9] $w(\cdot)$ is reminiscent of the probability weighting function in the rank-dependent utility theory (Quiggin, 1982). An advantage of this modeling approach is to capture a sender's intrinsic social preference without explicitly relying on the actual number of voters. Moreover, as shown in Section 4, there is a natural connection between the first order stochastic dominance ordering of $w(\cdot)$ and a sender's social preference.

[10] As is well known, the second-stage voting game has a plethora of equilibria in weakly dominated strategies. For example, whenever $n > 0$ it is an equilibrium for all voters to vote for reform regardless of their types or information, because no single vote is pivotal. Any information policy π can then be supported in equilibrium because it has no

3.1 Voting behavior and election outcome

Because voters have a common prior F and information transmission is public, they must share a common posterior about the state realization after hearing from the sender. Since voters' payoffs under reform are linear in state k, their expected payoffs depend only on their posterior expectation θ and are given by $\theta - v_i$ for all i. It is then a weakly dominant strategy for voter i to vote for reform if and only if $\theta \geq v_i$. Therefore, under the cutoff voting rule with threshold q, the election outcome is determined by the choice of the *pivotal* voter, whose type realization is $v^{(nq+1)}$. Note that $v^{(nq+1)}$ is a random variable and let $\hat{G}_n(\cdot;q)$ denote its cumulative distribution function. Since reform is adopted only if $v^{(nq+1)} \leq \theta$, $\hat{G}_n(\theta;q)$ gives the winning probability of reform. In Online Appendix A we derive the formal expression and useful properties of $\hat{G}_n(\theta,q)$.

Lemma 1. *Let $v_q^* := G^{-1}(q)$.*[11] *The following properties hold:*

1. $\hat{G}_n(\cdot;q)$ is strictly increasing and $v^{(nq+1)}$ converges in probability to v_q^*.

2. $\hat{g}_n(\cdot;q)$ is single-peaked for all $q \in (0,1)$ when n is sufficiently large. In addition, if g is log-concave, then $\hat{g}_n(\cdot;q)$ is strictly log-concave for all $n > 0$ and $q \in (0,1)$.

Part (1) of Lemma 1 says that the winning probability of reform strictly increases in θ. Moreover, as $n \to \infty$ the reform is adopted with probability one (zero) if $\theta > (<) v_q^*$. Part (2) is, as far as we know, novel and may be of independent interest. It establishes that regardless of the shape of G and voting rule q, for sufficiently large n the distribution of the pivotal type will always be single-peaked. Moreover, large n is not needed if g is already log-concave. This property plays a key role in establishing the optimality of interval revelation.

4 Indifference curves and the single-crossing property

In this section we present and derive useful properties for the statistical inference problem that drives the sender's optimal information policy. This exercise gives rise to a widely held single-crossing property – defined and formulated in terms of the sender's and the pivotal voter's indifference curves – that will drive our main results in subsequent sections. Omitted derivations and proofs for this section are relegated to Online Appendix B.

influence on voting anyway. The restriction to weakly undominated strategies rules out these uninteresting equilibria.

[11] Here, $G^{-1}(\cdot)$ is the inverse function of G, which is well defined because G is strictly increasing on its support $[\underline{v}, \overline{v}]$.

4.1 Indifference curves and the inference from the pivotal voter's choice

Given any realization of voter type profile v, it follows from equation (1) that the sender weakly prefers the reform if and only if

$$k \geq \varphi_n(v) := \rho \sum_{j=1}^{n+1} w_j \cdot v^{(j)} + (1-\rho)\chi \ .$$

Here $\varphi_n(v)$ is the sender's threshold of acceptance for the reform. Note that this depends on voters' realized type profile v whenever $\rho > 0$. Importantly, however, at the time of choosing his information policy, any sender with $\rho > 0$ cannot precisely observe $\varphi_n(v)$ because the realized types are voters' private information. Nevertheless, the election outcome, which is essentially the choice of the pivotal voter, is informative about the realization of $\varphi_n(v)$. This gives rise to a statistical inference problem based on the pivotal voter's choice that is key to understanding the misalignment of interests between the sender and the pivotal voter.

To clarify this inference problem, it is instructive to define and draw the indifference curves of the pivotal voter and the sender in a single plane, as in Figure 2. In each panel, the horizontal axis x represents the pivotal voter's type realization $v^{(nq+1)}$ and the vertical axis denotes the state k. The pivotal voter's indifference curve is then simply the 45-degree line; she is indifferent between alternatives if and only if $k = x$. The sender's indifference curve is derived as follows. For each $j \in \{1, \cdots, n+1\}$ and $x \in [\underline{v}, \overline{v}]$, let

$$\varphi_j(x;q,n) := \mathbb{E}\left[v^{(j)} \middle| v^{(nq+1)} = x; q, n\right]$$

denote the expectation of $v^{(j)}$ conditional on event $v^{(nq+1)} = x$, given q and n. The expectation of $\varphi_n(v)$ conditional on $v^{(nq+1)} = x$ is then

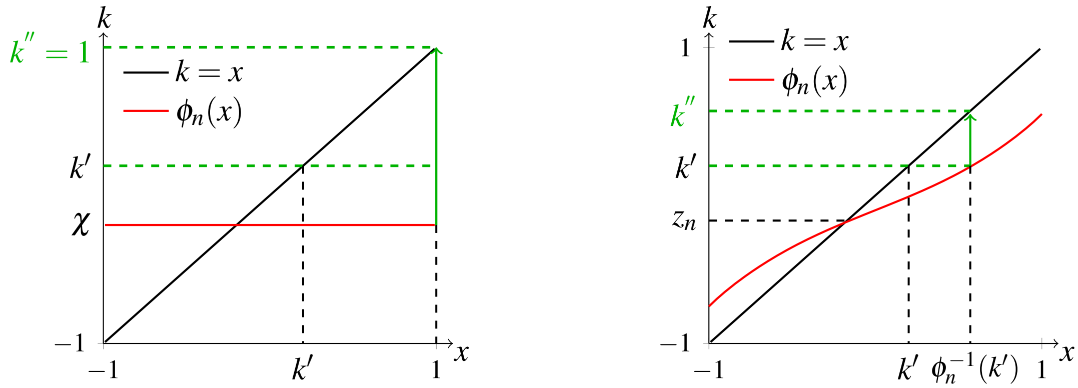
$$\phi_n(x) := \mathbb{E}\left[\varphi_n(v) \middle| v^{(nq+1)} = x\right] = \rho \sum_{j=1}^{n+1} w_j \cdot \varphi_j(x;q,n) + (1-\rho)\chi \ . \tag{3}$$

Therefore, conditional on event $v^{(nq+1)} = x$, the sender is indifferent between alternatives if and only if $k = \phi_n(x)$. For this reason, we refer to $\phi_n(x)$ as the sender's *indifference curve*. Computing $\phi_n(x)$ is essentially a problem of inferring the expectation of a weighted sum of order statistics conditional on the realization of a particular order statistic. This inference problem is at the heart of our analyses. We discuss a number of useful properties of $\phi_n(\cdot)$ in Subsection 4.3 below, but first intuitively explain the misalignment of interests between sender and pivotal voter based on their indifference curves.

Panel (a) of Figure 2 depicts the indifference curve of a self-interested sender with $\rho = 0$. In

Figure 2: Indifference Curves and the Single-Crossing Property

(a) Self-interested sender with $\rho = 0$ (b) Prosocial sender with $\rho > 0$



Note: In both panels the horizontal axis x denotes the pivotal voter's type realization $v^{(nq+1)}$, the black line denotes the pivotal voter's indifference curve, and the red line denotes the sender's indifference curve.

this case it is obvious from (3) that $\phi_n(x) = \chi$ for all x. The preference of a self-interested sender is thus independent of the pivotal voter's type realization.

Panel (b) of Figure 2 depicts the indifference curve of a prosocial sender with $\rho > 0$. It reveals two important and generic features (formally shown in Subsection 4.3). First, unlike the self-interested case, $\phi_n(x)$ is strictly increasing in x for all $n \geq 0$ whenever $\rho > 0$. This is because the pivotal voter's realized type $v^{(nq+1)}$ is positively correlated with all other order statistics $v^{(j)}$ for $j = 1, \cdots, n+1$. Therefore, no matter how the sender assigns his welfare weights, the pivotal voter's type is either *directly relevant* or *indirectly informative* about the sender's threshold for accepting the reform. It is in this way that the inference from the pivotal voter's choice is important for any prosocial sender with $\rho > 0$. Second, $\phi_n(x)$ generically differs from the 45-degree line. This suggests that even a prosocial sender's preference is generically different from that of the pivotal voter. This is because the pivotal voter decides the election outcome only based on his own payoff, while a prosocial sender will typically take other voters' welfare into account. Therefore, conflicts of interests between them are ubiquitous. As we shall see, this renders full information revelation generally suboptimal even for a benevolent social planner who wants to maximize voters' welfare.

Two remarks are in place. First, the inference problem formulated here is conceptually different from the inference about the state conditional on the event of being pivotal, which is central to the literature on information aggregation in voting (e.g., Feddersen and Pesendorfer 1996, 1997). In our model voters have no private information about state k, so the information aggregation issue is absent. In Section 8.3 we further clarify the connection between these two kinds of pivotal inferences via an application to common value voting. Second, for our inference problem to be interesting and relevant, it is necessary that there are multiple voters (i.e., $n > 0$) and their types are private information.[12] Therefore, our inference problem disappears in models where the sender has complete information about voters' preferences, such as in Alonso and Câmara (2016*a*).

4.2 Single-crossing property and its economic implications

To formally define our *single-crossing property* we need the following lemma, which characterizes the limit of $\phi_n(\cdot)$ as $n \to \infty$.

Lemma 2. *For $x \in [\underline{v}, \overline{v}]$, define*

$$\phi(x) := \rho \left[\int_0^q G^{-1}\left(\frac{y}{q}G(x)\right) dw(y) + \int_q^1 G^{-1}\left(\frac{y-q}{1-q} + \frac{1-y}{1-q}G(x)\right) dw(y) \right] + (1-\rho)\chi \ . \quad (4)$$

Then, as $n \to \infty$, $\phi_n(x)$ and $\phi_n'(x)$ uniformly converge to $\phi(x)$ and $\phi'(x)$, respectively, on $[\underline{v}, \overline{v}]$. Moreover, $\varphi_n(v)$ converges almost surely to

$$\phi^* := \phi(v_q^*) = \rho \int_0^1 G^{-1}(y) dw(y) + (1-\rho)\chi \ . \quad (5)$$

For any continuously differentiable function $h(\cdot)$, we say that $h(\cdot)$ is *single-crossing* on $[-1,1]$ if $h(x)$ crosses zero at most once on $[-1,1]$ and $h'(x) > 0$ whenever $h(x) = 0$ and $x \in [-1,1]$.

Definition 1. *We say that the single-crossing property holds for a sender if $x - \phi(x)$ is single-crossing on $[-1,1]$.*

By Lemma 2, $\phi_n(x)$ and $\phi_n'(x)$ converge uniformly to $\phi(x)$ and $\phi'(x)$, respectively, on $[\underline{v}, \overline{v}]$ (which contains $[-1,1]$ by assumption). Therefore, when the single-crossing property holds, there exists a threshold $\tilde{n} \geq 0$ such that for all $n > \tilde{n}$ function $x - \phi_n(x)$ is single-crossing on $[-1,1]$; that is, $\phi_n(x)$ crosses the pivotal voter's indifference curve $k = x$ at most once and if so only from above (cf. Figure 2). For such $\phi_n(\cdot)$ we can define a unique *switching state* z_n as follows:

$$z_n := \begin{cases} -1 & \text{if } x > \phi_n(x) \text{ for all } x \in [-1,1] \\ x & \text{if } x = \phi_n(x) \text{ for some } x \in [-1,1] \\ 1 & \text{if } x < \phi_n(x) \text{ for all } x \in [-1,1] \end{cases} . \quad (6)$$

The definition of z_n implies $k > \phi_n(k)$ for $k > z_n$ and $k < \phi_n(k)$ for $k < z_n$. Therefore, in any state $k > z_n$ the sender is more biased towards the reform than the pivotal voter in the following sense:

[12] With a single voter (i.e., $n = 0$) the inference problem becomes trivial: by (2) and (3), for $n = 0$ we have $\phi_n(x) = \rho x + (1-\rho)\chi$, which is independent of the sender's weighting function $w(\cdot)$ and voting rule q. In this case our setup reduces to a model of persuading a single and privately informed receiver, which is explored by, for instance, Kolotilin et al. (2017), Kolotilin (2018) and Kolotilin, Mylovanov and Zapechelnyuk (2022).

whenever the pivotal voter is indifferent (i.e., in event $k = x$) the sender must strictly prefer the reform to be adopted because $k > \phi_n(k)$. Similarly, in any state $k < z_n$ the sender is more biased towards the status quo than the pivotal voter in that he must strictly prefer the status quo to be maintained whenever the pivotal voter is indifferent.

An important economic implication of the single-crossing property is that the sender is tempted to manipulate voters' beliefs upwards (downwards) for state realizations above (below) the switching state z_n. Figure 2 illustrates this. Consider any state realization k' in $(z_n, 1)$. Under the single-crossing property $k' > \phi_n(k')$ must hold. Let $k'' = \phi_n^{-1}(k')$ if $k' \leq \phi_n(1)$ (right panel) or set $k'' = 1$ otherwise (left panel). As is evident in Figure 2, $k'' > k'$ must hold so that the sender and pivotal voter prefer different alternatives whenever $x \in (k', k'')$, with the sender strictly preferring the reform. The sender is then tempted to lie and let the pivotal voter believe that the realized state is k'', which is higher than the true state k'. It is in this sense that the sender is tempted to manipulate voters' beliefs about the state upwards. Following the same logic, if the state realization k' is below z_n then the sender is tempted to manipulate voters' beliefs about the state downwards.

4.3 Properties of the sender's indifference and sufficient conditions for the single-crossing property

In this subsection we formally present some important properties of the sender's indifference curve $\phi_n(\cdot)$ that allow us to establish two easy-to-check sufficient conditions for the single crossing property to hold. These properties are collected in Proposition 1.

Proposition 1. *The following properties about $\varphi_j(x;q,n)$ for each j and $\phi_n(x)$ hold:*

1. Each $\varphi_j(x;q,n)$ is strictly increasing in x.

2. If G is strictly log-concave[13], then $\varphi_j'(x;q,n) < 1$ for all $j < nq+1$.

3. If $1-G$ is strictly log-concave[14], then $\varphi_j'(x;q,n) < 1$ for all $j > nq+1$.

4. If $\rho > 0$, then for any $x \in (\underline{v}, \overline{v})$, $\phi_n(x)$ strictly decreases in q.

5. If $\rho > 0$, then for any $x \in (\underline{v}, \overline{v})$, $\phi_n(x)$ decreases as the weighting function $w(\cdot)$ decreases in first order stochastic dominance.

Observe from (3) that for $\rho > 0$ the sender's indifference curve $\phi_n(x)$ is an affine function of the expected order statistics $\varphi_j(x;q,n)$, which are conditional on the order statistic belonging to the pivotal voter being equal to x. For $j = nq+1$ we thus have $\varphi_j(x;q,n) = x$ by definition. The first part of Proposition 1 reveals that all these expected order statistics are increasing in x, and therefore so is $\phi_n(x)$ for all $n \geq 0$. How often the sender's indifference curve (potentially) crosses

[13] Strict log-concavity of G is equivalent to a strictly decreasing reversed hazard rate $g(x)/G(x)$.

[14] Strict log-concavity of $1-G$ is equivalent to a strictly increasing hazard rate $g(x)/(1-G(x))$.

the 45-degree line then hinges on the slope of $\phi_n(x)$. Intuitively, if the sender's indifference curve is flatter than the pivotal voter's, i.e. if $\phi_n'(x) < 1$ for all $x \in [-1,1]$, it can cross the 45-degree line at most once. More formally, since $\phi_n'(\cdot)$ converges uniformly to $\phi'(\cdot)$, if $\phi_n'(\cdot) < 1$ for all $n > \tilde{n}$ for some $\tilde{n} \geq 0$, our single crossing property holds. The second and third parts of Proposition 1 suggest sufficient conditions for this. In particular, when both G and $1-G$ are strictly log-concave, we immediately have from these two parts together with (3) that $\phi_n'(x) \leq 1$, with a strict inequality for all $n > 0$ if either (i) $\rho < 1$ or (ii) $\rho = 1$ and $w_{nq+1} < 1$.[15] Moreover, for generic G, q and $w(\cdot)$, it is evident from (3) that $\phi_n'(x) \to 0$ uniformly on $[-1,1]$ as $\rho \to 0$ from above. Lemma 3 summarizes these observations.

Lemma 3. *The single-crossing property holds if either (i) ρ is sufficiently close to* 0*, or (ii) both G and $1-G$ are strictly log-concave.*

Conditions (i) simply says that the sender is sufficiently self-interested. Condition (ii) is very mild; it holds for any log-concave density function g that satisfies a weak regularity condition.[16] A wide class of distributions that are frequently applied in economic theories indeed share log-concave densities (see Bagnoli and Bergstrom (2005) for examples).[17] Once condition (ii) is satisfied, the single-crossing property holds for all sender preferences represented by (1) and all super-majority voting rules. We note that condition (ii) is almost tight; suppose that either G or $1-G$ is strictly log-convex on some sub-interval within $[-1,1]$, then it is possible to construct a sender preference and voting rule under which the single-crossing property fails to hold.

The final two parts of Proposition 1 concern how a prosocial sender's indifference curve $\phi_n(\cdot)$ is affected by the voting rule q and his welfare weighting function $w(\cdot)$. These reveal that $\phi_n(\cdot)$ systematically shifts downwards – i.e., the sender becomes more biased towards passing the reform – as either q increases or the weighting function $w(\cdot)$ decreases in the sense of first order stochastic dominance. The comparative statics in $w(\cdot)$ are intuitively clear. If a pro-social sender starts to care relatively more about voters that like the reform better, he will do so as well. The comparative statics in q, however, are a priori less straightforward. These stem from the statistical inference problem that a pro-social sender faces. To illustrate, let q increase from q' to q''. The pivotal voter's type then shifts from $v^{(nq'+1)}$ to $v^{(nq''+1)}$. Under q'' the pivotal event $v^{(nq''+1)} = x$ necessarily implies $v^{(nq'+1)} \leq x$ (i.e., the pivotal voter's type must be lower than x under q'). Therefore, for any fixed

[15] With absolutely continuous $w(\cdot)$, it follows from (2) that $w_{nq+1} < 1$ can be ensured for all $n \geq \tilde{n}$ for some threshold $\tilde{n} \geq 0$. In particular, $\tilde{n} = 0$ if $w(\cdot)$ is strictly increasing on $[0,1]$.

[16] The regularity condition is: $\exists t < \bar{v}$ such that $g'(x)/g(x)$ is strictly decreasing on $[t, \bar{v}]$. This condition can be violated, for example, if g is the density of an exponential distribution. In this case G is still strictly log-concave, but $1-G$ is only weakly log-concave since $g(x)/(1-G(x))$ is a constant.

[17] Condition (ii) in Lemma 3 is in fact significantly less demanding than log-concavity of g in the following sense. A log-concave g must be single-peaked and thus G must be uni-modal. This need not be the case under our condition (ii). Our condition allows for some polarized distributions that put more weights on extreme types than on intermediate ones.

x, the event that the pivotal voter's type equals x implies that the entire realized type profile is systematically lower (and thus voters' ex-post payoffs from reform become systematically higher) as q increases. This makes any prosocial sender more inclined towards passing the reform.

5 The Single-crossing property and the optimality of interval revelation

This section presents our first main result, Theorem 1, which relates the single-crossing property to the optimality of interval revelation policies for a monopoly sender in sufficiently large elections (Section 5.1). Section 5.2 formally presents the persuasion problem and proves Theorem 1. Section 5.3 explores the robustness of the optimality of interval revelation to small scale elections (like committee voting) and the sender's commitment power.

5.1 Optimal information policy under monopolistic persuasion

Consider a monopoly sender and let $\phi_n(x)$ be his indifference curve. We assume that the single-crossing property holds, so there exists $\tilde{n} \geq 0$ such that for all $n \geq \tilde{n}$ the function $x - \phi_n(x)$ is single-crossing on $[-1,1]$ and the unique switching state z_n is identified by (6).

As explained in the Introduction, an interval revelation policy can be characterized by a *revelation interval* $[a,b]$ with $-1 \leq a \leq b \leq 1$ such that (i) all states $k \in [a,b]$ are precisely revealed, and (ii) states $k > b$ and $k < a$ are pooled under distinct messages as in Figure 1. Under such a policy, voters' (common) posterior expectation equals k for all states $k \in [a,b]$ due to full revelation, and equals $\mathbb{E}_F[k|k > b]$ (resp. $\mathbb{E}_F[k|k < a]$) for all states $k > b$ (resp. $k < a$). Note that both *full disclosure* (with $a = -1$ and $b = 1$) and *no disclosure* (with $a = b \in \{-1,1\}$) are special cases of interval revelation policies.

Our first main result, Theorem 1, relates the single-crossing property to the optimality of interval revelation policies in large elections under monopolistic persuasion.

Theorem 1. *Consider a monopoly sender for whom the single-crossing property holds. Then there exists an $N \geq 0$ such that for all $n > N$ any optimal information policy is outcome equivalent to an interval revelation policy whose revelation interval $[a_n, b_n]$ satisfies $-1 \leq a_n \leq z_n \leq b_n \leq 1$.*[18] *Moreover, the following holds:*

1. *If $-1 < z_n < 1$ then $a_n < z_n < b_n$ so that the revelation interval contains the switching state z_n in its interior.*

[18] Two information policies are *outcome equivalent* if their induced mappings from the realized state k to voters' posterior expectation of the state are equal almost everywhere.

2. *If $z_n = -1$, then $a_n = -1$ so that only sufficiently high states can be pooled.*

3. *If $z_n = 1$, then $b_n = 1$ so that only sufficiently low states can be pooled.*

Theorem 1 also shows how the location of the switching state z_n determines the structure of the optimal revelation interval. If $z_n \in (-1,1)$ so that $x - \phi_n(x)$ crosses zero at some interior state, then the optimal revelation interval must contain z_n in its interior, and both very high and very low states can be pooled (if so, under distinct messages). If instead $z_n = -1$, then $x > \phi_n(x)$ for all $x \in (-1,1)$ and the sender is uniformly more biased towards reform than the pivotal voter. In this case the optimal policy takes the form of *'upper censorship'* (Kolotilin, Mylovanov and Zapechelnyuk, 2022) in the sense that only sufficiently high states can be pooled. Finally, if $z_n = 1$, then $x < \phi_n(x)$ for all $x \in (-1,1)$ and the sender is uniformly more biased towards the status quo than the pivotal voter. In this case the optimal policy takes the form of *'lower censorship'* in that only sufficiently low states can be pooled.

Observe that Theorem 1 is general because it applies for all continuous prior F and for all G provided that the single-crossing property holds. By Lemma 3, this implies that Theorem 1 holds for generic G if the sender is sufficiently self-interested,[19] and it holds for all sender preferences characterized by (1) if both G and $1-G$ are strictly log-concave.

Now we sketch the proof of Theorem 1 and explain its intuition (Section 5.2 presents formal details). First, we show that any optimal information policy must precisely inform voters whether the realized state k is above, equal to, or below the switching point z_n (cf. Lemma 4 below). To understand why, first notice that the single-crossing property implies that the sender has no incentive to hide state realization $k = z_n$. This is because at the switching state $k = z_n$ the interests of the sender and the pivotal voter are perfectly aligned; their indifference curves cross at $k = z_n$. Moreover, it is always optimal for the sender to fully separate any pair of state realizations on different sides of z_n. To see why, consider any k_1 and k_2 such that $k_1 < z_n < k_2$ and suppose they are not fully separated. As explained above, the sender is tempted to manipulate voters' beliefs about state realizations upwards in state k_2 while downwards in k_1. By separating these two states, the induced posterior mean state will indeed be lower in k_1 and higher in k_2 than in any other case where they are not fully separated. The sender thus strictly benefits from such separation.

Next, since any optimal information policy must reveal whether state k is above or below z_n, we can decompose the sender's persuasion problem into two auxiliary problems on intervals $[-1, z_n]$ and $[z_n, 1]$, respectively. In Lemma 5 below we show that under our single-crossing property, there

[19] The result would be very different in a setup with only one representative voter (i.e., $n = 0$). This case is studied by Kolotilin, Mylovanov and Zapechelnyuk (2022). They show that G must be uni-modal (i.e., g is single-peaked) to ensure the optimality of an interval revelation policy for a self-interested sender with $\rho = 0$. In our setup this is not required because, as statement (2) of Lemma 1 shows, the density function of the pivotal voter's type distribution $\hat{g}_n(\cdot; q)$ is single-peaked for sufficiently large n for any g that is positive and twice-continuously differentiable.

exists $N \geq 0$ such that for all $n > N$ the sender's indirect utility (as a function of the posterior mean state) is *strictly S-shaped* (i.e., first strictly convex and then strictly concave) on $[z_n, 1]$. For such utility functions, Kolotilin, Mylovanov and Zapechelnyuk (2022) show that the sender's optimal strategy on $[z_n, 1]$ is featured by upper censorship: there is a unique $b_n \in [z_n, 1]$ such that all states $k \in [z_n, b_n]$ are revealed while all $k > b_n$ are pooled. Similarly, for all $n \geq N$ the sender's expected utility is *strictly inverse-S-shaped* (i.e., first strictly concave and then strictly convex) on $[-1, z_n]$ and the optimal strategy on this interval features lower censorship: there is a unique $a_n \in [-1, z_n]$ such that all states $k \in [a_n, z_n]$ are revealed while all $k < a_n$ are pooled. Putting the optimal strategies on the two intervals together, we obtain that the sender's optimal strategy is uniquely given by an interval revelation policy whose revelation interval $[a_n, b_n]$ must contain z_n in between.

What, finally, drives the optimal choices of boundaries a_n and b_n? Consider b_n first. Recall that the sender is tempted to manipulate voters' beliefs upwards for all states $k > z_n$. Suppose the sender increases b_n to some $b_n + \Delta$ with $\Delta > 0$ small. Then the sender loses the opportunity to manipulate voter's beliefs upwards in any state $k \in [b_n, b_n + \Delta]$ because these states are now fully revealed. Nevertheless, this expansion of b_n raises the induced posterior expectation from $\mathbb{E}_F[k|k > b_n]$ to $\mathbb{E}_F[k|k > b_n + \Delta]$ – so that the pivotal voter is now more likely to be convinced to pass the reform – in all states $k \in [b_n + \Delta, 1]$. The optimal choice of b_n therefore balances the marginal costs of losing the capability to manipulate voters' beliefs in some states with the marginal gains of an increased effectiveness of persuasion. The tradeoff governing the optimal choice of a_n is similar.

Next, we apply Theorem 1 to show the qualitative structures of the optimal revelation intervals in four examples of sender preferences. These are illustrated by the four panels of Figure 3.

Example 1. Self-interested sender. *Panel (a) depicts the indifference curve and the optimal revelation intervals for a self-interested sender with $\rho = 0$. By (3), $\phi_n(x) = \chi$ for all $x \in [\underline{v}, \overline{v}]$. His switching state z_n thus depends solely on χ. If $\chi \in (-1, 1)$, then $z_n = \chi$ and by Theorem 1 some interval revelation policy with $a_n < \chi < b_n$ is optimal for this sender in large elections. If instead $\chi \leq -1$ (resp. $\chi \geq 1$), then he is uniformly more biased towards the reform (resp. status quo) than the pivotal voter in all states. For these cases, Theorem 1 implies the sender's optimal information policy must be either upper (if $\chi \leq -1$) or lower (if $\chi \geq 1$) censorship in large elections.*

Example 2. Utilitarian social planner. *Panel (b) considers the case of a Utilitarian planner who aims at maximizing voters' ex-post average payoffs. His indifference curve $\phi_n(x)$ is given by*[20]

$$\phi_n(x) = \frac{n}{n+1}\left(q\mathbb{E}_G[v_i|v_i \leq x] + (1-q)\,\mathbb{E}_G[v_i|v_i \geq x]\right) + \frac{1}{n+1}x \tag{7}$$

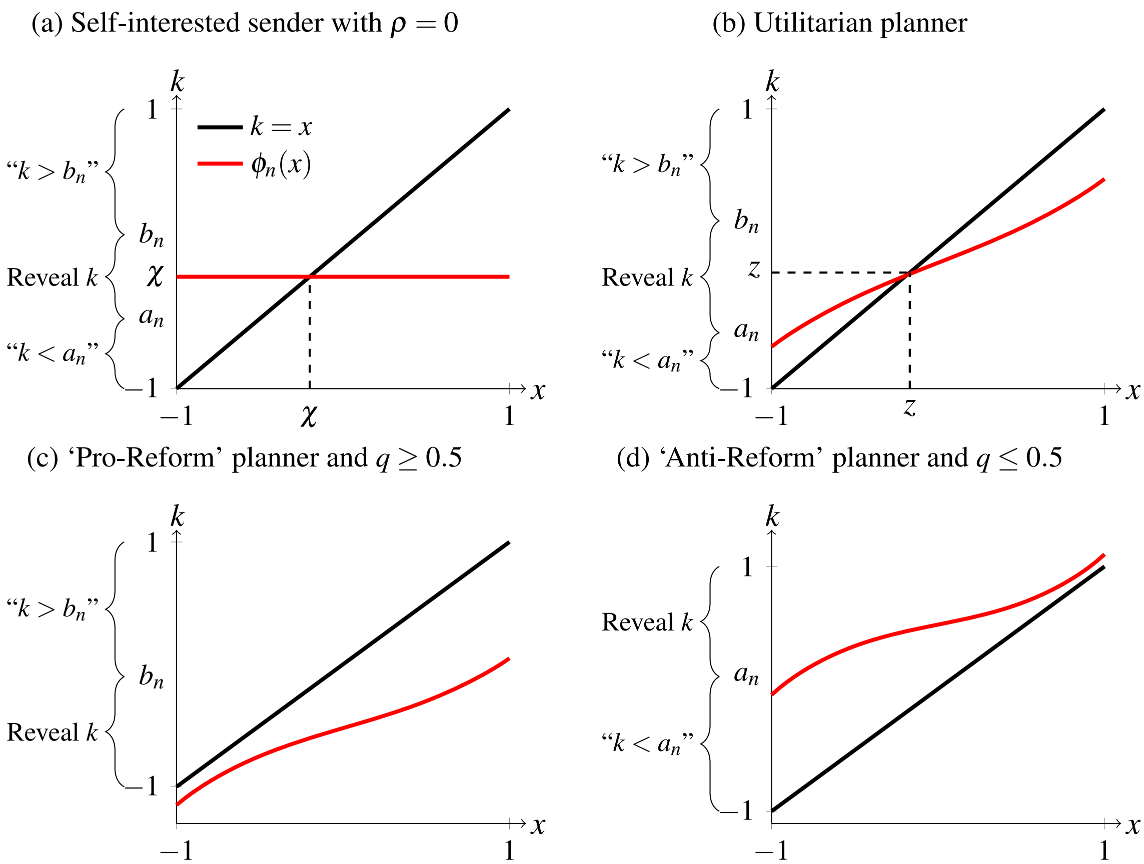
[20] To see why (7) is true, notice that for event $v^{(nq+1)} = x$ to hold, there must be one voter with type $v_i = x$, nq other voters with $v_i \leq x$, and the remaining $n(1-q)$ voters with $v_i \geq x$. Since each voter's type is independently drawn from G, the conditional expectation of any voter with $v_i \leq x$ (resp. $v_i \geq x$) equals $\mathbb{E}_G[v_i|v_i \leq x]$ (resp. $\mathbb{E}_G[v_i|v_i \geq x]$). Taking the average over the whole electorate size $n+1$ yields (7).

for $x \in [\underline{v}, \overline{v}]$. Therefore, $\phi_n(x) = x$ if and only if

$$q\mathbb{E}_G[v_i | v_i \leq x] + (1-q)\mathbb{E}_G[v_i | v_i \geq x] = x \; . \tag{8}$$

When both G and $1-G$ are strictly log-concave, the single crossing property holds by Lemma 3 and hence (8) admits a unique solution z on $(\underline{v}, \overline{v})$. A Utilitarian planner's switching point z_n thus depends only on z. If $z \in (-1,1)$ as depicted in panel (b), then $z_n = z$ and by Theorem 1 some interval revelation policy with $a_n < z < b_n$ is Utilitarian optimal in large elections. Interestingly, if $z \leq -1$ (resp. $z \geq 1$) then even a Utilitarian planner can be uniformly more biased towards reform (resp. status quo) than the pivotal voter. For these cases the Utilitarian optimal information policy is either upper (if $z \leq -1$) or lower (if $z \geq 1$) censorship in large elections.

Figure 3: The Structures of Optimal Revelation Intervals in Four Examples



Note: In each of these panels the horizontal axis x denotes the pivotal voter's type realization $v^{(nq+1)}$, the black line denotes the pivotal voter's indifference curve, and the red line denotes the sender's indifference curve.

Example 3. 'Pro-Reform' social planner. *In panel (c) we consider a non-Utilitarian social planner who aims at maximizing the average payoff of the subset of voters whose ex-post payoffs* *under reform are above the 50% percentile.*[21] *Suppose* $q \geq 0.5$ *so that a strict majority is required in order to pass the reform. In this case,* $x > \phi_n(x)$ *must hold for all* $x \in [-1,1]$*; that is, the sender must be uniformly more biased towards the reform than the pivotal voter in all states. This is because he assigns positive weights only to voters who always like the reform better than the pivotal voter does. The sender thus must prefer the reform if the pivotal voter is indifferent. Therefore, by Theorem 1, in large elections some upper censorship policy must be optimal.*

Example 4. 'Anti-Reform' social planner. *In panel (d) we consider a non-Utilitarian social planner who aims at maximizing the average payoff of the subset of voters whose ex-post payoffs under reform are below the 50% percentile.*[22] *Following the same logic as in Example 3, we can show that for all* $q \leq 0.5$ *(i.e., a strict majority is required to maintain the status quo)* $x < \phi_n(x)$ *must hold for all* $x \in [-1,1]$*; that is, such a sender must be uniformly more biased towards the status quo than the pivotal voter in all states. Theorem 1 then implies that some lower censorship policy must be optimal in large elections.*

We conclude this subsection with two remarks. First, if the sender could perfectly observe voters' types, our model would become a special case of Alonso and Câmara (2016*a*) and the optimal policy would be a binary strategy that only reports whether state k is above or below some cutoff. Therefore, the fact that our optimal information policy can have a more nuanced structure (i.e., with a non-trivial revelation interval) is due to our assumption that voters' types are not observable to the sender. Second, as Examples 2 to 4 illustrate, full disclosure can be suboptimal even when the sender's goal is to maximize voters' welfare. To understand why, observe that conditional on the pivotal voter's type being x, in all states k between x and $\phi_n(x)$ the sender's and pivotal voter's preferences disagree. Our single-crossing property implies that $x = \phi_n(x)$ can hold for at most one $x \in [-1,1]$ for sufficiently large n. The interim conflict of interests between the sender and the pivotal voter is thus ubiquitous. Consequently, full disclosure is generically suboptimal.

5.2 The persuasion problem and proof of Theorem 1

We start by formally presenting the persuasion problem faced by the monopoly sender. Let θ denote the common posterior expectation about the state realization, shared by all voters and the sender. Given $\phi_n(\cdot)$, the sender's (indirect) expected utility under θ is[23]

$$W_n(\theta) = \int_{\underline{v}}^{\theta} (\theta - \phi_n(x))\, \hat{g}_n(x;q)dx \tag{9}$$

[21] The welfare weighting function for such a 'pro-Reform' planner is $w(y) = \min\{2y, 1\}$ for $y \in [0,1]$.

[22] The welfare weighting function for such an 'anti-Reform' planner is $w(y) = \max\{2y-1, 0\}$ for $y \in [0,1]$.

[23] To see why (9) holds, recall that x is the type of the pivotal voter. By the discussion in Section 3.1, the reform is adopted if $\theta \geq x$ and in this case the sender's expected payoff is $\theta - \phi_n(x)$; otherwise the status quo is maintained and the sender's payoff is zero. This explains the integrand in (9).

where recall that $\hat{g}_n(\cdot;q)$ is the density function of the pivotal voter's type. Because the sender's expected payoff depends on posterior expectation θ only, it is convenient to present any information policy π by the distribution H_π of posterior means it induces. We say that a distribution of posterior means H is *feasible* if it can be induced by some information policy $\pi \in \Pi$. It is well known that given prior F, a distribution of posterior means H is feasible if and only if F is a *mean-preserving spread* of H (Gentzkow and Kamenica, 2016; Kolotilin et al., 2017; Dworczak and Martini, 2019).[24] In the sequel we write $F \succeq_{MPS} H$ if F is a mean-preserving spread of H.

The persuasion problem. For a monopoly sender, an information policy π is optimal if H_π solves

$$\max_{H \in \Delta([-1,1])} \int_{-1}^{1} W_n(\theta) dH(\theta), \quad \text{s.t. } F \succeq_{MPS} H \; . \tag{MP}$$

As we show in Online Appendix C.2, $W_n(\cdot)$ is twice-continuously differentiable and thus upper semi-continuous. Hence, (MP) admits at least one solution (Dworczak and Martini, 2019). For any $-1 \leq a \leq b \leq 1$, we define

$$H_{\mathscr{P}(a,b)}(\theta) := \begin{cases} F(a) \cdot \mathbb{1}\{\theta \geq \mathbb{E}_F[k|k<a]\}, & \text{if } \theta \in [-1,a) \\ F(\theta), & \text{if } \theta \in [a,b) \\ F(b) + [1-F(b)] \cdot \mathbb{1}\{\theta \geq \mathbb{E}_F[k|k>b]\}, & \text{if } \theta \in [b,1] \end{cases} \tag{10}$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. Note that $H_{\mathscr{P}(a,b)}(\theta)$ is the distribution of posterior means induced by any interval revelation policy with revelation interval $[a,b] \subseteq [0,1]$. We call any $\pi \in \Pi$ an *interval revelation policy* if H_π coincides with (10) for some $-1 \leq a \leq b \leq 1$ almost everywhere. In the sequel we slightly abuse notation and let $\mathscr{P}(a,b)$ denote both any specific interval revelation policy or the set of all policies with revelation interval $[a,b]$, whenever this does not cause confusion. For the special case $a = b$ we simply write $\mathscr{P}(a,b)$ as $\mathscr{P}(a)$ and refer to it as a *cutoff policy* because it only reveals whether the realize state is above, equal to, or below cutoff a.

Our proof for Theorem 1 relies on the following two lemmas, whose proofs are in Online Appendix C. Both lemmas establish important curvature properties of $W_n(\cdot)$ that help to pin down the solutions to the sender's problem (MP). In Lemma 4, we establish a novel '*increasing slope property*' and show that when this condition holds, any solution H to (MP) must be weakly more informative than a cutoff policy that only reveals whether the realized state is above, equal to, or below a certain threshold.

Lemma 4. *Suppose that $x - \phi_n(x)$ crosses zero only once and from below at an interior point*

[24] F is a mean-preserving spread of H if $\int_{-1}^{x} H(\theta)d\theta \leq \int_{-1}^{x} F(\theta)d\theta$ for all $x \in [-1,1]$ and the equality holds for $x = \pm 1$. An alternative, equivalent definition is that $\mathbb{E}_F[\omega(\cdot)] \geq \mathbb{E}_H[\omega(\cdot)]$ for any convex function $\omega(\cdot)$.

$z_n \in (-1,1)$. *Then $W_n(\cdot)$ satisfies the 'increasing-slope property' at point z_n, that is,*

$$\frac{W_n(x) - W_n(z_n)}{x - z_n} \leq \frac{W_n(y) - W_n(z_n)}{y - z_n}, \forall y > x$$

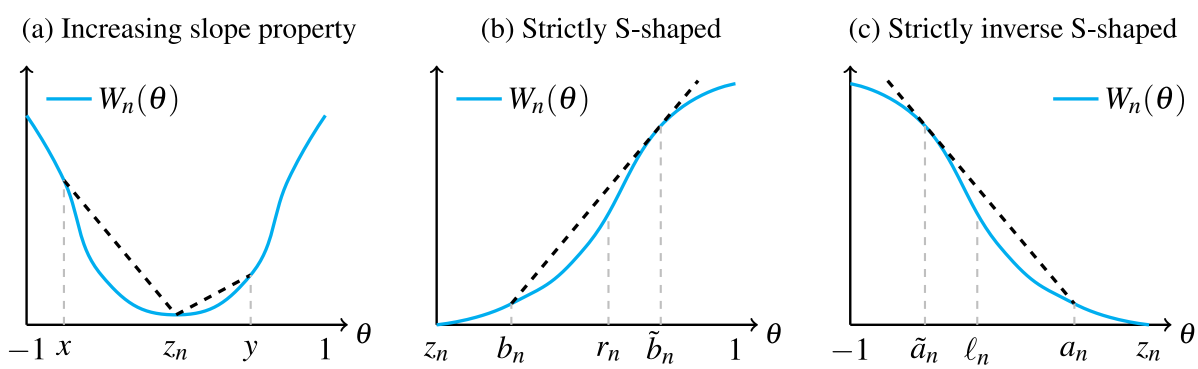
and strict inequality holds if $x < z_n < y$ (cf. panel (a) of Figure 4).[25] *Moreover, any solution H to problem (MP) must satisfy $H \succeq_{MPS} H_{\mathscr{P}(z_n)}$. In other words, any optimal information policy must reveal whether the state realization is above, equal to or below z_n.*

Next, in line with standard terminology (e.g., Kolotilin, Mylovanov and Zapechelnyuk (2022)), we say that $W_n(\cdot)$ is *strictly S-shaped* on some interval if it is strictly convex below some inflection point and strictly concave above it. Likewise, $W_n(\cdot)$ is *strictly inverse-S-shaped* on some interval if it is strictly concave below some inflection point and strictly concave above it. Notice that both definitions include strictly convex and concave functions as special cases.

Lemma 5. *Suppose that the single-crossing property holds. Then there exists an $N \geq 0$ such that for all $n > N$ there are ℓ_n and r_n with $-1 \leq \ell_n \leq z_n \leq r_n \leq 1$ such that:*[26]

1. *$W_n(\cdot)$ is strictly S-shaped on $[z_n, 1]$ with inflection point r_n (cf. panel (b) of Figure 4); and*
2. *$W_n(\cdot)$ is strictly inverse-S-shaped on $[-1, z_n]$ with inflection point ℓ_n (cf. panel (c) of Figure 4).*

Figure 4: Graphical illustrations of Lemmas 4, 5 and the proof of Theorem 1



With these ingredients we are ready to prove Theorem 1, directly using the quantities N, ℓ_n and r_n identified in Lemma 5.

[25] Geometrically, if a function $U(\cdot)$ satisfies the increasing-slope property at point z, then for all $x \neq z$ the line segment connecting $(x, U(x))$ and $(z, U(z))$ must lie above $U(\cdot)$ on (x,z) or (z,x), as demonstrated in panel (a) of Figure 4.

[26] In fact, our single-crossing property is almost necessary; suppose instead that $x - \phi(x)$ crosses zero from above at some interior point in $(-1,1)$, then for sufficiently large n there exists some interval $[x,y] \subset (-1,1)$ and $\varepsilon > 0$ such that $W_n(\cdot)$ is strictly concave on $[x,y]$ but is strictly convex on $[x-\varepsilon, x]$ and $[y, y+\varepsilon]$, respectively. In this case, it follows from the duality arguments in Dworczak and Martini (2019) and Kolotilin, Mylovanov and Zapechelnyuk (2022) that there exists some continuous prior F under which the optimal information policy does not feature our interval revelation.

Proof of Theorem 1. Depending on the value of z_n, we distinguish between three cases.

Case 1: $x-\phi_n(x)$ crosses zero from below at a unique interior point $z_n \in (-1,1)$. By Lemma 4, $W_n(\cdot)$ satisfies the increasing-slope property at z_n and any solution H to problem (MP) must satisfy $H \succeq_{MPS} H_{\mathscr{P}(z_n)}$. Consequently, the monopolistic persuasion problem can be divided into two auxiliary problems on intervals $[-1,z_n]$ and $[z_n,1]$, respectively:

$$\max_{H\in\Delta([z_n,1])} \int_{z_n}^{1} W_n(\theta)dH(\theta), \quad \text{s.t. } F_{\text{I}} \succeq_{MPS} H \tag{MP-I}$$

$$\max_{H\in\Delta([-1,z_n])} \int_{-1}^{z_n} W_n(\theta)dH(\theta), \quad \text{s.t. } F_{\text{II}} \succeq_{MPS} H \tag{MP-II}$$

In these problems, F_{I} is the truncated cdf of F on $[z_n,1]$ and F_{II} is the truncated cdf of F on $[-1,z_n]$.

Recall from Lemma 5 that, for all $n>N$, $W_n(\cdot)$ is strictly S-shaped on $[z_n,1]$ with inflection point r_n and strictly inverse S-shaped on $[-1,z_n]$ with inflection point ℓ_n. Then, by Kolotilin, Mylovanov and Zapechelnyuk (2022), the solution to problem (MP-I) is uniquely given by an interval revelation policy $\mathscr{P}(z_n,b_n)$ and threshold b_n satisfies the following complementary slackness condition

$$(\tilde{b}_n - b_n)W_n'(\tilde{b}_n) \le W_n(\tilde{b}_n) - W_n(b_n) \tag{FOC: b_n}$$

where $\tilde{b}_n = \mathbb{E}_F[k|k \ge b_n]$ and (FOC: b_n) is binding whenever $b_n \in (z_n,1)$ (cf. panel (b) of Figure 4).[27] Moreover, b_n and $\tilde{b}_n$ satisfy $b_n < r_n < \tilde{b}_n$ for $r_n \in (z_n,1)$, and $b_n = 1$ if $r_n = 1$. Similarly, the solution to (MP-II) is uniquely given by an interval revelation policy $\mathscr{P}(a_n,z_n)$ and threshold a_n satisfies

$$(a_n - \tilde{a}_n)W_n'(\tilde{a}_n) \le W_n(a_n) - W_n(\tilde{a}_n) \tag{FOC: a_n}$$

where $\tilde{a}_n = \mathbb{E}_F[k|k \le a_n]$ and (FOC: a_n) is binding whenever $a_n \in (-1,z_n)$ (cf. panel (c) of Figure 4). Moreover, a_n and $\tilde{a}_n$ satisfy $a_n > \ell_n > \tilde{a}_n$ for $\ell_n \in (-1,z_n)$, and $a_n = -1$ if $\ell_n = 1$. Taken together, these imply that the optimal solution is uniquely given by an interval revelation policy $\mathscr{P}(a_n,b_n)$.

Next we show that $a_n < z_n < b_n$ must hold whenever $z_n \in (-1,1)$. By (9) we have

$$\begin{aligned} W_n''(z_n) &= \hat{g}_n(z_n;q)\left(2-\phi_n{}'(z_n)\right) + (z_n - \phi_n(z_n))\,\hat{g}_n'(z_n;q) \\ &= \hat{g}_n(z_n;q)\left(2-\phi_n{}'(z_n)\right) > \hat{g}_n(z_n;q) > 0\,. \end{aligned}$$

The second step holds because $z_n - \phi_n(z_n) = 0$ by definition of z_n, and the third step holds because the single-crossing property requires $1-\phi_n'(z_n) > 0$ whenever $z_n - \phi_n(z_n) = 0$. Therefore, $W_n(\cdot)$ is

[27] If $(\tilde{b}_n - b_n)W_n'(\tilde{b}_n) > W_n(\tilde{b}_n) - W_n(b_n)$ for all $b_n \in [z_n,1]$ then $b_n = 1$. A similar result holds for a_n.

strictly convex in a neighborhood around z_n and thus $r_n > z_n$. This implies that

$$W_n'(z_n) < \frac{W_n(\theta) - W_n(z_n)}{\theta - z_n} \tag{11}$$

holds for all $\theta \in (z_n, r_n]$. Since $W_n(\cdot)$ satisfies the increasing-slope property at point z_n, the right-hand side of (11) is increasing in θ. Therefore, (11) must hold for all $\theta > z_n$. This implies that (FOC: b_n) cannot be binding at $b_n = z_n$ for any $z_n \in (-1,1)$ and thus $b_n > z_n$ must hold. $a_n < z_n$ can be established using similar arguments. This proves statement (1) of Theorem 1.

Case 2: $z_n = -1$ so that $x - \phi_n(x) > 0$ for all $x \in (-1,1)$. By Lemma 5, $W_n(\cdot)$ is S-shaped on $[-1,1]$ with inflection point $r_n \in [-1,1]$ for all $n \geq N$. The optimal information policy is therefore uniquely given by an upper-censorship policy $\mathscr{P}(-1, b_n)$, with b_n determined by condition (FOC: b_n). This proves statement (2) of Theorem 1.

Case 3: $z_n = 1$ so that $x - \phi_n(x) < 0$ for all $x \in (-1,1)$. By Lemma 5, $W_n(\cdot)$ is strictly inverse S-shaped on $[-1,1]$ with inflection point $\ell_n \in [-1,1]$ for all $n \geq N$. The optimal information policy is therefore uniquely given by a lower-censorship policy $\mathscr{P}(a_n, 1)$, with a_n determined by condition (FOC: a_n). This proves statement (3) of Theorem 1. □

5.3 Robustness of interval revelation: Electoral size and commitment

5.3.1 Optimality of interval revelation in small electorates

It is important to establish if and when Theorem 1 holds independently of the electorate size. When that is the case, our single-crossing property implies the optimality of interval revelation also for small-scale elections such as committee voting. Proposition 2 provides sufficient conditions for this to be the case.

Proposition 2. *Under either of the following two conditions, the single crossing property holds and the cutoff N in Theorem 1 equals zero so that interval revelation policies are optimal for all $n > 0$.*

1. g is log-concave and ρ is sufficiently close to 0;

2. g is log-concave, the sender is utilitarian (i.e., $\rho = 1$, $w(y) = y$ for $y \in [0,1]$, and $\phi_n(\cdot)$ is given by (7) in Example 2), and function

$$\frac{2 - \phi'(x)}{x - \phi(x)} + \frac{g'(x)}{g(x)} \tag{12}$$

is decreasing in x on $[-1, z)$ and $(z, 1]$, respectively.[28] *Here,*

$$\phi(x) = \lim_{n\to\infty} \phi_n(x) = q\mathbb{E}_G[v_i | v_i \leq x] + (1-q)\mathbb{E}_G[v_i | v_i \geq x]$$

[28] If $|z| \geq 1$ then this condition requires (12) to be decreasing on $[-1,1]$.

and z is the unique solution to $x - \phi(x) = 0$.

Condition (1) of Proposition 2 implies that, under any log-concave g, interval revelation policies are optimal independent of the electorate size whenever the sender is sufficiently self-interested. This directly generalizes the results of Alonso and Câmara (2016*b*) and Kolotilin, Mylovanov and Zapechelnyuk (2022) for persuading a single receiver to the case of publicly persuading multiple receivers under any super-majority rule. For generic sender preferences and voting rules, it is difficult to check whether log-concavity of g is still sufficient for $N = 0$ to hold. This is because in general functions $\phi_n(\cdot)$ and $\phi_n'(\cdot)$ may depend on n in complicated ways that impede tractability. Despite so, for a utilitarian sender, we can provide a simple way to verify whether $N = 0$ holds; this is condition (2) of Proposition 2. Given the functional form of g one can often easily check the monotonicity of (12) either analytically or numerically.[29] The conditions in Proposition 2 hold for interesting applications of our model. We provide examples in Section 8.

5.3.2 The sender's commitment power

In line with all Bayesian persuasion models, we assume that the sender can fully commit to any information policy that is feasible. While this assumption is often criticized for being too demanding, we argue that such concerns can be considerably alleviated for our purposes.

First of all, Bayesian persuasion solutions are often robust in many instances where some ex-post manipulation of signals is possible. For example, Gentzkow and Kamenica (2017*c*) show for generic Bayesian persuasion games that the possibility of ex-post manipulation does not affect the set of equilibrium outcomes if the sender must send verifiable messages but has to endogenously acquire the information first. For any equilibrium in which information is withheld, there exists an outcome equivalent equilibrium with (a different information acquisition strategy but) truthful disclosure. Furthermore, even in case the sender can covertly manipulate signals ex-post without being detected, the fact that our optimal information policy has a simple interval revelation structure suggests that it can often be credibly implemented without the sender's commitment.[30] In particular, in our setting the interval revelation policies satisfy the credibility notion in Lin and Liu (2023), meaning that the sender cannot profit from tampering with messages ex-post so long as he has to keep the promised

[29] This approach is particularly convenient when the truncated expectations $\mathbb{E}_G[v_i|v_i \leq x]$ and $\mathbb{E}_G[v_i|v_i \geq x]$ can be derived in closed form, such as for uniform and normal distributions. Indeed, we check analytically that (12) is decreasing for a uniform distribution. In Section 8 we present an application in which g is the density of a normal distribution and we verify numerically that (12) is decreasing for the parameters concerned. In fact, we conjecture that (12) is decreasing for all $q \in (0,1)$ and log-concave densities g, although we have not been able to formally prove it. If this conjecture is correct, then the log-concavity of g alone is sufficient to ensure $N = 0$ for a utilitarian sender.

[30] Our interval revelation policies belong to the broad class of *monotone partitional signals*, which partition a one-dimensional state space into disjoint pooling or revealing intervals. Previous papers have shown that in many contexts such monotone partitional signals are often imposed in practice or can be implemented without strong reliance on the sender's commitment power (Ivanov, 2020; Mensch, 2021; Kolotilin and Li, 2021)

distribution of messages unchanged.[31] This notion of credibility is most relevant when the sender is providing information about the quality distribution of a large population of objects. In Section 8 we present an application of our model that is consistent with this interpretation and formally show that our optimal interval revelation policies can indeed be credibly enforced by simple quota restrictions that do not rely on the sender's ability to commit.

Second, Bayesian persuasion models have been frequently used to study relevant research questions in media and politics (Besley and Prat, 2006; Gehlbach and Sonin, 2014; Shadmehr and Bernhardt, 2015; Kolotilin, Mylovanov and Zapechelnyuk, 2022; Gitmez and Molavi, 2022), which is one of the applications that our paper speaks to. In these contexts the commitment assumption seems plausible due to the senders' limited influence on the behavior of agents beyond their direct controls. This restricts their ability to manipulate signals ex-post. For example, to build its editorial policy, a media outlet can strategically decide which editors and journalists to hire (Chan and Suen, 2008; Denter, Dumav and Ginzburg, 2021). Nevertheless, it is often costly or impractical (at least in the short run) to change the editorial staff or to frequently intervene in their regular news production.

Third, an important goal of our paper is to characterize the welfare-optimal public information policies for a large variety of social welfare functions and voting rules. To study such normative questions, it is instructive to start with a simple benchmark in which the social planner can commit to any public information policy.[32]

6 Asymptotic properties and comparative statics

Our Theorem 1 shows that if the single-crossing property holds for a sender, then there exists a $N \geq 0$ such that for all $n > N$ interval revelation policies are optimal under monopolistic persuasion. In this section we further explore properties of the optimal revelation interval and the sender's payoff. In doing so we provide further intuition for the key drivers that shape the sender's optimal information policy, and how the latter responds to changes in model primitives such as the sender's preference and the voting rule. While many results in this section concern large n, in Section 8 we provide applications to show that most insights here carry over to small-scale elections as well.

[31] In addition, Guo and Shmaya (2021) and Lipnowski, Ravid and Shishkin (2022) obtain that Bayesian persuasion solutions are robust to small costs for the possibility of misreporting, while Mathevet, Pearce and Stacchetti (2022) and Best and Quigley (2023) show that senders' reputation incentives can also help to generate commitment power.

[32] After establishing this benchmark, interesting extensions may include the possibility where the sender can introduce a mechanism to elicit voters' types before sending the public information (in the spirit of Kolotilin et al. (2017)), or even to provide private messages that target distinct voters. A formal exploration of these possibilities is beyond the scope of our paper, and hence we leave it for future research.

6.1 Asymptotically optimal revelation interval and the sender's payoff

In this subsection we characterize the sender's asymptotically optimal revelation interval as $n \to \infty$ and compare his asymptotic payoff to two natural benchmarks. Omitted proofs are in Online Appendix D. Our main result here is Theorem 2, which shows that under the single-crossing property the asymptotically optimal revelation interval can be characterized with only three variables: $v_q^* = G^{-1}(q)$, $\phi^* = \phi(v^*) = \rho \int_0^1 G^{-1}(y)dw(y) + (1-\rho)\chi$, and

$$z^* := \lim_{n\to\infty} z_n = \begin{cases} -1 & \text{if } x > \phi(x) \text{ for all } x \in [-1,1] \\ x & \text{if } x = \phi(x) \text{ for some } x \in [-1,1] \\ 1 & \text{if } x < \phi(x) \text{ for all } x \in [-1,1] \end{cases}. \tag{13}$$

Before stating Theorem 2, let us recall the economic implications of the three variables. By Lemmas 1 and 2, we have $v^{(nq+1)} \xrightarrow{p} v_q^*$ and $\varphi_n(v) \xrightarrow{a.s.} \phi^*$, respectively. Therefore, as $n \to \infty$, the pivotal voter prefers reform (status quo) almost surely if $k > (<) v_q^*$, while the sender prefers reform (status quo) almost surely if $k > (<)\phi^*$. The difference between v_q^* and ϕ^* thus reflects the *ex-ante* conflict of interests between the sender and the pivotal voter, i.e. *before the pivotal voter's type is drawn*. Finally, z^* is the limit of switching states z_n and, following the discussion in Section 4.2, in all states $k > (<)z^*$ the sender is more biased towards the reform (status quo) than the pivotal voter as $n \to \infty$. This is due to the inference problem from the pivotal voter's choice explained in Section 4.1. Unlike the ex-ante conflict of interest, this inference procedure determines the wedge between preferences of the pivotal voter and the sender *conditional on each possible realization of the pivotal voter's type*. Importantly, z^* can differ from ϕ^* only when the sender is prosocial (i.e., $\rho > 0$). Lemma 6 summarizes useful properties about the relationship between the three quantities.

Lemma 6. *Suppose $\phi^* \in (-1,1)$. Then v_q^*, ϕ^* and z^* satisfy the following properties:*

1. *If $\phi^* = v_q^*$ then $z^* = \phi^* = v_q^*$.*
2. *If $\phi^* \neq v_q^*$ then (i) $z^* = \phi^*$ if $\rho = 0$ and (ii) either $z^* < \phi^* < v_q^*$ or $z^* > \phi^* > v_q^*$ if $\rho > 0$.*

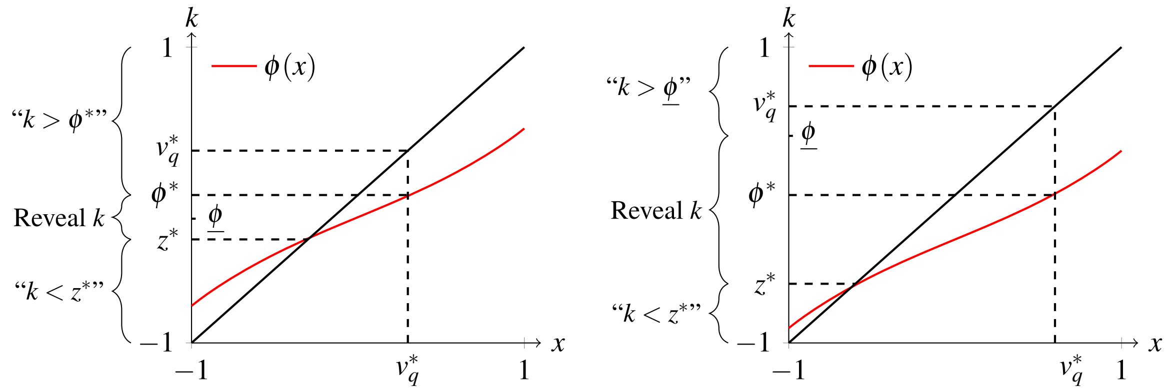
Figure 5 illustrates Lemma 6 for the case $\rho > 0$ and $\phi^* < v_q^*$. As is evident graphically and from (13), z^* is the intersection point of $\phi(\cdot)$ and the 45-degree line (the pivotal voter's indifference curve) on $[-1,1]$, provided that they have one. Because $\phi(\cdot)$ is non-decreasing and $\phi^* = \phi(v_q^*)$ by definition (cf. (5)), v_q^* and z^* must lie on different sides of ϕ^* whenever $\phi^* \in [-1,1]$ and $\phi^* \neq v_q^*$.

To simplify exposition, we assume $v_q^*, \phi^* \in [-1,1]$ in what follows.[33] Let $\{a_n\}_{n\geq N}$ and $\{b_n\}_{n\geq N}$ denote the sequences of the boundaries of the optimal revelation intervals. Define $a^* := \lim_{n\to\infty} a_n$ and $b^* := \lim_{n\to\infty} b_n$. Theorem 2 characterizes them as functions of v_q^*, ϕ^* and z^*.

[33] Any $\phi^* < -1$ (resp. $\phi^* > 1$) is equivalent to the case $\phi^* = -1$ (resp. $\phi^* = 1$). The same applies for v_q^*.

Figure 5: Relationship between v_q^*, ϕ^* and z^*, and the Asymptotically Optimal Revelation Intervals

(a) $\rho > 0$ and $\underline{\phi} \leq \phi^* < v_q^*$ (b) $\rho > 0$ and $\phi^* < \underline{\phi} < v_q^*$



Theorem 2. *Suppose that the single-crossing property holds and $v_q^*, \phi^* \in [-1,1]$. Define*

$$\overline{\phi} := \sup\left\{y \in [-1,1] : \mathbb{E}_F\left[k|k \leq y\right] \leq v_q^*\right\} \text{ and} \tag{14}$$

$$\underline{\phi} := \inf\left\{y \in [-1,1] : \mathbb{E}_F[k|k \geq y] \geq v_q^*\right\} \tag{15}$$

such that $\underline{\phi} \leq v_q^* \leq \overline{\phi}$ always holds. Then a^* and b^* are characterized as follows:

1. *If $v_q^* = \phi^* = z^*$, then $a^* = b^* = \phi^*$.*
2. *If $v_q^* > \phi^* \geq z^*$, then $a^* = z^*$ and $b^* = \max\{\phi^*, \underline{\phi}\} \leq v_q^*$.*
3. *If $v_q^* < \phi^* \leq z^*$, then $a^* = \min\left\{\phi^*, \overline{\phi}\right\} \geq v_q^*$ and $b^* = z^*$.*

The boundaries $\overline{\phi}$ and $\underline{\phi}$ defined in this theorem jointly determine the limit extent to which a monopoly sender can manipulate outcomes of large elections. More specifically, notice that under any interval revelation policy the induced election outcome in the limit can be characterized by an implementation threshold $t^* \in [-1,1]$ such that the reform (status quo) is implemented almost surely if $k > (<)t^*$. To implement any interior $t^* \in (-1,1)$ it is necessary that $a^* \leq t^* \leq b^*$. Given the pivotal voter's type v_q^*, the set of feasible implementation thresholds in the limit is exactly $[\underline{\phi}, \overline{\phi}]$.

By Theorem 2, a^* and b^* must satisfy the following chain of inequalities:

$$\min\left\{v_q^*, z^*\right\} \leq a^* \leq \min\left\{\phi^*, z^*\right\} \leq \max\left\{\phi^*, z^*\right\} \leq b^* \leq \max\left\{v_q^*, z^*\right\} . \tag{16}$$

Three implications are immediate from (16). First, full disclosure (i.e., $a_n = -1$ and $b_n = 1$) is suboptimal for sufficiently large n if either $v_q^* \in (-1,1)$ so that the pivotal voter's ex-ante preference is state-dependent, or $z^* \in (-1,1)$ so that the sender is not uniformly biased towards any alternative relative to the pivotal voter as $n \to \infty$. Second, no disclosure (i.e., $a_n = b_n \in \{-1,1\}$) is never optimal for sufficiently large n whenever $\phi^* \in (-1,1)$, that is, the sender's ex-ante preference is state-dependent. Third, if both $z^*, v_q^* \in (-1,1)$, then $-1 < a_n < b_n < 1$ must hold – i.e., the optimal policy pools both high and low states – for n large enough. We summarize these in Corollary 1.

Corollary 1. *Suppose that the single-crossing property holds. If either $v_q^* \in (-1,1)$ or $z^* \in (-1,1)$, then full disclosure is suboptimal for sufficiently large n. If $\phi^* \in (-1,1)$, then no disclosure is suboptimal for sufficiently large n. If both $z^*, v_q^* \in (-1,1)$, then the optimal information policy will pool sufficiently high and low states with distinct messages for sufficiently large n.*

As we explained in Section 5, with finite n a necessary condition for the optimal information policy to have a non-trivial revelation interval (i.e., $a_n < b_n$) is that the sender is uncertain about the pivotal voter's type. As $n \to \infty$ such uncertainty vanishes due to the law of large numbers and one might expect $a^* = b^*$ to hold in some circumstances. Indeed, for the case $v_q^* = \phi^*$ – i.e., there is no ex-ante conflict of interests between the sender and the pivotal voter – Theorem 2 immediately implies $a^* = b^* = \phi^*$. It is perhaps surprising that even in the absence of any ex-ante conflict of interests the asymptotically optimal information policy is not full disclosure, but instead a binary cutoff policy that only reveals whether the state realization is above, equal to, or below ϕ^*. Such a cutoff policy is indeed asymptotically optimal because it induces the pivotal voter to implement the reform (status quo) almost surely as $n \to \infty$ whenever $k > (<)\phi^*$, which perfectly coincides with the sender's favored outcome ex-ante.[34]

Now suppose $\phi^* \neq v_q^*$ so that the ex-ante preferences of the sender and the pivotal voter are not aligned. Corollary 2 gives sufficient and necessary conditions for $a^* = b^*$ to hold in this case.

Corollary 2. *Suppose that the single-crossing property holds, $v_q^* \in (-1,1)$ and $v_q^* \neq \phi^*$. If $\rho = 0$, then $a^* = b^*$ if and only if $\chi \in [\underline{\phi}, \overline{\phi}]$. If $\rho > 0$ then $a^* < b^*$.*

First consider the case $\rho = 0$, that is, the sender is self interested with $\phi^* = \chi$. For this case Corollary 2 claims that $a^* = b^* = \chi$ if and only if χ lies in the feasible set of implementation thresholds $[\underline{\phi}, \overline{\phi}]$. This is because by simply revealing whether k is above or below χ the sender can already sway the pivotal voter's decision in his preferred direction in all states as $n \to \infty$. Any further expansion of the revelation interval is weakly harmful in the limit as it can only make the pivotal voter choose the sender's less preferred alternative. If instead χ lies outside $[\underline{\phi}, \overline{\phi}]$, then $a^* < b^*$ so that the optimal revelation interval is non-degenerate even in the limit. For instance,

[34] Kamenica and Gentzkow (2011) make a similar observation when they note that aligning the interests between the sender and the receiver does not necessarily imply more information disclosure by the sender. Specifically, they write that "*On the one hand, the more Receiver responds to information in a way consistent with what Sender would do, the more Sender benefits from providing information. On the other hand, when preferences are more aligned Sender can provide less information and still sway Receiver's action in a desirable direction. Hence, making preferences more aligned can make the optimal signal either more or less informative*." (pp. 2604-2605). Recently, Curello and Sinander (2023) deliver a more comprehensive analysis for the comparative statics of aligning interests between a sender and a receiver in linear persuasion games.

if $\chi < \underline{\phi}$ we will have $a^* = \chi$ and $b^* = \underline{\phi} > \chi$ by Theorem 2. The expansion of b^* from χ to $\underline{\phi}$ is driven by the demand to effectively persuade the pivotal voter. The resulting structure is very similar to the judge example in Kamenica and Gentzkow (2011) in that the pooling message "$k > \underline{\phi}$" just makes the pivotal voter indifferent ex-ante. It is therefore the difficulty to persuade voters that produces a non-trivial revelation interval in the sender's asymptotically optimal information policy.

If the sender is prosocial with $\rho > 0$ (while still supposing $v_q^* \neq \phi^*$), then by Lemma 6 we have $\phi^* \neq z^*$ and hence $a^* < b^*$ by (16). Therefore, unlike a self-interested sender, a prosocial sender's optimal information policy will contain a non-trivial revelation interval as $n \to \infty$ even when the ex-ante conflicts of interests are arbitrarily small. Figure 5 illustrates the optimal revelation interval for a prosocial sender as $n \to \infty$ when $\phi^* < v_q^*$. In contrast to the previous case with $\rho = 0$, here the emergence of a non-trivial revelation interval in the limit is not driven by the difficulty of persuading the pivotal voter. It instead stems from the fact that the sender is uncertain about his preference (as it depends on voters' private types) and he must infer this through the pivotal voter's choice.

Finally, we study the sender's payoff. Let W^* denote the sender's asymptotic payoff under the optimal information policy as $n \to \infty$.[35] We compare W^* with two important benchmarks $\overline{W}$ and W^{Full}. Here, $\overline{W}$ is the sender's payoff under his *omniscient* control – i.e., he directly observes state k and voters' type profile v and dictates the election outcome – as $n \to \infty$. W^{Full} is the sender's payoff under full information disclosure as $n \to \infty$. Proposition 3 gives the rankings of $\overline{W}$, W^* and W^{Full}.

Proposition 3. *Suppose $v_q^* \in (-1,1)$. Then $\overline{W}$, W^* and W^{Full} are ranked as follows:*

1. *If $v_q^* = \phi^*$, then $\overline{W} = W^* = W^{Full}$;*
2. *If $v_q^* \neq \phi^*$ and $\underline{\phi} \leq \phi^* \leq \overline{\phi}$, then $\overline{W} = W^* > W^{Full}$;*
3. *If $\phi^* < \underline{\phi}$ or $\phi^* > \overline{\phi}$, then $\overline{W} > W^* > W^{Full}$.*

To understand Proposition 3, compare first W^* and W^{Full}. Note that $W^* \geq W^{\text{Full}}$ necessarily holds because a sender can always opt for full disclosure, under which the pivotal voter of type v_q^* implements the reform if $k > v_q^*$ and maintains the status quo otherwise. Hence, whenever an ex-ante conflict of interests exists (i.e. $v_q^* \neq \phi^*$), the sender is strictly worse off under full disclosure than under his optimal information policy (i.e., $W^* > W^{\text{Full}}$). Intuitively, full disclosure implies that the conflicting states between ϕ^* and v_q^* are fully revealed, while the optimal information policy should avoid doing so as much as possible.

Next, we relate W^* to the 'omniscient control' benchmark $\overline{W}$. Recall that $[\underline{\phi}, \overline{\phi}]$ is the set of feasible implementation thresholds t^* in the limit. Whenever ϕ^* falls within this set, the sender can secure his preferred outcome with probability one as $n \to \infty$ by setting $t^* = \phi^*$. In that case he receives payoff $\overline{W}$, which is what he could secure under omniscient control. Note that $\phi^* \in [\underline{\phi}, \overline{\phi}]$ holds if either (i) ϕ^* is close to v_q^* so that the ex-ante conflict of interests between the sender and the

[35] That is, $W^* := \lim_{n\to\infty} W_n$, where W_n is the value of persuasion problem (MP) with electorate size n.

pivotal voter is low, or (ii) v_q^* is close to $\mathbb{E}_F[k]$ so that, a priori, the pivotal voter is almost indifferent between the reform and the status quo. In the latter case very weak evidence is already sufficient to persuade the pivotal voter (and thus $[\underline{\phi}, \overline{\phi}]$ is wide). If ϕ^* lies below the feasible set ($\phi^* < \underline{\phi}$), the sender's preferred alternative will (as $n \to \infty$) almost surely not be elected when $k \in (\phi^*, \underline{\phi})$. The sender therefore gets strictly less than $\overline{W}$. A similar intuition applies when $\phi^* > \overline{\phi}$.

Theorem 2 and Proposition 3 together yield interesting policy implications for a social planner who can affect both the voting rule and public information. These results suggest that, as $n \to \infty$, the planner's optimal strategy is to choose q such that $v_q^* = \phi^*$ and in the meanwhile adopt a cutoff policy that only announces whether k is above or below ϕ^*. These imply that, somewhat contrary to conventional wisdom, in large elections even a benevolent social planner would optimally opt for a very coarse information policy despite that he can completely eliminate the ex-ante conflict of interests between the pivotal and average voters by wisely setting the voting rule. Importantly however, this by no means suggests that the planner should suppress other entities, such as mass media outlets, from reporting additional public information about k. This is because under $v_q^* = \phi^*$ while the welfare-optimal information policy in the limit is coarse, $W^* = W^{\text{Full}}$ holds so that additional information revelation cannot strictly lower asymptotic welfare. Therefore, to maximize welfare, the planner should only ensure that voters are informed about whether the state is above or below the critical cutoff ϕ^*. So long as this is done, any extra public information disclosure by others cannot substantially harm welfare in large elections and hence should not be suppressed.

6.2 Comparative statics

In this subsection we derive comparative statics for how the boundaries a_n and b_n of the optimal revelation interval vary with the sender's preference and the voting rule. Omitted proofs for this subsection are in Online Appendix E.

We first study the effects of shifting the sender's preference towards the reform, holding ρ fixed. Such a shift can occur, for instance, if $\rho < 1$ and χ decreases. In that case the sender's personal payoffs from reform increase and his threshold for accepting reform decreases. Proposition 4 shows how such a preference shift towards reform affects the sender's optimal revelation interval.

Proposition 4. *Suppose $\rho < 1$ and either condition (i) or (ii) in Lemma 3 holds.*[36] *Consider any $\chi_I > \chi_{II}$. Then, for sufficiently large n, as χ decreases from χ_I to χ_{II} the following holds:*

1. *a_n weakly decreases, strictly so if $a_n \in (-1, 1)$ under $\chi = \chi_I$.*
2. *b_n weakly decreases, strictly so if $b_n \in (-1, 1)$ under $\chi = \chi_I$.*

In words, as the sender's personal payoff from reform increases, he will pool fewer states downwards but more states upwards.

[36] Recall that these conditions are (i) ρ is sufficiently close to 0, or (ii) both G and $1-G$ are strictly log-concave.

The intuition of Proposition 4 is as follows. On the one hand, as the sender's personal payoff from reform increases, he becomes more tempted to persuade voters to pass the reform. Therefore, b_n decreases because the sender is now tempted to manipulate voters' beliefs upwards in some states that he would previously have been willing to reveal truthfully. On the other hand, such a preference shift makes the sender less tempted to persuade voters to maintain the status quo. Consequently, a_n also decreases because the sender is now willing to truthfully reveal some states he would previously pool to manipulate voters' beliefs downwards.

For a prosocial sender with $\rho > 0$, a similar preference shift towards the reform could also occur if his welfare weighting function $w(\cdot)$ decreases in first order stochastic dominance (cf. Proposition 1.4). In this way the sender systematically puts more weights on voters whose ex-post types are lower and hence receive higher payoffs under reform. Such a change also makes the sender more tempted to persuade voters to pass the reform. Following the intuition discussed above, one may expect that the result in Proposition 4 continues to hold in this case. In Online Appendix E we confirm that, under mild conditions, this is indeed true (cf. Proposition E.1 therein).

Next we turn to the effects on a_n and b_n of changing voting rule q, the required vote share to pass the reform. The results depend critically on whether the sender is self-interested ($\rho = 0$) or prosocial ($\rho > 0$). We consider the self-interested case first.

Proposition 5. *Suppose $\rho = 0$ and consider any $q_I, q_{II} \in (0,1)$ with $q_{II} > q_I$. Then, for sufficiently large n, as q rises from q_I to q_{II} the following holds:*

1. *a_n weakly increases, strictly so if $a_n \in (-1,1)$ under $q = q_I$.*
2. *b_n weakly increases, strictly so if $b_n \in (-1,1)$ under $q = q_I$.*

In words, if the sender is self-interested, then increasing the required vote share to pass the reform makes him pool more states downwards but fewer states upwards.

Proposition 5 is entirely driven by a *stringency effect*: as q increases it becomes harder to persuade the pivotal voter to pass the reform while easier to convince her to keep the status quo (Alonso and Câmara, 2016*a*; Curello and Sinander, 2023).[37] This is because the pivotal voter's threshold of acceptance $v^{(nq+1)}$ increases in q. For a self-interested sender who does not care about voter welfare, his best response would be to shift up both thresholds a_n and b_n. By raising b_n the sender makes message "$k > b_n$" more effective in persuading the pivotal voter – who is now harder to convince – to pass the reform. At the same time, the demand for the effectiveness of message "$k < a_n$" is lower because it is now easier to convince the pivotal voter to keep the status

[37] Despite the similarity in intuition, Alonso and Câmara (2016*a*) study a model where voters have no private types. Closer to ours, Curello and Sinander (2023) allow voters to have private types but assume that the sender is self-interested with state-independent preference and that g is strictly log-concave. For this special case they obtain similar comparative statics for raising the threshold to pass the reform. Our results show that the stringency effect holds for generic g (so long as n is large enough) and is relevant also for a sender who cares about voters' welfare (see below).

quo. Therefore, by increasing a_n the sender can expand the set of states in which he can successfully persuade the pivotal voter to maintain the status quo at minor costs of reduced effectiveness.

An important implication of Proposition 5 is that, under the stringency effect alone, both a_n and b_n are non-decreasing in q. Proposition 6 shows that, perhaps surprisingly, such monotone effects of q on a_n and b_n no longer hold for a prosocial sender that cares about voter welfare. The intuition here is that, besides a stringency effect, a shift in q then has an additional effect because it also shifts the prosocial sender's indifference curve. This second effect stems from the inference problem based on the pivotal voter's choice explained in Section 4.

Proposition 6. *Suppose $\rho > 0$, both G and $1-G$ are strictly log-concave, n is sufficiently large, and $-1 < a_n < b_n < 1$ holds under $q = q_I$. Then there exist $q_I, q_{II} \in (0,1)$ with $q_{II} > q_I$ such that, as q increases from q_I to q_{II}, any one of the following may happen:*

1. *a_n strictly decreases and b_n strictly increases.*
2. *a_n strictly increases and b_n strictly decreases.*
3. *both a_n and b_n strictly decrease.*
4. *both a_n and b_n strictly increase.*

In words, if the sender is prosocial, then raising the required vote share to pass the reform may make him reveal more states both upwards and downwards, pool more states in both directions, or reveal more states in one direction and pool more states in the other direction.

Proposition 6 shows that for a prosocial sender an increase in the vote share required to pass the reform may shift a_n and b_n both downwards or in opposite directions, which are impossible under the stringency effect alone. This result is driven by a novel *sender-preference effect*; increasing the vote share q required to pass the reform induces a shift of a prosocial sender's preference towards the reform (cf. Proposition 1.3). Therefore, following the intuition discussed above for the effects of sender-preference shifts, this induced sender-preference effect *per se* drives both a_n and b_n downwards as q increases. The net effect of an increase in q then depends on the relative strengths of the stringency and sender-preference effects. For instance, if the sender-preference effect dominates in driving a_n while the stringency effect dominates in driving b_n, then the net effect would be a strict expansion of the revelation interval $[a_n, b_n]$ on both sides.[38] Conversely, if the sender-preference effect dominates in driving b_n while the stringency effect dominates in driving a_n, then the net effect would be a strict reduction of the revelation interval $[a_n, b_n]$ on both sides.

[38] This case can be illustrated by the shift from panel (a) to (b) in Figure 5. There, an increase in q raises both v_q^* and $\underline{\phi}$ but lowers z^* through shifting down $\phi(\cdot)$. This strictly expands the (limit) revelation interval in both directions.

7 Competition in persuasion with multiple senders

In this section we extend our model to allow for competition in persuasion with multiple senders. We show that our first main result – that the single-crossing property ensures the optimality of interval revelation in sufficiently large elections – continues to hold under competition. Moreover, we establish our second main result: if the single-crossing property holds for all senders, then under a mild regularity condition the minimally informative equilibrium outcome under competition simply follows from each sender's optimal information policy under monopoly. Therefore, to pin down the equilibrium outcome under competition, it suffices to solve for each individual sender's monopoly problem. This also yields an intuitive sufficient condition for full disclosure being the unique competitive equilibrium outcome.

Specifically, we consider a setup with multiple senders competing in persuading voters a la Gentzkow and Kamenica (2017*b*). Let there be a set M of senders with $|M| \geq 2$. For each sender $m \in M$, his preference is characterized by utility function (1) with parameters $\rho_m \in [0,1]$, $\chi_m \in \mathbb{R}$, and weighting function $w_m(\cdot)$. In this way, for each sender m we can obtain his indifference curve $\phi_n^m(\cdot)$ via (3). We assume that all senders' preferences are commonly known among themselves. Each sender m simultaneously chooses an information policy π_m from the feasible set Π, prior to observing the realization of k. Given profile $\{\pi_m\}_{m\in M}$, we denote by $\pi := \left\langle \{\pi_m\}_{m\in M} \right\rangle$ the *joint information policy* induced by observing the signal realizations from all π_m's. Notice that $\pi \in \Pi$ necessarily because it is clearly feasible. In this way, our information environment is *Blackwell-connected*; given any strategy profile $\pi_{-m} := \left\{\pi_j\right\}_{j\in M\setminus\{m\}}$ of other senders, each sender m can unilaterally deviate to any feasible joint information policy that is Blackwell more informative than $\langle \pi_{-m} \rangle$.[39] We focus on equilibria in pure and weakly undominated strategies.[40] The equilibrium derivation and proofs for all results in this section are in Online Appendix F.

Suppose the single-crossing property holds for a sender $m \in M$ and let z_n^m and $W_n^m(\cdot)$ denote, respectively, the implied switching state and indirect utility function.[41] Then, by Theorem 1, there exists $N_m \geq 0$ such that for all $n \geq N_m$ sender m's monopolistically optimal information policy is characterized by a revelation interval $[a_n^m, b_n^m]$ that satisfies $a_n^m < z_n^m < b_n^m$. Moreover, by Lemma 5 in Section 5.2, for any $n \geq N_m$ there are two cutoffs $\ell_n^m \in [-1, a_n^m]$ and $r_n^m \in [b_n^m, 1]$ such that $W_n^m(\cdot)$ is strictly convex on $[\ell_n^m, r_n^m]$ and is strictly concave on $[-1, \ell_n^m]$ and $[r_n^m, 1]$. Theorem 3 shows that the optimality of interval revelation policies generalizes to competition in persuasion.

[39] Formally, for any $\pi, \pi' \in \Pi$, π is Blackwell more informative than π' if and only if $H_\pi \succeq_{MPS} H_{\pi'}$.

[40] For competition with multiple senders, the restriction to pure strategies is often made in the literature (Gentzkow and Kamenica, 2017*a*,*b*; Mylovanov and Zapechelnyuk, 2021), but may nevertheless have substantive consequences. As noted in Gentzkow and Kamenica (2017*b*) (page 318), when senders can use mixed strategies the information environment is no longer Blackwell-connected, which is a key property we use to derive equilibria under competition. Li and Norman (2018) show by means of examples that allowing for mixed strategies indeed changes the set of equilibria.

[41] Namely, z_n^m and $W_n^m(\cdot)$ are respectively defined by (6) and (9), with $\phi_n(\cdot)$ therein replaced by $\phi_n^m(\cdot)$.

Theorem 3. *Suppose the single-crossing property holds for sender m. Then for all $n \geq N_m$, there exists a subset of interval revelation policies $\mathscr{P}_n^m$ such that for any pure strategy profile π_{-m} of senders other than m (which need not be interval revelation), there is some $\pi_m \in \mathscr{P}_n^m$ that is sender m's best response to π_{-m}. Moreover, the best response set $\mathscr{P}_n^m$ is characterized by*

$$\mathscr{P}_n^m := \{\mathscr{P}(c,d) : [a_n^m, b_n^m] \subseteq [c,d] \subseteq [\ell_n^m, r_n^m]\} \ . \tag{BR}$$

In words, $\mathscr{P}_n^m$ is the set of all interval revelation policies whose revelation interval (i) contains sender m's optimal revelation interval under monopolistic persuasion, and (ii) is contained by the maximal interval on which the sender's indirect utility function $W_n^m(\cdot)$ is strictly convex.

An important implication of Theorem 3 is that the single-crossing property ensures the optimality of interval revelation policies for a sender even when voters can get additional public information from arbitrary sources other than the sender.

For the remainder of this section we impose the following assumption:

Assumption 1. *The following conditions hold:*

1. *The single-crossing property holds for each sender $m \in M$.*
2. *For all $m \in M$ and $n \geq 0$, $\phi_n^{m\prime}(x) < 2$ holds on $[-1,1]$.*

By Lemma 3, both conditions in Assumption 1 hold – independently of the senders' preferences and the voting rule – if both G and $1-G$ are strictly log-concave. Following Gentzkow and Kamenica (2017*b*), we say an equilibrium is *minimally informative* if the joint information policy it induces is no more Blackwell informative than any information policy that can be induced by some other equilibrium. Theorem 4 characterizes the (essentially unique) outcome induced by the minimally informative equilibria under competition in persuasion.

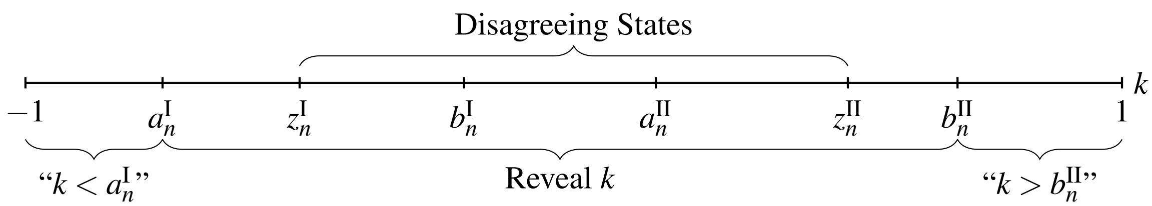
Theorem 4. *Suppose Assumption 1 holds and let $N := \max_{m \in M} N_m$. Then, for all $n \geq N$, the following properties hold:*

1. *In any minimally informative equilibrium the joint information policy induced by all senders is outcome equivalent to an interval revelation policy with revelation interval $\left[a_n^{\min}, b_n^{\max}\right]$, where $a_n^{\min} = \min_{m \in M}\{a_n^m\}$ and $b_n^{\max} = \max_{m \in M}\{b_n^m\}$.*

2. *If each sender $m \in M$ is restricted to use interval revelation policies from his best-response set $\mathscr{P}_n^m$ given by (BR), then the minimal informative equilibrium is the unique equilibrium in pure and weakly undominated strategies.*

Theorem 4 suggests that, under a weak regularity condition (i.e., part (2) of Assumption 1), if the single-crossing property holds for all senders then there is a simple two-step algorithm to derive the unique outcome of the minimally informative equilibria. First, solve the monopolistic persuasion problem for each sender m and obtain the optimal revelation interval $[a_n^m, b_n^m]$. Second, take the convex hull of all these optimal intervals. The interval revelation policy with this resulting revelation interval is then the unique outcome of the minimally informative equilibria when these senders compete in persuading voters. Figure 6 illustrates this for the case with two senders.

Figure 6: Minimally Informative Equilibrium with Two Senders



Note: $[a_n^m, b_n^m]$ denotes the optimal revelation interval under monopolistic persuasion for sender $m \in \{\mathrm{I}, \mathrm{II}\}$.

It is well known that multiple equilibria exist under competition in public Bayesian persuasion. The literature typically focuses on minimally informative equilibria that are Pareto-optimal among senders (Gentzkow and Kamenica, 2017*b*). Building on part (2) of Theorem 4, we provide an additional argument in favor of the minimal informative equilibria by restricting senders to use interval revelation policies from their best response sets. Under these restrictions, the minimally informative equilibrium is the unique equilibrium in pure and weakly undominated strategies. Due to this favorable equilibrium selection, all senders would indeed prefer these restrictions to be enforced. This would help them to avoid the risk of coordinating on equilibrium outcomes that are excessively informative and thereby would make all senders strictly worse off.

Finally, we discuss an interesting corollary of Theorem 4, which gives a neat sufficient condition for full information disclosure to be the unique equilibrium outcome. To do so we introduce the notion of '*disagreeing states*'. State k is a disagreeing state if there exist at least two senders $\mathrm{I}, \mathrm{II} \in M$ who are weakly biased towards different alternatives relative to the pivotal voter (formally, there exist senders $\mathrm{I}, \mathrm{II} \in M$ with $\phi_n^{\mathrm{I}}(k) \leq k \leq \phi_n^{\mathrm{II}}(k)$). These two senders thus have incentives to manipulate voters' beliefs in opposite directions. Disclosing more information then always strictly benefits at least one of these senders. This in the end leads to full revelation of all such states. When the single-crossing property holds for all senders $m \in M$, the set of disagreeing states is precisely the interval $[z_n^{\min}, z_n^{\max}]$, where $z_n^{\min} = \min_{m \in M}\{z_n^m\}$ and $z_n^{\max} = \max_{m \in M}\{z_n^m\}$. Because $a_n^{\min} \leq z_n^{\min}$ and $b^{\max} \geq z_n^{\max}$, Theorem 4 implies that all disagreeing states must be revealed in any equilibrium. The observation above yields the following corollary.

Corollary 3. *Suppose Assumption 1 holds. If there exist two senders $I, II \in M$ with $z_n^I = -1$ and $z_n^{II} = 1$, then full disclosure is the unique equilibrium outcome.*

This corollary says that full disclosure must hold in all equilibria whenever there are two senders who are uniformly more biased towards different alternatives than the pivotal voter. This holds, for example, if the competition is between two self-interested senders who always favor opposite alternatives (i.e., $\rho_{\text{I}} = \rho_{\text{II}} = 0$, $\chi_{\text{I}} \leq -1$ and $\chi_{\text{II}} \geq 1$), or between a 'pro-Reform' planner and an 'anti-Reform' planner under simple majority rule (cf. Examples 3 and 4 in Section 5). In the latter case both senders aim at maximizing voters' payoffs and they just differ in their welfare weights. Therefore, contrary to many prior papers on competition in Bayesian persuasion (Gentzkow and Kamenica, 2017*a*,*b*; Cui and Ravindran, 2020), we show that a stark conflict of interests between senders is far from necessary to induce full disclosure as the unique equilibrium.

8 Applications

In this section we discuss three applications of our model, including both large-scale elections (Section 8.1) and small-scale committee voting (Sections 8.2 and 8.3). In all these applications we focus on welfare implications, which best reflect the novelty and relevance of our results.

8.1 The welfare effect of media competition in large elections

Building on Corollary 3 and the asymptotic results in Section 6.1, we present a straightforward but insightful application to study how media competition affects voter welfare. Let $M = \{\text{I}, \text{II}\}$ and interpret these two senders as public mass media outlets. Suppose that both outlets are partisan and opposite-minded; that is, their preferred alternatives are opposite and state-independent. We model this by assuming $\rho_{\text{I}} = \rho_{\text{II}} = 0$, $\chi_{\text{I}} \leq -1$ and $\chi_{\text{II}} \geq 1$. It is then immediate from Corollary 3 that competition between these two media outlets leads to full disclosure in all equilibria.[42]

Now consider a utilitarian social planner who wants to maximize voters' welfare in large elections as $n \to \infty$. His ex-ante threshold to accept the reform is then $\phi^* = \mathbb{E}_G[v]$, the expected type of the average voter.[43] The asymptotic welfare under media competition is W^{Full} because full disclosure is the unique equilibrium outcome. Under the second-best benchmark (i.e., the planner implements his own optimal information policy), the asymptotic welfare is W^*. By Proposition 3, $W^* \geq W^{\text{Full}}$ and the strict inequality holds whenever $v_q^* \neq \phi^*$, that is, when there is an ex-ante conflict of interests between the average and the pivotal voters. As such conflict of interests increases,

[42] As noted in the discussion below Corollary 3, the same result can hold even if media outlets are not purely partisan, but intrinsically care about voters' welfare like the pro-reform and anti-reform senders in Examples 3 and 4, respectively. Therefore, all results in this application continue to hold when media outlets have prosocial preferences, so long as they are uniformly more biased towards different alternatives than the pivotal voter.

[43] To see this, recall that a utilitarian planner's welfare weighting function is $w(y) = y$ for $y \in [0,1]$. Together with $\rho = 1$, this implies $\phi^* = \int_0^1 G^{-1}(y)dw(y) = \int_{\underline{v}}^{\overline{v}} x dG(x) = \mathbb{E}_G[v]$. All analyses here apply similarly to any non-utilitarian social planner with a different weighting function $w(\cdot)$ (because this affects asymptotic welfare only via ϕ^*).

media competition may result in greater welfare losses due to the excessive information revelation it induces. Therefore, to correctly evaluate the welfare effect of media competition, one must account for broader characteristics of the electoral and institutional background – such as the distribution of voter preferences and voting rules – that are important for determining v_q^* and ϕ^*.

Despite its straightforwardness, our result sheds important lights to the debate over the welfare effect of media competition. Most studies in the related literature assume from the outset that more information is better for welfare and thus focus their analyses on whether media competition improves information revelation.[44] This would lead one to conclude that media competition is ideal for welfare if it induces full disclosure, and the regulation of media market should simply aim at boosting information revelation. Our result suggests that this seemingly appealing wisdom is incomplete, as it overlooks the fact that the actual election process cannot always implement the welfare-optimal outcome even under full information. When such friction exists, media competition may harm welfare by inducing too much, rather than too little, information revelation. To maximally avoid the perverse welfare effect of excessive information revelation, the voting rule must be carefully designed to align the interests of the average and pivotal voters.

8.2 Optimal information provision to a committee: Private values

In this subsection and the next we move away from large-scale elections and apply our model to small committees. In doing so, we show that most of our insights and results derived in previous sections are also relevant in non-political contexts with small voting bodies.

We consider a 'population' interpretation of our model as inspired by recent works in Bayesian persuasion (Lin and Liu, 2023; Doval and Smolin, 2023). To fix ideas, suppose a science foundation received a large number (say a unit mass) of research proposals, which are heterogeneous in quality. Let $F(\cdot)$ be the quality distribution of these proposals; $F(x)$ is then the fraction of proposals with quality $k \leq x$. Each proposal is independently evaluated by its own single committee. The committee consists of $n+1$ members, who apply some super-majority rule q to jointly decide whether or not to advise in favor of funding a proposal; the proposal is positively evaluated if and only if it receives more than nq votes of approval. While all members see higher quality as a merit, their thresholds of approval – v_i for each member i – may differ. One way to interpret this is that each member is specialized in evaluating the proposal in a distinct dimension (e.g., economic impact,

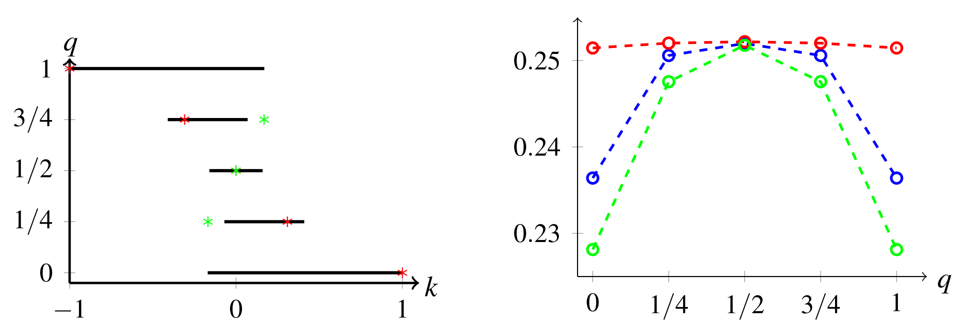
[44] For example, it has been argued that media competition can benefit voters and improve political accountability by increasing the costs of media capture that suppresses disclosure of unfavorable information to the politician (Besley and Prat, 2006). Competition may also discipline media outlets to provide information that aligns better with the interests of their audiences (Gentzkow and Shapiro, 2006; Chan and Suen, 2008). These papers conclude that media competition is welfare-improving because it induces better information disclosure. On the other hand, the literature also identifies channels through which media competition can deteriorate voter welfare by reducing information disclosure. For instance, competition can drive profit-maximizing media to invest fewer resources in the provision of political news or topics of common interests (Chen and Suen, 2018; Cagé, 2019; Perego and Yuksel, 2022).

costs, profitability) and evaluates the quality of the proposal against this dimension. The committee takes all these dimensions into account with equal weight. It is therefore socially optimal to fund a proposal of quality k if and only if $k > \tilde{v}$, where $\tilde{v}$ is the sample mean of $(v_1, \cdots, v_{n+1})$. Each v_i is independently drawn from a common distribution G with log-concave density g. For ease of exposition, we impose the following parametric assumptions.

Example 5. *The committee consists of five members, that is $n = 4$. The feasible super-majority voting rules are then $q \in \{0, 1/4, 1/2, 3/4, 1\}$. g is the density of a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 0.25$. F is the cdf of a uniform distribution on $[-1, 1]$.*

Committee members cannot directly observe the quality of the research proposals. An information intermediary, instead, can observe and publicly release information about the quality of all proposals. We first consider a utilitarian intermediary and study the welfare-optimal public information policy. In practice, such a utilitarian intermediary can be, for instance, a division within the science foundation that assesses the quality of all proposals and then assigns each proposal a rating that will be subsequently reported to the decision committee. Recall that condition (2) of Proposition 2 provides us an easy way to check whether interval revelation is optimal for all $n > 0$ for a utilitarian sender by examining the monotonicity of function (12), which depends on only g and the voting rule q. In Online Appendix G we verify that this condition is indeed satisfied for all the combinations of g and q in Example 5. Therefore, the utilitarian optimal information policy is characterized by a revelation interval $[a_n, b_n]$ whose boundaries are pinned down by the complementary slackness conditions (FOC: b_n) and (FOC: a_n) in Section 5.2.

Figure 7: Welfare-Optimal Revelation Intervals (left) and Welfare (right)



Note: Model parameters are taken from Example 5. In the left panel the solid black lines represent the utilitarian optimal revelation intervals. The red and green stars denote, respectively, z^* and v_q^* as functions of q. In this example, $v_q^* = z^* = \phi^* = 0$ if and only if $q = 1/2$. In the right panel, the blue and green circles depict the committee's welfare under the optimal interval revelation policy and under full disclosure, respectively. The red circles depict the committee's welfare with rational voters in a common value setting (cf. Section 8.3).

The left panel of Figure 7 presents the welfare-optimal revelation intervals for the environment in Example 5 for each feasible voting rule. Consistent with our central insight, full transparency is generically suboptimal for welfare. In fact, many results and properties about the optimal revelation intervals established in Section 6 for large elections continue to hold here, despite the small electorate size. For example, in line with the comparative statics derived in Section 6.2, both boundaries of the optimal revelation intervals vary non-monotonically with q. This suggests that both the stringency and sender-preference effects are relevant even with only five voters. Moreover, consistent with Theorem 2, even in the absence of ex-ante conflict (i.e., $q = 1/2$ and $v_q^* = \phi^*$) the optimal information policy is very coarse because the revelation interval is quite narrow.

The right panel of Figure 7 illustrates how the committee's welfare varies with voting rule q under both optimal information policies (blue) and full disclosure (green). In both regimes, welfare is single-peaked in q and maximized at $q = 1/2$, where $v_q^* = \phi^*$ holds. Consistent with Proposition 3.1, at this peak the welfare gap is quite small, confirming that extra information disclosure cannot substantially harm welfare when the ex-ante interests of the average and pivotal voters are aligned. If instead $q \neq 1/2$ so that $v_q^* \neq \phi^*$, then full transparency can substantially reduce welfare and such harm increases as q deviates further away from $1/2$. This suggests that scrutiny with information provision is most relevant when the voting rule is not optimally set in that it does not eliminate the ex-ante conflict of interests between the average and pivotal voters.

Our exercise here contributes to the literature on information intermediaries by providing a relevant yet under-explored argument for why an intermediary may optimally withhold valuable information. The literature proposes several explanations for this. These include, among other things, the need to entice participation by upstream clients (Lerner and Tirole, 2006; Farhi, Lerner and Tirole, 2013; Harbaugh and Rasmusen, 2018), conflicts of interests between the intermediary and evaluators (Ostrovsky and Schwarz, 2010), and the need to motivate costly efforts by agents to improve their quality (Dubey and Geanakoplos, 2010; Boleslavsky and Cotton, 2015; Onuchic and Ray, 2023). None of these frictions appear in our setup. Unlike these earlier works, in our model the intermediary is utilitarian and it provides information to optimally guide decision making by a committee of members with heterogeneous private preferences. It is the ex-post conflict between pivotal and average voters that renders full transparency suboptimal.

Finally, we use the 'population' interpretation discussed in this subsection to formally show in what sense our full commitment assumption can be relaxed. We thus shift away from the specific case in which the information intermediary is utilitarian and allow it to have a more general preference as given by (1) in Section 3. This specification requires that, all else equal, the intermediary is better off if a proposal with higher quality is funded. This seems reasonable in many relevant cases in practice.[45] We also remove the parametric assumptions of Example 5.

[45] For example, consider a university who can observe and provide information about qualities of the research proposals proposed by its affiliated scholars. The university may prefer its proposals to get funded regardless of their underlying qualities. However, conditional on a proposal being funded, the university typically enjoys a higher (material or reputational) payoff if that proposal indeed has a better quality. Such self-interested preference can be captured by (1) with $\rho = 0$ and $\chi < -1$. Since g is log-concave, it follows from Theorem 1 and part (1) of Proposition 2 that

The general underlying idea is that interval revelation policies can be effectively enforced by simple quota restrictions that do not rely on the intermediary's ability to commit. With quota restrictions in place the intermediary may still deviate ex-post from the message she originally intended to send, yet the resulting overall message distribution should always stay in keeping with the quota restrictions. This idea is formalized by the notion of credibility proposed by Lin and Liu (2023). Specifically, an information policy π is *credible* if, given that voters update their posterior beliefs based on π via Bayes rule, the sender has no incentive to deviate to any other π' that (i) shares the same message space S with π and (ii) its marginal distribution on messages coincides with that of π. That is, π is credible if the sender have no incentive to covertly manipulate its signals ex-post so long as he cannot change the marginal distribution of messages. Formally, in Online Appendix G we show

Proposition 7. *If the intermediary's payoff function is given by* (1) *in Section 3, then all interval revelation policies (i.e., all* $\mathscr{P}(a,b)$ *with* $[a,b] \subseteq [-1,1]$*) are credible.*

To see why this proposition holds, first observe that our interval revelation policy is equivalent to a *quality rating scheme* that induces the following classification of proposals according to their quality: proposals with high quality $k > b$ are rated as "*good*", those with low quality $k < a$ are rated as "*bad*", and each project with a mediocre quality $k \in [a,b]$ receives a more precise rating "*quality is k.*" Under this scheme, a proposal with a higher actual quality k is strictly more likely to be approved for funding. This is indeed credible because if the intermediary covertly deviates by giving a low or mediocre quality proposal a higher rating, then he must assign some high quality proposal a lower rating to preserve the marginal distribution of messages. This inevitably lowers the funding chance of the project with a higher quality, which is undesirable for any intermediary with preference given by (1). Finally, we note that in practice the science foundation can enforce any marginal distribution of messages through a simple quota allocation that assigns a fixed budget to each message while keeping the aggregate number of all messages equal to that of all proposals. Together with Proposition 7, this implies that the science foundation can credibly implement the welfare-optimal information policy by simple quota restrictions, despite that the intermediary can have misaligned preferences (given by (1)) and may covertly manipulate signals ex-post.

8.3 Optimal information provision to a committee: Common values

In this subsection we continue to study the optimal information provision to a committee. Now, we consider an environment with common values and private signals. To do so, we modify our

independently of the committee size the optimal information policy for a such self-interested intermediary is featured by pooling high quality levels: the optimal revelation interval takes the form $[-1,\tilde{b}_n]$ for all $n > 0$. This in general differs from the utilitarian optimal interval revelation policies as in the left panel of Figure 7.

setup by making all voters' payoffs identical and equal to $k-\tilde{v}$ if a proposal with quality k is funded, where $\tilde{v}$ still equals the sample mean of $(v_1,\cdots,v_{n+1})$. It is thus again socially optimal to fund a proposal if and only if $k>\tilde{v}$. In this context, v_i's can be interpreted as committee members' private signals, which are independently drawn from distribution G, and these signals jointly determine a common value component $\tilde{v}$.[46] We still study the optimal information policy by a utilitarian intermediary, but our analyses distinguish between whether voters are rational or not.

Boundedly rational voters. We first assume that voters are boundedly rational in that they erroneously take their own private signals as the optimal estimate of the common value. Such a cognitive mistake could arise from a (strong) projection bias, which is a well-documented psychologically phenomenon. To model projection bias, we follow the related literature (Loewenstein, O'Donoghue and Rabin, 2003; Breitmoser, 2019; Gagnon-Bartsch, Pagnozzi and Rosato, 2021) by assuming that each voter i assigns weight $1-\sigma$ to her prior that other voters' types are independently drawn from G, and weight σ to that all other voters' private types are equal to her own type v_i.[47] Here $\sigma\in[0,1]$ measures the *degree of projection*. For ease of exposition we consider the extreme case $\sigma=1$, that is, every member i believes that all others share exactly her own signal v_i. Given this erroneous belief, she would conclude that $\tilde{v}=v_i$ and hence vote in favor of a proposal if and only if $k>v_i$.

This setup is mathematically equivalent to the one studied in the previous subsection, and thus all our results there continue to hold. Consider again the model primitives in Example 5 but now in this common value environment. It follows that the utilitarian optimal information policies are characterized by the revelation intervals in the left panel of Figure 7. The committee's welfare under the optimal information policy and full disclosure are, respectively, given by the blue and green circles in the right panel of Figure 7. We again see that the committee's welfare is sensitive to voting rules, and excessive information revelation can substantially reduce welfare when q is far away from $1/2$ so that the ex-ante conflict between the average and pivotal voters is large.

More generally, our results suggest that full disclosure can be suboptimal, and thus the optimal design of information provision and voting rule is important and technically non-trivial, even in an environment where voters share common preferences. This stems from their bounded rationality, which prevents voters from optimally using their private signals to guide their votes. These insights are important because, except for the extreme projection bias studied here, bounded rationality can also emerge from various other sources and is ubiquitous in practice.

Rational voters. Now we turn to the case where voters are perfectly rational and show that, in

[46] Modeling a commonly valued state as a function (usually the sum, average, or maximum) of the profile of agents' iid private signals is convenient for its tractability. This modeling approach has been adopted by various papers to study common value auctions (Bulow and Klemperer, 2002; Goeree and Offerman, 2002; Bergemann, Brooks and Morris, 2020) and information cascades in social learning (Gale, 1996; Çelen and Kariv, 2004). Recently, He, Sandomirskiy and Tamuz (2022) refer to such signals as '*private private information*' and study their properties.

[47] That is, voter i with realized signal v_i believes that any other voter j's signal is $\sigma\cdot v_i+(1-\sigma)\cdot v_j$.

stark contrast to the previous case, full disclosure is optimal for welfare, independent of voting rule q. Unlike the boundedly rational voter discussed above, a rational voter would (correctly) infer the committee's average signals conditional on pivotal events and use this as her estimate of the common value component $\tilde{v}$. This is the familiar kind of pivotal inference that is central to the Condorcet jury voting literature (Feddersen and Pesendorfer, 1996, 1997). More specifically, conditional on being pivotal and $v_i = x$, a rational voter prefers to fund a proposal if and only if its expected quality $\theta \geq \phi_n(x)$, where $\phi_n(x)$ is the posterior expectation of $\tilde{v}$ conditional on $v^{(nq+1)} = x$ given n (cf. (7) in Example 2).[48] Note that $\phi_n(\cdot)$ is strictly increasing so its inverse exists. Therefore, given expected quality θ, only voters with type $v_i \leq \phi_n^{-1}(\theta)$ will vote for approval. The proposal is thus funded if and only if the pivotal voter's type $x \leq \phi_n^{-1}(\theta)$ and the committee's welfare is

$$\widehat{W}_n(\theta) = \int_{\underline{v}}^{\phi_n^{-1}(\theta)} (\theta - \phi_n(x))\, \hat{g}_n(x;q) dx \,.$$

Simple calculus shows that, for all $q \in [0,1]$, $\widehat{W}_n''(\theta;q) = \frac{\hat{g}_n\left(\phi_n^{-1}(\theta)\right)}{\phi_n'\left(\phi_n^{-1}(\theta)\right)} > 0$ so that $\widehat{W}_n(\cdot)$ is strictly convex on $[-1,1]$. It then follows from standard arguments (Kamenica and Gentzkow, 2011; Kolotilin, Mylovanov and Zapechelnyuk, 2022) that full disclosure is uniquely optimal regardless of the voting rule q. To see why this is true, note that in this common value environment the rational voter's inference based on pivotal events precisely coincides with our inference problem for a utilitarian planner were the pivotal voter irrational and approves the proposal if and only if $x \leq \theta$. By internalizing the planner's inference problem, rational voters now use the socially optimal threshold $\phi_i^{-1}(\theta)$ to make approval decisions and hence eliminate any misalignment of interests between the planner and the pivotal voter. Full disclosure is therefore unambiguously optimal for welfare, and more generally the committee is strictly better off with more information revelation.

Finally, we discuss implications for welfare. The red circles in the right panel of Figure 7 depict the committee's welfare under full disclosure, which we have shown is socially optimal with rational voters, for all feasible voting rules q under the primitives given in Example 5. It shows that the optimal welfare is still maximized at $q = 1/2$ (where $v_q^* = \phi^*$ holds) but it varies little with the voting rule. This suggests that, in stark contrast to the previous case with behavioral voters, with rational voters the choice of voting rules is not so important provided that information about qualities are fully disclosed. Another interesting observation is that when $q \neq 1/2$ so that $v_q^* \neq \phi^*$, there can be a substantial gap in welfare between the case with rational voters and full disclosure (red circles) and the case in which voters have strong projection biases and the public information is optimally provided to maximize welfare (blue circles). This gap in welfare is mainly caused by the difference in information supplied about proposals' quality. This happens because the pivotal

[48] This claim implicitly relies on that each voter's threshold to approve a proposal strictly increases with her private signal. This can be proved true using standard arguments; see, for instance, Feddersen and Pesendorfer (1997).

voter is much more likely to make mistakes ex-post under full information when she has a strong projection bias and adopts a wrong threshold of approval. The utilitarian planner is thus forced to withhold some valuable information about qualities to maximally avoid these mistakes. In contrast, with rational voters this problem is absent and the planner can safely provide all information.

9 Conclusion

This paper studies public Bayesian persuasion in elections with two alternatives. We do so in a general framework that simultaneously allows for heterogeneous voters, a rich set of sender preferences, any super-majority voting rule, and an arbitrary number of senders. In our model, sender(s) can strategically provide public information about a payoff-relevant state. Voters' payoffs depend on both the state and their own private types, which are unobservable to sender(s). Our first main result establishes a widely held condition that ensures the optimality of interval revelation policies, which reveal intermediate states but pool extreme ones, in sufficiently large elections. We also show that in many circumstances, this result carries over to voting in small committees. This condition features a single-crossing property over a sender's and the pivotal voter's indifference curves. Our second main result uncovers a simple way to relate the structure of the minimally informative equilibrium outcome under competitive persuasion to each sender's optimal information policy under monopolistic persuasion.

When there is only one sender, interval revelation is uniquely optimal in sufficiently large elections if the single-crossing property holds. When multiple senders compete in persuading voters, the single-crossing property ensures the optimality for a sender to restrict attention to a subset of interval revelation policies, provided that other senders use pure strategies only. Moreover, under a weak regularity condition, the outcome of any minimally informative pure-strategy equilibrium can be reproduced by an interval revelation policy whose revelation interval is simply the convex hull of each sender's optimal revelation interval under monopoly. This produces a neat sufficient condition to check when senders' competition in persuasion must induce full disclosure in all equilibria.

An important value of our framework is its ability to tractably characterize the welfare-optimal information policies for a rich set of social welfare functions and voting rules. Core to this is our analyses for the statistical inference problem based on the pivotal voter's choice, which determines the conflict of interests between a prosocial sender and the pivotal voter. This inference problem also gives rise to a novel sender-preference effect: ceteris paribus, raising the required vote share to pass an alternative makes any prosocial sender more leaning towards passing it. This effect yields new comparative static predictions for how the welfare-optimal information policy should vary with voting rules. Importantly, our inference problem is relevant independent of the electorate size. Therefore, our results provide economic insights to applications concerning both large-scale elections and small-scale committee decision making.

We conclude by discussing some limitations of our model and suggesting some avenues for future research. First, throughout the paper we focus on public persuasion and assume that voters' private types are unknown to any sender. This excludes the possibilities of *targeted persuasion* (i.e., sending different information to different voters) and *eliciting voters' private information* (e.g., by offering a menu of signals for voters). It would be interesting to extend our analyses to incorporate either or both possibilities in a general model of voting.[49] This would shed light on the strategic values of micro-targeting and screening for a monopoly sender in publicly persuading voters. Whereas this first limitation is applicable to any size of the voting body, the following two are related more to large-scale elections.

Second, we assume that the welfare weights senders assign to each voter depend on the voter's ex-post payoff ranking, but not on her identity or any other characteristic. Under our assumption that voters are ex-ante homogeneous this restriction is without loss of generality. In practice, however, voters do differ in many characteristics that might be observable to senders; e.g., gender, age, education, occupation, party affiliation, etc. These characteristics often systematically influence how voters fare under policy reforms; e.g., drivers are more likely to experience negative income shocks from a car fuel levy. In these scenarios it would be relevant to study the optimal information policy when the welfare weights can depend on such observable characteristics.

Third, in our model the policy reform affects voters homogeneously by shifting their payoffs under reform by a same amount. We impose this assumption for tractability, because in this way our information design problem can be solved using established linear programming techniques. In many real-life situations, however, policy reforms affect voters in heterogeneous, and sometimes even opposite, ways. For instance, citizens living in more polluted areas may benefit more from an environmental protection policy. Consequently, some citizens may prefer a more thorough policy reform while others may instead desire a softer one. In these scenarios policy reforms can affect the distribution of voters' payoffs, and sometimes may even induce preference changes in opposite directions among citizens and thus lead to polarized attitudes towards reforms. All these features are important issues in discourses of contemporary distributive politics. We therefore believe that exploring the implications of heterogeneous policy effects for information design in elections is a promising and highly relevant area for future research.

[49] Heese and Lauermann (2021) study in a binary-state model targeted persuasion by a monopoly sender whose preference is independent of voters' private types. They show that the sender can ensure his preferred alternative to win with probability one in equilibrium as the electorate size goes to infinity. It is unclear, however, whether their results continue to hold if the sender's preference can depend on voters' private types as in our model.

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Online Appendices for
"Public Persuasion in Elections: Single-Crossing Property and the Optimality of Interval Revelation"

Contents

A Derivation of $\hat{G}_n(\cdot;q)$ and the Proof of Lemma 1

In this appendix we derive $\hat{G}_n(\cdot;q)$ and prove Lemma 1 in Section 3. For any $y, q \in [0,1]$, define

$$\tau_n(y;q) := \frac{(n+1)!}{\lfloor nq \rfloor! \cdot \lceil n(1-q) \rceil!} y^{\lfloor nq \rfloor}(1-y)^{\lceil n(1-q) \rceil} \tag{A.1}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote, respectively, the floor and ceiling functions. In fact, $\tau_n(\cdot;q)$ is the density function of a Beta distribution $B(\alpha, \beta)$ with parameters $\alpha = \lfloor nq \rfloor + 1$ and $\beta = \lceil n(1-q) \rceil + 1$. The following properties about $\tau_n(y;q)$ are useful.

Lemma A.1. *Suppose nq is an integer. Then the following properties hold:*

(a) $\tau_n'(y;q) = \tau_n(y;q)\frac{n(q-y)}{y(1-y)}$.

(b) $\tau_n(y;q)$ *is increasing on* $[0,q)$ *and decreasing on* $(q,1]$.

(c) $\lim_{n\to\infty} \tau_n(y;q) = \infty$ *if* $y = q$ *and* $\lim_{n\to\infty} \tau_n(y;q) = 0$ *if* $y \neq q$.

Proof of Lemma A.1. Since nq is an integer, we can drop the floor and ceiling functions in (A.1). Taking the natural logarithm of $\tau_n(y;q)$ and computing its derivative then yields

$$\frac{\tau_n'(y;q)}{\tau_n(y;q)} = \frac{n(q-y)}{y(1-y)} .$$

Hence, $\tau_n'(y;q) > (<) 0$ for $y < (>) q$. This proves (a) and (b). To show (c), we use Stirling's formula to approximate $n!$ for all positive integer n: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.[1] With this approximation, we obtain

$$\tau_n(y;q) \approx \sqrt{\frac{n}{2\pi q(1-q)}} \left(\frac{y}{q}\right)^{nq} \left(\frac{1-y}{1-q}\right)^{n(1-q)} . \tag{A.2}$$

If $y = q$, then $\tau_n(y;q) \approx \sqrt{\frac{n}{2\pi q(1-q)}} \to \infty$. If $y \neq q$, we take the natural logarithm of (A.2) and get

$$\ln \tau_n(y;q) \approx \frac{1}{2}\ln n + n\psi(y;q) - \frac{1}{2}\ln 2\pi q(1-q) . \tag{A.3}$$

where

$$\psi(y;q) := q \ln \frac{y}{q} + (1-q)\ln\frac{1-y}{1-q} .$$

It holds that (i) $\psi(q;q) = 0$, and (ii) $\psi'(y;q) > (<) 0$ for $y < (>) q$. Therefore, if $y \neq q$, then $\psi(y;q) < 0$ and the right hand side of (A.3) converges to $-\infty$ as $n \to \infty$. This implies $\lim_{n\to\infty} \tau_n(y;q) = 0$ for $y \neq q$ and thus completes the proof for part (c). $\square$

[1] The expression $l_n \approx r_n$ denotes $\lim_{n\to\infty} \frac{l_n}{r_n} = 1$, where l_n and r_n are sequences of real numbers.

Lemma A.1 immediately extends to other values of q in which nq is not an integer; we can just replace q by $\hat{q} := \frac{\lfloor nq \rfloor}{n}$. In this way, (a) and (b) of Lemma A.1 hold with $\hat{q}$. Part (c) of Lemma A.1 also holds for $\hat{q}$ because $\hat{q}$ converges to q as $n \to \infty$. In the remainder of this appendix and all subsequent appendices we assume nq to be an integer for ease of exposure, with the understanding that this is without loss of generality.

Now we are ready to derive $\hat{G}_n(\cdot;q)$, the distribution of the pivotal voter's type $v^{(nq+1)}$. Let $\hat{g}_n(\cdot;q)$ denote the density function. Note that for $v^{(nq+1)} = x$ to hold, there must be nq voters with $v_i \leq x$ and $n(1-q)$ others with $v_i \geq x$, with the remaining pivotal voter having $v_i = x$. Because voters' types are independently drawn from G, we have

$$\hat{g}_n(x;q) = \frac{(n+1)!}{(nq)![n(1-q)]!}(G(x))^{nq}(1-G(x))^{n(1-q)}g(x) = \tau_n\left(G(x);q\right)g(x) \qquad \text{(A.4)}$$

and

$$\hat{G}_n(x;q) = \int_{\underline{v}}^{x} \tau_n\left(G(x);q\right)g(x)dx = \int_0^{G(x)} \tau_n(y;q)dy \; . \qquad \text{(A.5)}$$

We are now ready to prove Lemma 1 in Section 3.1.

Proof of Lemma 1. We first show part (1). That $\hat{G}_n(\cdot;q)$ is strictly increasing follows immediately from $\hat{g}_n(x;q) = \tau_n(G(x);q)g(x) > 0$. To show $v^{(nq+1)} \xrightarrow{p} v_q^*$ it suffices to establish

$$\lim_{n\to\infty} \hat{G}_n(x;q) \to \begin{cases} 0, & \text{if } x < v_q^* \\ 1/2, & \text{if } x = v_q^* \\ 1, & \text{if } x > v_q^* \end{cases} . \qquad \text{(A.6)}$$

For $x < v_q^*$ we have $G(x) < q$ and

$$\hat{G}_n(x;q) = \int_0^{G(x)} \tau_n(y;q)dy < G(x)\tau_n(G(x);q) \to 0 \; .$$

The second and third steps follow from (b) and (c) of Lemma A.1, respectively. If instead $x > v_q^*$, then $G(x) > q$ and $\int_{G(x)}^1 \tau_n(y)dy < (1-G(x))\tau_n(G(x);q) \to 0$. Therefore, $\hat{G}_n(x) = 1 - \int_{G(x)}^1 \tau_n(y)dy \to 1$. Finally, if $x = v_q^*$, then $G(x) = G(v_q^*) = q$ and $\hat{G}_n(x) = \int_0^q \tau_n(y;q)dy$. Below we show $\lim_{n\to\infty} \int_0^q \tau_n(y;q)dy = 1/2$.

Recall that $\tau_n(y;q)$ is the density function of a random variable Y following Beta distribution $B(\alpha,\beta)$ with parameters $\alpha = nq+1$ and $\beta = n(1-q)+1$. Let q_n denote the median of Y; that is, $\int_0^{q_n} \tau_n(y;q)dy = 1/2$. We show that the sequence of medians q_n converges to q and thus $\lim_{n\to\infty} \int_0^q \tau_n(y;q)dy = \lim_{n\to\infty} \int_0^{q_n} \tau_n(y;q)dy = 1/2$. For a Beta-distributed random variable $Y \sim Beta(\alpha,\beta)$, Groeneveld and Meeden (1977) show that its median q_n must be bounded between its mean μ_n and mode m_n. For a Beta distribution, it is also well known that $\mu_n = \frac{\alpha}{\alpha+\beta}$ and $m_n = \frac{\alpha-1}{\alpha+\beta-2}$. Since $\alpha = nq+1$ and $\beta = n(1-q)+1$, both μ_n and m_n converge to q as $n \to \infty$. This implies that the median q_n must converge to q as well. This establishes part (1) of this proposition.

Next we prove part (2). By (A.4) and part (a) of Lemma A.1, we have

$$\hat{g}_n'(x;q) = \tau_n'(G(x);q)g^2(x) + \tau_n(G(x);q)g'(x) = \hat{g}_n(x;q)\left(n\frac{g(x)}{G(x)}\frac{q-G(x)}{1-G(x)} + \frac{g'(x)}{g(x)}\right)$$

or equivalently

$$\frac{\hat{g}_n'(x;q)}{\hat{g}_n(x;q)} = n\frac{g(x)}{G(x)}\frac{q-G(x)}{1-G(x)} + \frac{g'(x)}{g(x)}\,. \tag{A.7}$$

We show that $\hat{g}_n(\cdot;q)$ is single-peaked for sufficiently large n. By (A.7),

$$\hat{g}_n'(x;q) > 0 \Longleftrightarrow \lambda_n(x) := q - G(x) + \frac{1}{n}\frac{G(x)\,(1-G(x))}{g(x)}\frac{g'(x)}{g(x)} > 0\,.$$

Recall that g is strictly positive and twice-continuously differentiable on $[\underline{v},\overline{v}]$. These imply that (i) there exists some $\varepsilon > 0$ such that $g(x) > \varepsilon$ for all x, and (ii) both $\frac{G(x)(1-G(x))}{g(x)}\frac{g'(x)}{g(x)}$ and its first order derivative are uniformly bounded. Therefore, as $n \to \infty$, $\lambda_n(x)$ and $\lambda_n'(x)$ converge uniformly to $q - G(x)$ and $-g(x)$, respectively. Hence, for sufficiently large n, $\lambda_n(x)$ is strictly decreasing and its root $\hat{x}_n$ converges to v_q^*. This implies that $\hat{g}_n(x;q)$ is single-peaked for sufficiently large n.

Finally, we assume g is log-concave and show that in this case $\hat{g}_n(\cdot;q)$ is also log-concave for all $n \geq 0$ (strictly so for $n > 0$) and $q \in (0,1)$. Our starting point is that log-concavity of g implies that both G and $1-G$ are log-concave (Bagnoli and Bergstrom, 2005). Therefore, $\frac{g'(\cdot)}{g(\cdot)}$ is decreasing and by (A.7) it suffices to show that function $\eta(x) := \frac{g(x)}{G(x)}\frac{q-G(x)}{1-G(x)}$ is strictly decreasing. Differentiating $\eta(x)$ and simplifying the expression, we obtain

$$\eta'(x) = \frac{g(x)}{G(x)}\frac{1-q}{1-G(x)}\left\{\left(\frac{g(x)}{G(x)} - \frac{g'(x)}{g(x)}\right)\frac{G(x)-q}{1-q} - \frac{g(x)}{1-G(x)}\right\}\,.$$

Therefore, $\eta'(x) < 0$ if and only if

$$\left(\frac{g(x)}{G(x)} - \frac{g'(x)}{g(x)}\right)\frac{G(x)-q}{1-q} < \frac{g(x)}{1-G(x)}\,. \tag{A.8}$$

Note that $\frac{g(x)}{G(x)} - \frac{g'(x)}{g(x)} \geq 0$ because G is log-concave. (A.8) thus holds for all $q \geq G(x)$. Now we establish (A.8) for $q < G(x)$. Using the fact that $\frac{G(x)-q}{1-q}$ strictly decreases in q for $q < G(x)$, we get

$$\left(\frac{g(x)}{G(x)} - \frac{g'(x)}{g(x)}\right)\frac{G(x)-q}{1-q} < \left(\frac{g(x)}{G(x)} - \frac{g'(x)}{g(x)}\right)G(x)\,. \tag{A.9}$$

Thus, it suffices to show that the right-hand side of (A.9) is less than $\frac{g(x)}{1-G(x)}$. This is true because

$$\left(\frac{g(x)}{G(x)}-\frac{g'(x)}{g(x)}\right)G(x)\leq\frac{g(x)}{1-G(x)}\Longleftrightarrow\frac{g^2(x)-g'(x)G(x)}{g(x)}\leq\frac{g(x)}{1-G(x)}$$
$$\Longleftrightarrow g^2(x)\left(1-G(x)\right)-g'(x)G(x)\left(1-G(x)\right)\leq g^2(x)$$
$$\Longleftrightarrow 0\leq g^2(x)+g'(x)\left(1-G(x)\right)$$
$$\Longleftrightarrow 1-G \text{ is log-concave} \Longleftarrow g \text{ is log-concave}$$

This completes the proof. □

B Useful Properties of $\phi_n(\cdot)$ and Omitted Proofs for Section 4

In this appendix we derive and establish some important properties for $\phi_n(\cdot)$ – the indifference curve of the sender – and its limit as $n\to\infty$. We also prove Lemmas 2 and 3 in Section 4.

Recall from (3) in Section 4 that

$$\phi_n(x)=\mathbb{E}\left[\varphi_n(v)\middle|v^{(nq+1)}=x\right]=\rho\sum_{j=1}^{n+1}w_j\varphi_j(x;q,n)+(1-\rho)\chi \tag{B.1}$$

where

$$\varphi_j(x;q,n)=\mathbb{E}\left[v^{(j)}\middle|v^{(nq+1)}=x;q,n\right]$$

is the expectation of the j-th order statistic $v^{(j)}$ conditional on event $v^{(nq+1)}=x$. If $\rho=0$, it is obvious that $\phi_n(x)=\chi$ is a constant. If $\rho>0$, the properties of $\phi_n(x)$ depend closely on $\varphi_j(x;q,n)$. For any $j\neq nq+1$, let $\widetilde{g}_j(\cdot|x;q,n)$ denote the density function for the distribution of $v^{(j)}$ conditional on $v^{(nq+1)}=x$ given parameters q and n. We show that

$$\widetilde{g}_j(y|x;q,n)=\begin{cases}\tau_{nq-1}\left(\frac{G(y)}{G(x)};\frac{j-1}{nq}\right)\frac{g(y)}{G(x)}, & \text{if } j<nq+1\\ \tau_{n(1-q)-1}\left(\frac{G(y)-G(x)}{1-G(x)};\frac{j-nq-2}{n(1-q)}\right)\frac{g(y)}{1-G(x)}, & \text{if } j>nq+1\end{cases}. \tag{B.2}$$

To see why, first consider $j<nq+1$. Conditional on $v^{(nq+1)}=x$, $v^{(j)}$ is the j-th lowest order statistic from nq independent random draws from a truncated distribution with cdf $\frac{G(y)}{G(x)}$ for $y\in[\underline{v},x]$. (B.2) for $j<nq+1$ thus follows from (A.4). Now consider $j>nq+1$. Conditional on $v^{(nq+1)}=x$, $v^{(j)}$ is the $(j-nq-1)$-th lowest order statistic from $n(1-q)$ independent random draws from a truncated distribution with cdf $\frac{G(y)-G(x)}{1-G(x)}$ for $y\in[x,\overline{v}]$. This implies (B.2) for $j<nq+1$ through (A.4). Lemma B.1 explicitly characterizes $\varphi_j(x;q,n)$.

Lemma B.1. *For all* $x \in [\underline{v}, \overline{v}]$,

$$\varphi_j(x;q,n) = \begin{cases} \int_0^1 \underline{t}(x,y)\tau_{nq-1}\left(y; \frac{j-1}{nq}\right) dy, & \text{if } j < nq+1 \\ x, & \text{if } j = nq+1 \\ \int_0^1 \overline{t}(x,y)\tau_{n(1-q)-1}\left(y; \frac{j-nq-2}{n(1-q)}\right) dy, & \text{if } j > nq+1 \end{cases} \tag{B.3}$$

where

$$\underline{t}(x,y) := G^{-1}\left(yG(x)\right) \tag{B.4}$$

$$\overline{t}(x,y) := G^{-1}\left(y + (1-y)G(x)\right) \tag{B.5}$$

for all $x \in [\underline{v}, \overline{v}]$ *and* $y \in [0,1]$.

Proof of Lemma B.1. $\varphi_j(x;q,n) = x$ for $j = nq+1$ is immediate from its definition. For $j < nq+1$, it follows from the upper part (B.2) that

$$\begin{aligned} \varphi_j(x;q,n) &= \int_{\underline{v}}^{x} y\tilde{g}_j(y|x;q,n)dy = \int_{\underline{v}}^{x} y\tau_{nq-1}\left(\frac{G(y)}{G(x)}; \frac{j-1}{nq}\right)\frac{dG(y)}{G(x)} \\ &= \int_0^1 G^{-1}\left(yG(x)\right)\tau_{nq-1}\left(y; \frac{j-1}{nq}\right) dy = \int_0^1 \underline{t}(x,y)\tau_{nq-1}\left(y; \frac{j-1}{nq}\right) dy\,. \end{aligned}$$

Finally, for all $j > nq+1$ it follows from the lower part (B.2) that

$$\begin{aligned} \varphi_j(x;q,n) &= \int_{x}^{\overline{v}} y\tilde{g}_j(y|x;q,n)dy = \int_{x}^{\overline{v}} y\tau_{n(1-q)-1}\left(\frac{G(y)-G(x)}{1-G(x)}; \frac{j-nq-2}{n(1-q)}\right)\frac{dG(y)}{1-G(x)} \\ &= \int_0^1 G^{-1}\left(y+(1-y)G(x)\right)\tau_{n(1-q)-1}\left(y; \frac{j-nq-2}{n(1-q)}\right) dy \\ &= \int_0^1 \overline{t}(x,y)\tau_{n(1-q)-1}\left(y; \frac{j-nq-2}{n(1-q)}\right) dy\,. \end{aligned}$$

This completes the proof. □

Lemma B.2 summarizes useful properties about the functions $\underline{t}(x,y)$ and $\overline{t}(x,y)$ defined above. (Subscripts for these two functions denote taking partial derivatives.)

Lemma B.2. *Both* $\underline{t}(x,y)$ *and* $\overline{t}(x,y)$ *are three times continuously differentiable and they satisfy the following properties:*

1. $\underline{t}(x,y) < x < \overline{t}(x,y)$ *for all* $y \in (0,1)$.
2. *Both* $\underline{t}(x,y)$ *and* $\overline{t}(x,y)$ *are strictly increasing in* x *and* y.
3. *If* G *is strictly log-concave, then* $\underline{t}_x(x,y) < 1$ *for all* $y \in (0,1)$.
4. *If* $1-G$ *is strictly log-concave, then* $\overline{t}_x(x,y) < 1$ *for all* $y \in (0,1)$.

Proof of Lemma B.2. The fact that both $\underline{t}(x,y)$ and $\bar{t}(x,y)$ are three times continuously differentiable for all $(x,y) \in [\underline{v},\bar{v}] \times [0,1]$ follows from our assumption that G is twice continuously differentiable on $[\underline{v},\bar{v}]$. Parts (1) and (2) of this lemma follow immediately from the definitions of $\underline{t}(x,y)$ and $\bar{t}(x,y)$. To show part (3), note from (B.4) that

$$G\left(\underline{t}(x,y)\right) = yG(x) \ .$$

Taking the first order derivative with respect to x on both sides and rearranging terms yields

$$g\left(\underline{t}(x,y)\right)\underline{t}_x(x,y) = yg(x) \Longleftrightarrow \underline{t}_x(x,y) = y\frac{g(x)}{g\left(\underline{t}(x,y)\right)} = \frac{g(x)}{G(x)}\Bigg/\frac{g\left(\underline{t}(x,y)\right)}{G\left(\underline{t}(x,y)\right)} \ . \tag{B.6}$$

If G is strictly log-concave, then $\frac{g(\cdot)}{G(\cdot)}$ is strictly decreasing. Since $\underline{t}_x(x,y) < x$ for $y \in (0,1)$, it follows from (B.6) that $\underline{t}_x(x,y) < 1$. To show part (4), note from (B.5) that

$$G\left(\bar{t}(x,y)\right) = y + (1-y)\left(1-G(x)\right)$$

holds for all x and y. Simple algebra reveals that

$$\bar{t}_x(x,y) = (1-y)\frac{g(x)}{g\left(\bar{t}(x,y)\right)} = \frac{g(x)}{1-G(x)}\Bigg/\frac{g(\bar{t}(x,y))}{1-G(\bar{t}(x,y))} \ . \tag{B.7}$$

If $1-G$ is strictly log-concave, then $\frac{g(\cdot)}{1-G(\cdot)}$ is strictly increasing. Since $\bar{t}(x,y) > x$ for $y \in (0,1)$, it follows from (B.7) that $\bar{t}_x(x,y) < 1$. $\square$

Notice that parameters j and q affect $\varphi_j(x;q,n)$ only through their impacts on $\widetilde{g}_j(\cdot|x;q,n)$. The next lemma shows that $\widetilde{g}_j(\cdot|x;q,n)$ increases in strict monotone likelihood-ratio dominance order as j increases and q decreases. For two probability density functions $l(\cdot)$ and $r(\cdot)$, we write $l(\cdot) \succ_{LR} r(\cdot)$ if the likelihood ratio $\frac{l(\cdot)}{r(\cdot)}$ is strictly increasing.

Lemma B.3. *The following properties for $\widetilde{g}_j(\cdot|x;q,n)$ hold:*

1. *Suppose $j' > j$, then $\widetilde{g}_{j'}(\cdot|x;q,n) \succ_{LR} \widetilde{g}_j(\cdot|x;q,n)$ holds if $j > nq+1$ or $j' < nq+1$.*
2. *Suppose $q' > q$, then $\widetilde{g}_j(\cdot|x;q,n) \succ_{LR} \widetilde{g}_j(\cdot|x;q',n)$ holds if $j < nq+1$ or $j > nq'+1$.*

Proof of Lemma B.3. We first show part (1). Using (B.2) and (A.1), we obtain

$$\frac{\widetilde{g}_{j'}(y|x;q,n)}{\widetilde{g}_j(y|x;q,n)} \propto \begin{cases} \left(\frac{G(y)}{G(x)-G(y)}\right)^{j'-j} \text{for } y \in [\underline{v},x], & \text{if } nq+1 > j' > j \\ \left(\frac{G(y)-G(x)}{1-G(y)}\right)^{j'-j} \text{for } y \in [x,\bar{v}], & \text{if } j' > j > nq+1 \end{cases} .$$

In both cases, the likelihood ratio $\frac{\widetilde{g}_{j'}(y|x;q,n)}{\widetilde{g}_j(y|x;q,n)}$ is strictly increasing in y since $j' > j$. To show part (2), suppose $q' > q$ and note that

$$\frac{\widetilde{g}_j(y|x;q,n)}{\widetilde{g}_j(y|x;q',n)} \propto \begin{cases} \left(\frac{G(x)}{G(x)-G(y)}\right)^{n(q'-q)} \text{ for } y \in [\underline{v}, x], & \text{if } j < nq+1 \\ \left(\frac{G(y)-G(x)}{1-G(y)}\right)^{n(q'-q)} \text{ for } y \in [x, \overline{v}], & \text{if } j > nq'+1 \end{cases} .$$

In both cases, the likelihood ratio $\frac{\widetilde{g}_j(y|x;q,n)}{\widetilde{g}_j(y|x;q',n)}$ is strictly increasing in y when $q' > q$. □

We now establish Lemma B.4, which collects important properties of $\varphi_j(x;q,n)$.

Lemma B.4. *Let $j \in \{1, \cdots, n+1\}$. $\varphi_j(x;q,n)$ satisfies the following properties:*

1. *Each $\varphi_j(x;q,n)$ is strictly increasing in x;*
2. *If G is strictly log-concave, then $\varphi_j'(x;q,n) < 1$ for all $j < nq+1$;*
3. *If $1-G$ is strictly log-concave, then $\varphi_j'(x;q,n) < 1$ for all $j > nq+1$.*
4. *$\varphi_j(x;q,n)$ is strictly increasing in index j and decreasing in q.*

Proof of Lemma B.4. We start with part (1). Note that both $\underline{t}(x,y)$ and $\bar{t}(x,y)$ strictly increase in x for all $y \in (0,1)$ (cf. Lemma B.2). By (B.3), each $\varphi_j(x;q,n)$ must be strictly increasing in x.

To show part (2), suppose that G is strictly log-concave so that $\frac{g(\cdot)}{G(\cdot)}$ is strictly decreasing. By Lemmas B.1 and B.2, for $j < nq+1$ we have

$$\varphi_j'(x;q,n) = \int_0^1 \underline{t}_x(x,y)\tau_{nq-1}\left(y;\frac{j-1}{nq}\right)dy < \int_0^1 \tau_{nq-1}\left(y;\frac{j-1}{nq}\right)dy = 1 .$$

The second step follows from statement (3) of Lemma B.2. The proof for part (3) is analogous.

Finally we establish part (4). To show that $\varphi_j(x;q,n)$ strictly increases in j, note first that $\varphi_{nq+1}(x;q,n) = x$ follows immediately from the definition. Moreover, (B.3) and the fact that $\underline{t}(x,y) < x < \bar{t}(x,y)$ for $y \in (0,1)$ imply $\varphi_j(x;q,n) > (<)x$ for $j > (<)nq+1$. Hence, $\varphi_{j'}(x;q,n) > \varphi_j(x;q,n)$ holds for $j' \geq nq+1 \geq j$ with at least one inequality holding strictly. Now consider $j' > j > nq+1$ or $nq+1 > j' > j$. Observe that both $\underline{t}(x,y)$ and $\bar{t}(x,y)$ are strictly increasing functions of y, and $\varphi_j(x;q,n)$ equals the expectation of $\underline{t}(x,y)$ or $\bar{t}(x,y)$ for random variable y under distribution $\widetilde{g}_j(\cdot|x;q,n)$. By Lemma B.3, $\tilde{g}_{j'}(\cdot|x;q,n) \succ_{LR} \tilde{g}_j(\cdot|x;q,n)$ and strict likelihood ratio dominance implies $\varphi_{j'}(x;q,n) > \varphi_j(x;q,n)$ (see, for instance, Appendix B of Krishna (2009)).

To show that $\varphi_j(x;q,n)$ decreases in q, consider two different q' and q'' with $q' < q''$. If $nq'+1 \leq j \leq nq''+1$ then by (B.3) and Lemma B.2 we have $\varphi_j(x;q',n) \leq x \leq \varphi_j(x;q'',n)$ with at least one inequality strict. Now consider $j < nq'+1$ or $j > nq''+1$. In this case it follows from Lemma B.3 that $\widetilde{g}_j(\cdot|x;q',n) \succ_{LR} \widetilde{g}_j(\cdot|x;q'',n)$ so that $\varphi_j(x;q',n) < \varphi_j(x;q'',n)$ holds. □

B.1 Proof of Proposition 1

Parts (1) to (3) of this proposition follow directly from Lemma B.4. Next we show parts (4) and (5). Since each $\varphi_j(x;q,n)$ is strictly increasing in x and decreasing in q, $\phi_n(x)$ must inherit these properties whenever $\rho > 0$. This implies part (4). Finally, to show part (5), note that

$$\begin{aligned}\sum_{j=1}^{n+1} w_j \varphi_j(x;q,n) &= \sum_{j=2}^{n+1}\left[\sum_{l=j}^{n+1} w_l\right]\big(\varphi_j(x;q,n)-\varphi_{j-1}(x;q,n)\big)+\varphi_1(x;q,n) \\ &= \sum_{j=2}^{n+1}\left[1-w\left(\frac{j-1}{n+1}\right)\right]\big(\varphi_j(x;q,n)-\varphi_{j-1}(x;q,n)\big)+\varphi_1(x;q,n)\ .\end{aligned}$$

Consider two weighting functions $w^{\mathrm{I}}(\cdot)$ and $w^{\mathrm{II}}(\cdot)$. Let $\phi_n^{\mathrm{I}}(\cdot)$ and $\phi_n^{\mathrm{II}}(\cdot)$ denote function $\phi_n(\cdot)$ when $w(\cdot)$ equals $w^{\mathrm{I}}(\cdot)$ and $w^{\mathrm{II}}(\cdot)$, respectively. Using the above equation we obtain

$$\phi_n^{\mathrm{I}}(x)-\phi_n^{\mathrm{II}}(x)=\rho\sum_{j=2}^{n+1}\left[w^{\mathrm{II}}\left(\frac{j-1}{n+1}\right)-w^{\mathrm{I}}\left(\frac{j-1}{n+1}\right)\right]\big(\varphi_j(x;q,n)-\varphi_{j-1}(x;q,n)\big)\ .$$

By Lemma B.4, $\varphi_j(x;q,n)-\varphi_{j-1}(x;q,n) > 0$ holds for all $j > 1$ and $x \in (\underline{v},\overline{v})$. Suppose $w^{\mathrm{II}}(\cdot)$ first order stochastically dominates $w^{\mathrm{I}}(\cdot)$. Then $w^{\mathrm{II}}(y)-w^{\mathrm{I}}(y) \le 0$ holds for all $y \in (0,1)$ and this implies $\phi_n^{\mathrm{I}}(x)-\phi_n^{\mathrm{II}}(x) \le 0$ for $x \in (\underline{v},\overline{v})$. This proves part (5).

B.2 Asymptotic properties of $\phi_n(\cdot)$ and the proofs of Lemmas 2 and 3

In this subsection we derive asymptotic properties of $\phi_n(\cdot)$ as $n \to \infty$ and prove Lemmas 2 and 3 in Section 4. We also establish some additional properties in Lemmas B.5 and B.6 below; these are relevant for proofs in Appendices D and E.

Given a sender's preference parameters ρ, $w(\cdot)$ and χ, we define

$$\phi(x) := \rho\left[\int_0^q \underline{t}\left(x,\frac{y}{q}\right)dw(y)+\int_q^1 \overline{t}\left(x,\frac{y-q}{1-q}\right)dw(y)\right]+(1-\rho)\chi \tag{B.8}$$

for $x \in [\underline{v},\overline{v}]$, where $\underline{t}(\cdot)$ and $\overline{t}(\cdot)$ are given (B.4) and (B.5), respectively. Taking derivative yields

$$\phi'(x)=\rho\left[\int_0^q \underline{t}_x\left(x,\frac{y}{q}\right)dw(y)+\int_q^1 \overline{t}_x\left(x,\frac{y-1}{1-q}\right)dw(y)\right]\ . \tag{B.9}$$

Moreover, using (B.4), (B.5) and the fact that $v_q^* = G^{-1}(q)$, we obtain

$$\underline{t}\left(v_q^*,\frac{y}{q}\right)=G^{-1}\left(\frac{y}{q}G(v_q^*)\right)=G^{-1}(y)$$

and

$$\bar{t}\left(v_q^*, \frac{y-q}{1-q}\right) = G^{-1}\left(\frac{y-q}{1-q} + (1-\frac{y-q}{1-q})G(v_q^*)\right) = G^{-1}(y) \ .$$

These together imply

$$\phi^* := \phi(v_q^*) = \rho \int_0^1 G^{-1}(y) dw(y) + (1-\rho)\chi \ . \tag{B.10}$$

Lemma 2 in Section 4 is then equivalent to that $\phi_n(\cdot)$ and $\phi_n'(\cdot)$ uniformly converge to (B.8) and (B.9), respectively, and $\varphi_n(v)$ converges almost surely to (B.10) as $n \to \infty$.

B.2.1 Proof of Lemma 2

Consider any $z \in (0,1)$. By (B.3) in Lemma B.1 we have

$$\varphi_{\lfloor (n+1)z \rfloor}(x;q,n) = \begin{cases} \int_0^1 \underline{t}(x,y)\tau_{nq-1}\left(y; \frac{\lfloor (n+1)z \rfloor - 1}{nq}\right) dy, & \text{if } z < \frac{nq+1}{n+1} \\ \int_0^1 \bar{t}(x,y)\tau_{n(1-q)-1}\left(y; \frac{\lfloor (n+1)z \rfloor - nq - 2}{n(1-q)}\right) dy, & \text{if } z > \frac{nq+1}{n+1} \end{cases} .$$

By Lemma A.1c, as $n \to \infty$, $\tau_{nq-1}\left(y; \frac{\lfloor (n+1)z \rfloor - 1}{nq}\right)$ concentrates all its probability mass on $\frac{z}{q}$ and $\tau_{n(1-q)-1}\left(y; \frac{\lfloor (n+1)z \rfloor - nq - 2}{n(1-q)}\right)$ concentrates all its mass on $\frac{z-q}{1-q}$ for all $z \neq q$. Therefore,

$$\lim_{n\to\infty} \varphi_{\lfloor (n+1)z \rfloor}(x;q,n) = \begin{cases} \underline{t}\left(x, \frac{z}{q}\right), & \text{if } z < q \\ \bar{t}\left(x, \frac{z-q}{1-q}\right), & \text{if } z > q \end{cases} . \tag{B.11}$$

Using the definition of $\phi_n(x)$ and the fact that $w_j = w\left(\frac{j}{n+1}\right) - w\left(\frac{j-1}{n+1}\right)$, we get

$$\phi_n(x) = \rho \sum_{j=1}^{n+1} \left[w(z_j) - w(z_{j-1})\right] \varphi_{(n+1)z_j}(x;q,n) + (1-\rho)\chi \tag{B.12}$$

where $z_j := \frac{j}{n+1}$. Taking $n \to \infty$ and using the definition of a Riemann integral, we have

$$\begin{aligned} \lim_{n\to\infty} \sum_{j=1}^{n+1} \left[w(z_j) - w(z_{j-1})\right] \varphi_{(n+1)z_j}(x;q,n) &= \int_0^1 \lim_{n\to\infty} \varphi_{\lfloor (n+1)z \rfloor}(x;q,n) dw(z) \\ &= \int_0^q \underline{t}\left(x, \frac{y}{q}\right) dw(y) + \int_q^1 \bar{t}\left(x, \frac{y-q}{1-q}\right) dw(y) \end{aligned}$$

where the last step follows from (B.11). Combining this with (B.12) yields

$$\lim_{n\to\infty} \phi_n(x) = \rho \left[\int_0^q \underline{t}\left(x, \frac{y}{q}\right) dw(y) + \int_q^1 \bar{t}\left(x, \frac{y-q}{1-q}\right) dw(y)\right] + (1-\rho)\chi \ . \tag{B.13}$$

Therefore, $\phi_n(x)$ converges point-wise to (B.8) on $[\underline{v},\overline{v}]$. Observe that $\phi_n(x)$ for each $n\geq 0$ is uniformly L-Liptschitz continuous on $[\underline{v},\overline{v}]$ for sufficient large L.[2] Hence, $\phi(x)$ given by (B.8) is also L-Liptschitz continuous for sufficiently large L. These together imply that the convergence of $\phi_n(x)$ to (B.8) is uniform.[3] That $\phi_n'(x)$ converges uniformly to (B.9) can be proved analogously.

Finally, we prove that $\varphi_n(v) := \rho\sum_{j=1}^{n+1} w_j v^{(j)} + (1-\rho)\chi$ converges in probability to ϕ^* given by (B.10). It suffices to show that $\sum_{j=1}^{n+1} w_j v^{(j)}$ converges in probability to $\int_0^1 G^{-1}(y)dw(y)$. To prove this, we use a strong law of large numbers for linear combinations of order statistics established by Van Zwet (1980). Let $U_1, U_2, \cdots, U_{n+1}$ be $n+1$ random variables drawn from a uniform distribution on $(0,1)$, and $U_{1:n+1}\leq U_{2:n+1}\leq\cdots\leq U_{n+1:n+1}$ denote the ordered $U_1, U_2, \cdots, U_{n+1}$. We can therefore rewrite $v^{(j)}$ as $G^{-1}(U_{j:n+1})$ for each $j=1,\cdots,n+1$. For $t\in(0,1)$, define $\gamma_n(t) := G^{-1}(U_{\lceil (n+1)t\rceil:n+1})$ and $\xi_n(t) := (n+1)\cdot\left[w\left(\frac{\lceil (n+1)t\rceil}{n+1}\right) - w\left(\frac{\lceil (n+1)t\rceil-1}{n+1}\right)\right]$. We can then rewrite $\sum_{j=1}^{n+1} w_j v^{(j)}$ in an integral form as

$$\sum_{j=1}^{n+1} w_j v^{(j)} = \sum_{j=1}^{n+1}\left[w\left(\frac{j}{n+1}\right) - w\left(\frac{j-1}{n+1}\right)\right] G^{-1}(U_{j:n+1}) = \int_0^1 \gamma_n(t)\xi_n(t)dt\ . \tag{B.14}$$

Our assumptions for G and $w(\cdot)$ ensure that $G^{-1}(\cdot), w(\cdot)\in L_1$, $\sup_n\|\xi_n\|_\infty<\infty$, and $\lim_{n\to\infty}\int_0^t \xi_n(x)dx = \int_0^t w'(x)dx = w(t)$ for all $t\in(0,1)$.[4] It follows from Theorem 2.1 and Corollary 2.1 of Van Zwet (1980) that the integral in (B.14) converges almost surely to $\int_0^1 G^{-1}(y)dw(y)$ as $n\to\infty$.

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[2] This is because both $\underline{t}_x(x,y)$ and $\overline{t}_x(x,y)$ are uniformly bounded on $[\underline{v},\overline{v}]$. This uniform boundedness also holds for $\underline{t}_{xx}(x,y)$, $\overline{t}_{xx}(x,y)$, $\underline{t}_{xxx}(x,y)$, $\overline{t}_{xxx}(x,y)$, and these imply uniform L-Liptschitz continuity for $\phi_n'(\cdot)$ and $\phi_n''(\cdot)$ on $[\underline{v},\overline{v}]$ for sufficiently large L.

[3] This stems from the following general observation: *For any $L>0$, let $\{f_n(\cdot)\}_{n\geq 0}$ be a sequence of L-Lipschitz continuous functions on a closed interval $[a,b]$ that converges point-wise to a L-Lipschitz continuous function $f(\cdot)$. Then $f_n(\cdot)$ converges uniformly to $f(\cdot)$ on $[a,b]$.* The proof is as follows. Given any $\varepsilon>0$, consider a pair of $\delta,\eta>0$ such that $\varepsilon>2L\delta+\eta$. Partition interval $[a,b]$ into $K+1$ intervals with cutoffs $a=x_0<x_1<\cdots<x_{K+1}=b$ such that $|x_i-x_{i-1}|<\delta$ for all $i\in\{1,\cdots,K+1\}$. For this finite set $\{x_i\}_{i=1}^{K+1}$ point-wise convergence implies that there exists a threshold N such that for all $n>N$ we have $|f_n(x_i)-f(x_i)|<\eta$ for all $i\in\{1,\cdots,K\}$. Now consider any $x\in[a,b]$ and let i be such that $x\in[x_{i-1},x_i]$. We then obtain

$$\begin{aligned}|f_n(x)-f(x)| &= |f_n(x)-f_n(x_i)+f_n(x_i)-f(x_i)+f(x_i)-f(x)|\\ &\leq |f_n(x)-f_n(x_i)|+|f_n(x_i)-f(x_i)|+|f(x_i)-f(x)|\\ &< 2L\delta+\eta<\varepsilon\end{aligned}$$

for all $n>N$. This proves uniform convergence on $[a,b]$.

[4] Here L_1 refers to the space of Lebesgue measurable functions $f:(0,1)\mapsto\mathbb{R}$ with finite $\|\cdot\|_1$ norm. $\|\cdot\|_\infty$ denotes the essential supremum norm. Moreover, the absolute continuity of $w(\cdot)$ ensures that its derivative $w'(\cdot)$ exists almost everywhere and $\int_0^t w'(x)dx = w(t)$ for all $t\in(0,1)$.

B.2.2 Proof of Lemma 3

We only need to show that for all $x \in [\underline{v}, \overline{v}]$ that $\phi'(x) < 1$ holds under either condition (i) and (ii) of Lemma 3. Because both $\underline{t}_x\left(x, \frac{y}{q}\right)$ and $\overline{t}_x\left(x, \frac{y-1}{1-q}\right)$ are uniformly bounded, it follows from (B.9) that $\phi'(x)$ must uniformly converge to zero as $\rho \to 0$. This implies $\phi'(x) < 1$ for all $x \in [\underline{v}, \overline{v}]$ if ρ is sufficiently close to zero. If instead both G and $1-G$ are strictly log-concave, it follows from Lemma B.2 that $\underline{t}_x\left(x, \frac{y}{q}\right) < 1$ for $y < q$ and $\overline{t}_x\left(x, \frac{y-q}{1-q}\right) < 1$ for $y > q$. Therefore, by (B.9), $\phi'(x) < 1$ holds uniformly on $[\underline{v}, \overline{v}]$ if either $\rho < 1$ or $w(\cdot)$ does not put all weights on $y = q$ (i.e., $w(\cdot)$ is not a step function with threshold q). The latter condition for $w(\cdot)$ is always satisfied due to our assumption that $w(\cdot)$ is absolutely continuous.

B.2.3 Additional useful properties of $\phi(\cdot)$ and ϕ^*

Here we establish several additional properties for $\phi(\cdot)$ and ϕ^* summarized in Lemmas B.5 and B.6. The results are relevant for the proofs in Appendix E.

Lemma B.5. *For any $x \in [\underline{v}, \overline{v}]$, the following properties hold:*

1. If $\rho > 0$, then $\phi(x)$ is strictly decreasing in q.

2. If $\rho > 0$, then $\phi(x)$ strictly decreases as $w(\cdot)$ shifts from $w^I(\cdot)$ to $w^{II}(\cdot)$, where $w^I(\cdot), w^{II}(\cdot) \in \Delta([-1,1])$ and $w^I(\cdot) \succeq_{FOSD} w^{II}(\cdot)$.[5]

Proof of Lemma B.5. For all $y \in [0,1]$ let

$$t(x,y;q) := \begin{cases} \underline{t}\left(x, \frac{y}{q}\right), & \text{if } y \leq q \\ \overline{t}\left(x, \frac{y-q}{1-q}\right), & \text{if } y > q \end{cases} \tag{B.15}$$

where $\underline{t}(\cdot)$ and $\overline{t}(\cdot)$ are given by (B.4) and (B.5), respectively. Using (B.8) and (B.15), we can rewrite $\phi(\cdot)$ as $\phi(x) = \rho \int_0^1 t(x,y;q)dw(y) + (1-\rho)\chi$ so that $\phi(x)$ depends on x, q and $w(\cdot)$ only through the integral $\int_0^1 t(x,y;q)dw(y)$. Because both $\underline{t}(x,y)$ and $\overline{t}(x,y)$ are strictly increasing in y (cf. Lemma B.2), it follows from (B.15) that $t(x,y;q)$ is strictly decreasing in q. $\int_0^1 t(x,y;q)dw(y)$ must inherit the same property and therefore part (1) holds. Next, consider two weighting functions $w^{\mathrm{I}}(\cdot)$ and $w^{\mathrm{II}}(\cdot)$ such that $w^{\mathrm{I}}(\cdot) \succeq_{FOSD} w^{\mathrm{II}}(\cdot)$. Because $t(x,y;q)$ is strictly increasing in y, $\int_0^1 t(x,y;q)dw^{\mathrm{I}}(y) > \int_0^1 t(x,y;q)dw^{\mathrm{II}}(y)$ must hold. Therefore, $\phi(x)$ must be strictly higher under $w^{\mathrm{I}}(\cdot)$ than under $w^{\mathrm{II}}(\cdot)$. This proves part (2). □

Building on Lemma B.5, we establish our next Lemma B.6, which characterizes how ϕ^* varies with model primitives $w(\cdot)$ and q.

[5] We use notation $\succeq_{FOSD}$ to denote the partial order implied by first order stochastic dominance.

Lemma B.6. *Suppose $\rho > 0$. Then ϕ^* is invariant in q and it strictly decreases as $w(\cdot)$ shifts from $w^I(\cdot)$ to $w^{II}(\cdot)$, where $w^I(\cdot) \succeq_{FOSD} w^{II}(\cdot)$.*

Proof of Lemma B.6. Recall that $\phi^* = \phi(v_q^*) = \rho \int_0^1 G^{-1}(y)dw(y) + (1-\rho)\chi$. It is clear from this expression that ϕ^* is invariant in q. Consider any $w^{\mathrm{I}}(\cdot)$ and $w^{\mathrm{II}}(\cdot)$ with $w^{\mathrm{I}}(\cdot) \succeq_{FOSD} w^{\mathrm{II}}(\cdot)$. Then $\int_0^1 G^{-1}(y)dw^{\mathrm{I}}(y) > \int_0^1 G^{-1}(y)dw^{\mathrm{II}}(y)$ must hold because $G^{-1}(y)$ is strictly increasing. Since $\rho > 0$, it follows that ϕ^* strictly decreases as $w(\cdot)$ shifts from $w^{\mathrm{I}}(\cdot)$ to $w^{\mathrm{II}}(\cdot)$. $\square$

C Omitted Proofs for Section 5

C.1 Proof of Lemma 4

Our proof for Lemma 4 proceeds in two steps. In Step 1 we establish a general property (cf. Observation C.1) that a sender's utility function satisfying the increasing-slope property at any interior point z implies $H \succeq_{MPS} H_{\mathscr{P}(z)}$ for any H that solves his monopolistic persuasion problem. In Step 2 we subsequently show that $W_n(\cdot)$ indeed satisfies the increasing slope property at switching point z_n whenever it is interior in $(-1,1)$. These together establish our Lemma 4.

Step 1. Let $U(\cdot)$ be a generic utility function (of voters' posterior expected state θ) defined on $[-1,1]$. Then, for any prior $F \in \Delta([-1,1])$ (which need not be continuous and fully supported), let

$$\mathscr{U}_\pi(U,F) := \mathbb{E}_{H_\pi}[U(\cdot)] = \int_{-1}^1 U(\theta)dH_\pi(\theta)$$

denote the sender's expected payoff under any feasible information policy $\pi \in \Pi$.[6] Let $\underline{\pi}$ denote the *null* information policy that reveals no information. Then $H_{\underline{\pi}}$ is a degenerate distribution with all mass on prior mean $\mu_F := \mathbb{E}_F[k]$ and therefore

$$\mathscr{U}_{\underline{\pi}}(U,F) = U(\mu_F)\ . \tag{C.1}$$

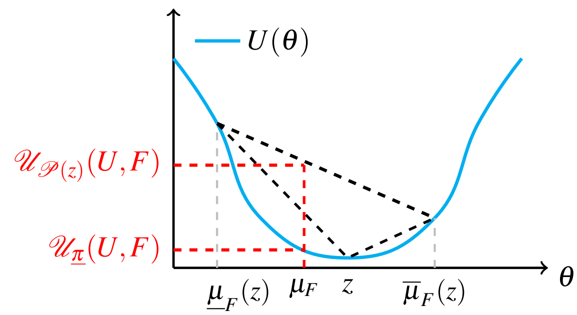
On the other hand, for a cutoff policy $\mathscr{P}(z)$ with $z \in (-1,1)$, we have

$$\mathscr{U}_{\mathscr{P}(z)}(U,F) := F^-(z)U(\underline{\mu}_F(z)) + \big(F(z) - F^-(z)\big)\,U(z) + (1-F(z))\,U(\overline{\mu}_F(z)) \tag{C.2}$$

where $F^-(z) := \lim_{x\uparrow z} F(z)$, $\underline{\mu}_F(z) := \mathbb{E}_F[k|k<z]$ and $\overline{\mu}_F(z) := \mathbb{E}_F[k|k>z]$. Figure C.1 illustrates $\mathscr{U}_{\mathscr{P}(z)}(U,F)$ and $\mathscr{U}_{\underline{\pi}}(U,F)$ for a function $U(\cdot)$ that satisfies the increasing slope property at some $z \in (-1,1)$ and a prior F with no mass point at z.

[6] Recall that H_π denotes the distribution of posterior expectations induced by π under prior F.

Figure C.1: $\mathscr{U}_{\mathscr{P}(z)}(U,F)$ and $\mathscr{U}_{\underline{\pi}}(U,F)$



Claim C.1. *Suppose $U(\cdot)$ satisfies the increasing slope property at some point $z \in (-1,1)$, then $\mathscr{U}_{\mathscr{P}(z)}(U,F) > \mathscr{U}_{\underline{\pi}}(U,F)$ for any $F \in \Delta([-1,1])$ that satisfies $0 < F^-(z) \le F(z) < 1$.*

Proof of Claim C.1. By (C.1) and (C.2), we obtain

$$\begin{aligned}\mathscr{U}_{\mathscr{P}(z)}(U,F) - \mathscr{U}_{\underline{\pi}}(U,F) =& F^-(z)\left(\underline{\mu}_F(z)-\mu_F\right)\frac{U(\underline{\mu}_F(z))-U(\mu_F)}{\underline{\mu}_F(z)-\mu_F}\\ &+\left(F(z)-F^-(z)\right)(z-\mu_F)\frac{U(z)-U(\mu_F)}{z-\mu_F}\\ &+(1-F(z))\left(\overline{\mu}_F(z)-\mu_F\right)\frac{U(\overline{\mu}_F(z))-U(\mu_F)}{\overline{\mu}_F(z)-\mu_F}\ .\end{aligned}$$

On the other hand, by the law of iterated expectations, we have

$$\begin{aligned}&F^-(z)\underline{\mu}_F(z)+\left(F(z)-F^-(z)\right)z+(1-F(z))\overline{\mu}_F(z)=\mu_F\\ \Longrightarrow\ &\left(F(z)-F^-(z)\right)(z-\mu_F) = -F^-(z)\left(\underline{\mu}_F(z)-\mu_F\right)-(1-F(z))\left(\overline{\mu}_F(z)-\mu_F\right)\ .\end{aligned}$$

These together imply

$$\begin{aligned}\mathscr{U}_{\mathscr{P}(z)}(U,F) - \mathscr{U}_{\underline{\pi}}(U,F) =& F^-(z)\left(\mu_F-\underline{\mu}_F(z)\right)\left(\frac{U(z)-U(\mu_F)}{z-\mu_F}-\frac{U(\mu_F)-U(\underline{\mu}_F(z))}{\mu_F-\underline{\mu}_F(z)}\right)\\ &+(1-F(z))\left(\overline{\mu}_F(z)-\mu_F\right)\left(\frac{U(\overline{\mu}_F(z))-U(\mu_F)}{\overline{\mu}_F(z)-\mu_F}-\frac{U(z)-U(\mu_F)}{z-\mu_F}\right)\ .\end{aligned}$$

Since $\underline{\mu}_F(z) < x < \overline{\mu}_F(z)$ for $x \in \{z, \mu_F\}$, the increasing slope property at z implies $\frac{U(z)-U(\mu_F)}{z-\mu_F} - \frac{U(\mu_F)-U(\underline{\mu}_F(z))}{\mu_F-\underline{\mu}_F(z)} \ge 0$ and $\frac{U(\overline{\mu}_F(z))-U(\mu_F)}{\overline{\mu}_F(z)-\mu_F} - \frac{U(z)-U(\mu_F)}{z-\mu_F} \ge 0$, with at least one of these holding with strict inequality.[7] This implies $\mathscr{U}_{\mathscr{P}(z)}(U,F) - \mathscr{U}_{\underline{\pi}}(U,F) \ge 0$ for all F. Finally, notice that if

[7] In case $z = \mu_F$, we can let $\frac{U(z)-U(\mu_F)}{z-\mu_F}$ be any number between $U'_-(z)$ and $U'_+(z)$, which are the left and right derivatives of $U(\cdot)$ at point z, respectively. Notice that the increasing slope property at point z implies the existence of both $U'_-(z)$ and $U'_+(z)$ (through the monotone convergence theorem) and that $U'_-(z) \le U'_+(z)$.

$0 < F^{-}(z) \le F(z) < 1$ holds, then both $F^{-}(z)\left(\mu_F - \underline{\mu}_F(z)\right)$ and $(1-F(z))\left(\overline{\mu}_F(z) - \mu_F\right)$ are strictly positive so that $\mathscr{U}_{\mathscr{P}(z)}(U,F) - \mathscr{U}_{\underline{\pi}}(U,F) > 0$ must hold. $\square$

We are now ready to establish the following general observation.

Observation C.1. *Suppose $U(\cdot)$ satisfies the increasing slope property at point $z \in (-1,1)$. Then $H \succeq_{MPS} H_{\mathscr{P}(z)}$ for any H (if it exists) that solves*

$$\max_{H\in\Delta([-1,1])} \int_{-1}^{1} U(\theta)dH(\theta), \quad \text{s.t. } F \succeq_{MPS} H \ . \tag{C.3}$$

Proof of Observation C.1. We show this by contradiction. Suppose $U(\cdot)$ satisfies the increasing slope property at point z and there exists some $H \not\succeq_{MPS} H_{\mathscr{P}(z)}$ that solves (C.3). Let $\pi = (S,\sigma)$ be an information policy that induces H. For each $s \in S$, let γ_s denote the posterior distribution induced by s and h_s be the mean of γ_s. Let $\delta \in \Delta(S)$ denote the ex-ante distribution of messages $s \in S$ induced by π. With these we have

$$\int_{-1}^{1} U(\theta)dH(\theta) = \int_{s\in S} U(h_s)d\delta(s) = \int_{s\in S} \mathscr{U}_{\underline{\pi}}(U,\gamma_s)d\delta(s) \ .$$

Since $H \not\succeq_{MPS} H_{\mathscr{P}(z)}$, there exists $s \in S$ such that $0 < \gamma_s^{-}(z) \le \gamma_s(z) < 1$ holds. Denote by $\widetilde{S} \subseteq S$ the set of all such s. Then $\widetilde{S}$ must have positive probability measure under δ. Consider the joint information policy $\widetilde{\pi}$ induced by π and the cutoff policy $\mathscr{P}(z)$, and let $\widetilde{H} = H_{\widetilde{\pi}}$. For all events $s \in \widetilde{S}$, it follows from Claim C.1 that $\mathscr{U}_{\mathscr{P}(z)}(U,\gamma_s) > \mathscr{U}_{\underline{\pi}}(U,\gamma_s)$. For $s \notin \widetilde{S}$, $\mathscr{U}_{\mathscr{P}(z)}(U,\gamma_s) = \mathscr{U}_{\underline{\pi}}(U,\gamma_s)$ holds trivially. Therefore, we have

$$\begin{aligned}
\int_{-1}^{1} U(\theta)d\widetilde{H}(\theta) &= \int_{s\in\widetilde{S}} \mathscr{U}_{\mathscr{P}(z)}(U,\gamma_s)d\delta(s) + \int_{s\in S/\widetilde{S}} \mathscr{U}_{\mathscr{P}(z)}(U,\gamma_s)d\delta(s) \\
&= \int_{s\in\widetilde{S}} \mathscr{U}_{\mathscr{P}(z)}(U,\gamma_s)d\delta(s) + \int_{s\in S/\widetilde{S}} \mathscr{U}_{\underline{\pi}}(U,\gamma_s)d\delta(s) \\
&> \int_{s\in\widetilde{S}} \mathscr{U}_{\underline{\pi}}(U,\gamma_s)d\delta(s) + \int_{s\in S/\widetilde{S}} \mathscr{U}_{\underline{\pi}}(U,\gamma_s)d\delta(s) = \int_{-1}^{1} U(\theta)dH(\theta) \ .
\end{aligned} \tag{C.4}$$

This contradicts that H is a solution to (C.3) and thus completes the proof. $\square$

Step 2. Now we shown that $W_n(\cdot)$ satisfies the increasing slope property at z_n for all $z_n \in (-1,1)$. Recall from (9) in Section 5 that $W_n(\theta) = \int_{\underline{v}}^{\theta} (\theta - \phi_n(x))\, \hat{g}_n(x;q)dx$. For any $\theta \neq z_n$ we have

$$\begin{aligned}
W_n(\theta) - W_n(z_n) &= \int_{\underline{v}}^{\theta} (\theta - \phi_n(x))\, \hat{g}_n(x;q)dx - \int_{\underline{v}}^{z_n} (z_n - \phi_n(x))\, \hat{g}_n(x;q)dx \\
&= \int_{z_n}^{\theta} (\theta - \phi_n(x))\, \hat{g}_n(x;q)dx + (\theta - z_n) \int_{\underline{v}}^{z_n} \hat{g}_n(x;q)dx
\end{aligned}$$

Therefore,

$$\lambda_n(\theta;z_n) := \frac{W_n(\theta)-W_n(z_n)}{\theta-z_n} = \int_{z_n}^{\theta} \frac{\theta-\phi_n(x)}{\theta-z_n}\hat{g}_n(x;q)dx + \int_{\underline{v}}^{z_n} \hat{g}_n(x;q)dx \ .$$

Taking the derivative of $\lambda_n(\theta;z_n)$ with respect to θ yields

$$\lambda_n{}'(\theta;z_n) = \frac{\theta-\phi_n(\theta)}{\theta-z_n}\hat{g}_n(\theta;q) + \int_{z_n}^{\theta} \frac{\phi_n(x)-z_n}{(\theta-z_n)^2}\hat{g}_n(x;q)dx \ . \tag{C.5}$$

Recall from the premise of this lemma that $\phi_n(\cdot)$ crosses zero only once and from above at z_n. For any $\theta > z_n$, $x > \phi_n(x) \geq z_n$ holds for all $x \in (z_n, \theta]$. Therefore, the first term on the right-hand side of (C.5) must be strictly positive and the second term is non-negative. This implies $\lambda_n{}'(\theta;z_n) > 0$ for $\theta > z_n$. For any $\theta < z_n$, $x < \phi_n(x) \leq z_n$ holds for all $x \in [\theta, z_n)$. So the first term on the right-hand side of (C.5) is strictly positive, and the second term equals

$$\int_{z_n}^{\theta} \frac{\phi_n(x)-z_n}{(\theta-z_n)^2}\hat{g}_n(x;q)dx = \int_{\theta}^{z_n} \frac{z_n-\phi_n(x)}{(\theta-z_n)^2}\hat{g}_n(x;q)dx$$

and is non-negative. This implies $\lambda_n{}'(\theta;z_n) > 0$ for $\theta < z_n$ as well. Taken together, $\lambda_n{}'(\theta;z_n) > 0$ holds for all $\theta \neq z_n$. Finally, since $W_n(\theta)$ is differentiable, $\lambda_n(\theta;z_n)$ is continuous at $\theta = z_n$ and $\lim_{\theta\to z_n} \lambda_n(\theta;z_n) = W_n{}'(z_n)$. These together implies that $\lambda_n(\theta;z_n)$ is strictly increasing in θ and hence satisfies the increasing slope property at point z_n. By Observation C.1, this implies Lemma 4.

C.2 Proof of Lemma 5

By (9), the second order derivative of $W_n(\theta)$ is given by

$$\begin{aligned} W_n''(\theta) &= \hat{g}_n(\theta;q)\left(2-\phi_n'(\theta)\right) + \hat{g}_n'(\theta;q)\left(\theta-\phi_n'(\theta)\right) \\ &= \hat{g}_n(\theta;q)\left\{2-\phi_n{}'(\theta) + (\theta-\phi_n(\theta))\frac{\hat{g}_n'(\theta;q)}{\hat{g}_n(\theta;q)}\right\} \\ &= \hat{g}_n(\theta;q)\left\{2-\phi_n'(\theta) + (\theta-\phi_n(\theta))\left(n\frac{g(\theta)}{G(\theta)}\frac{q-G(\theta)}{1-G(\theta)} + \frac{g'(\theta)}{g(\theta)}\right)\right\} \ . \end{aligned} \tag{C.6}$$

The last step follows from (A.7). Because G is strictly positive and twice-continuously differentiable on $[\underline{v},\bar{v}]$, $W_n''(\cdot)$ is continuous. By (C.6) and the fact that $q = G(v_q^*)$, $W_n''(\theta) > 0$ if and only if

$$(\theta-\phi_n(\theta))\left(G(\theta)-G(v_q^*)\right) < \frac{G(\theta)\left(1-G(\theta)\right)}{ng(\theta)}\left(2-\phi_n{}'(\theta) + (\theta-\phi_n(\theta))\frac{g'(\theta)}{g(\theta)}\right) \ . \tag{C.7}$$

Because $\phi_n(\cdot)$, $\phi_n'(\cdot)$ and $\phi_n''(\cdot)$ are uniformly Lipschitz continuous and $g(\cdot)$ is positive and twice continuously differentiable on $[\underline{v}, \overline{v}]$, both the value and the first order derivative of the right-hand side of (C.7) converge to zero uniformly for all $\theta \in [\underline{v}, \overline{v}]$.[8] Let $\zeta_n(\theta) := (\theta - \phi_n(\theta))\left(G(\theta) - G(v_q^*)\right)$ denote the left-hand side of (C.7) and

$$\zeta(\theta) := \lim_{n\to\infty} \zeta_n(\theta) = (\theta - \phi(\theta))\left(G(\theta) - G(v_q^*)\right). \tag{C.8}$$

Both the value and derivative of $\zeta_n(\cdot)$ converge uniformly to $\zeta(\cdot)$ and $\zeta'(\cdot)$, respectively. Therefore, $\lim_{n\to\infty} W_n''(\theta) > (<) 0$ if and only if $\zeta(\theta) < (>) 0$. On the one hand, $G(\theta) - G(v_q^*)$ is increasing in θ and admits a unique root v_q^* at which its derivative equals $g(v_q^*) > 0$. On the other hand, under our definition of the single-crossing property (cf. Definition 1), $\theta - \phi(\theta)$ crosses zero at most once and from below on $[-1,1]$. Recall that $z^* = \lim_{n\to\infty} z_n$ (cf. (13)) and the single-crossing property requires $1 - \phi'(z^*) > 0$ whenever $\phi(z^*) = z^*$ and $z^* \in [-1,1]$. Let

$$\ell^* := \max\left\{\min\left\{z^*, v_q^*\right\}, -1\right\} \quad \text{and} \quad r^* := \min\left\{\max\left\{z^*, v_q^*\right\}, 1\right\}. \tag{C.9}$$

It follows from (C.8) that $\zeta(\theta) < 0$ for all $\theta \in (\ell^*, r^*)$ and $\zeta(\theta) > 0$ for all $\theta \in [-1,1]/[\ell^*, r^*]$. We distinguish between three cases.

Case 1: $z^* = -1$ and $\theta - \phi(\theta) > 0$ for all $\theta \in [-1,1]$. In this case, $\ell^* = -1$ and $\zeta(\theta) > (<) 0$ for $\theta > (<) r^*$ on $[-1,1]$. It then follows that there exists some $N \geq 0$ such that for all $n \geq N$ we have (i) $\theta - \phi_n(\theta) > 0$ for all $\theta \in [-1,1]$ so that $z_n = -1$, and (ii) there is some $r_n \in [-1,1]$ with $r_n \to r^*$ such that $W_n''(\theta) > (<) 0$ for $\theta < (>) r_n$. This implies that $W_n(\cdot)$ is strictly S-shaped on $[z_n, 1]$ with inflection point r_n for all $n \geq N$.

Case 2: $z^* = 1$ and $\theta - \phi(\theta) < 0$ for all $\theta \in [-1,1]$. In this case, $r^* = 1$ and $\zeta(\theta) < (>) 0$ for $\theta > (<) \ell^*$ on $[-1,1]$. There then exists $N \geq 0$ such that for all $n \geq N$ we have (i) $\theta - \phi_n(\theta) < 0$ for all $\theta \in [-1,1]$ so that $z_n = 1$, and (ii) there is some $\ell_n \in [-1,1]$ with $\ell_n \to \ell^*$ such that $W_n''(\theta) > (<) 0$ for $\theta > (<) \ell_n$. This implies that $W_n(\cdot)$ is strictly inverse S-shaped on $[-1, z_n]$ with inflection point ℓ_n for all $n \geq N$.

Case 3: $z^* \in [-1,1]$ and $z^* = \phi(z^*)$. Recall that the single-crossing property requires $1 - \phi'(z^*) > 0$ whenever $\phi(z^*) = z^* \in [-1,1]$. It follows that there exists some $N \geq 0$ and $\varepsilon > 0$ such that for all $n \geq N$ we have that

(i) there exists a unique $\tilde{z}_n \in \delta(z^*; \varepsilon) := (z^* - \varepsilon, z^* + \varepsilon)$ such that both $\tilde{z}_n = \phi_n(\tilde{z}_n)$ and $1 - \phi_n'(\tilde{z}_n) > 0$ hold, and

[8] This is because both $1 - {\phi_n}'(\theta)$ and $\frac{g'(\theta)}{G'(\theta)}$ are uniformly bounded on $[\underline{v}, \overline{v}]$ under our assumption for G.

(ii) there are $\tilde{\ell}_n, \tilde{r}_n \in \mathbb{I} := [-1,1] \cup \delta(z^*;\varepsilon)$ such that $W_n''(\theta) > 0$ if $\theta \in (\tilde{\ell}_n, \tilde{r}_n)$ and $W_n''(\theta) < 0$ if $\theta \in \mathbb{I}/[\tilde{\ell}_n, \tilde{r}_n]$, where $\tilde{\ell}_n$ and $\tilde{r}_n$ satisfy $\tilde{\ell}_n \le \tilde{r}_n$, $\tilde{\ell}_n \to \max\{\min\{z^*, v_q^*\}, \min\{-1, z^*-\varepsilon\}\}$, and $\tilde{r}_n \to \min\{\max\{z^*, v_q^*\}, \max\{1, z^*+\varepsilon\}\}$.

Let $\ell_n = \max\{\tilde{\ell}_n, -1\}$ and $r_n = \min\{\tilde{r}_n, 1\}$. It follows from (ii) that $W_n''(\theta) > 0$ if $\theta \in (\ell_n, r_n)$ and $W_n''(\theta) < 0$ if $\theta \in [-1,1]/[\ell_n, r_n]$. Moreover, $\ell_n \to \ell^*$ and $r_n \to r^*$. Finally, (i) implies

$$\begin{aligned} W_n''(\tilde{z}_n) &= \hat{g}_n(\tilde{z}_n;q)\left(2-\phi_n'(\tilde{z}_n)\right) + \hat{g}_n'(\tilde{z}_n;q)\left(\tilde{z}_n - \phi_n(\tilde{z}_n)\right) \\ &= \hat{g}_n(\tilde{z}_n;q)\left(2-\phi_n'(\tilde{z}_n)\right) > \hat{g}_n(\tilde{z}_n;q) > 0\,. \end{aligned}$$

Therefore $\tilde{z}_n \in (\tilde{\ell}_n, \tilde{r}_n)$ must hold. If $\tilde{z}_n \in [-1,1]$, then $\tilde{z}_n = z_n$ and we obtain $\ell_n \le z_n \le r_n$ with at least one inequality holding strictly. If $\tilde{z}_n < -1$ (resp. $\tilde{z}_n > 1$) then $-1 = z_n = \ell_n \le r_n$ (resp. $1 = z_n = r_n \ge \ell_n$). Taken together, for all $n \ge N$, $W_n(\cdot)$ is S-shaped on $[z_n, 1]$ with inflection point $r_n \ge z_n$ and is inverse S-shaped on $[-1, z_n]$ with inflection point $\ell_n \le z_n$.

These three cases together imply that there exists an $N \ge 0$ such that for all $n \ge N$ it holds that $W_n(\cdot)$ is strictly S-shaped on $[z_n, 1]$ with inflection point $r_n \in [z_n, 1]$ while strictly inverse S-shaped on $[-1, z_n]$ with inflection point $\ell_n \in [-1, z_n]$. Moreover, in the limit we have $\lim_{n\to\infty} \ell_n = \ell^*$ and $\lim_{n\to\infty} r_n = r^*$. This concludes the proof.

C.3 Proof of Proposition 2

Let N be the threshold for n in Lemma 5 and Theorem 1. We show that $N = 0$ holds – or equivalently, $W_n(\cdot)$ is strictly S-shaped on $[z_n, 1]$ and strictly inverse-S-shaped on $[-1, z_n]$ for all $n \ge 0$ – under either of the two conditions in Proposition 2. To start, by (C.6) we have

$$W_n{}''(\theta) = \hat{g}_n(\theta;q)\left(\theta - \phi_n(\theta)\right)\left\{\frac{2-\phi_n{}'(\theta)}{\theta - \phi_n(\theta)} + \frac{\hat{g}_n'(\theta;q)}{\hat{g}_n(\theta;q)}\right\}\,. \tag{C.10}$$

We only consider interval $(z_n, 1]$ and show that $W_n(\cdot)$ is strictly S-shaped on it for all $n > 0$.[9] By definition of z_n, $\theta - \phi_n(\theta) > 0$ for all $\theta > z_n$. It therefore suffices to show that the term in the curly bracket in (C.10) is strictly decreasing. In what follows we show that this is indeed true under either of the two conditions in Proposition 2.

Consider condition (1) first. By Lemma 1, when g is log-concave, $\hat{g}_n(\cdot;q)$ is strictly log-concave so that $\frac{\hat{g}_n'(\theta;q)}{\hat{g}_n(\theta;q)}$ is strictly decreasing for all $n > 0$. Hence, we only need $\frac{2-\phi_n{}'(\theta)}{\theta-\phi_n(\theta)}$ to be decreasing in θ on $(z_n, 1]$ for all $n > 0$. Let $\xi_n(\theta) := \frac{2-\phi_n{}'(\theta)}{\theta-\phi_n(\theta)}$. Simple algebra reveals that $\xi_n'(\theta) \le 0$ if and only if

$$\phi_n{}''(\theta)\left(\theta - \phi_n(\theta)\right) + \left(2 - \phi_n{}'(\theta)\right)\left(1 - \phi_n{}'(\theta)\right) \ge 0\,. \tag{C.11}$$

[9] The proof for the inverse S-shape property on $[-1, z_n]$ is analogous and hence omitted.

Note that ${\phi_n}'(\theta) = \rho \sum_{j=1}^{n+1} w_j \cdot \varphi_j'(\theta; q, n)$ and ${\phi_n}''(\theta) = \rho \sum_{j=1}^{n+1} w_j \cdot \varphi_j''(\theta; q, n)$. As $\rho \to 0$ we have ${\phi_n}'(\theta) \to 0$ and ${\phi_n}''(\theta) \to 0$ uniformly for all $\theta \in [\underline{v}, \overline{v}]$. Therefore, the left-hand side of (C.11) converges to 2 for all $\theta \in [-1,1]$ as $\rho \to 0$. This implies $\xi_n'(\theta) < 0$ for all $\theta \in [-1,1]$ and $n > 0$ when ρ is sufficiently small. Therefore, condition (1) in Proposition 2 indeed ensures $N = 0$.

Now we check condition (2). For a utilitarian sender, recall from (7) in Example 2 that

$$\phi_n(\theta) = \frac{n}{n+1}\left(q\mathbb{E}_G\left[v_i|v_i \le \theta\right] + (1-q)\,\mathbb{E}_G\left[v_i|v_i \ge \theta\right]\right) + \frac{1}{n+1}\theta = \frac{n}{n+1}\phi(\theta) + \frac{\theta}{n+1}\ .$$

Therefore,

$$\theta - \phi_n(\theta) = \frac{n}{n+1}\left(\theta - \phi(\theta)\right) \quad \text{and} \quad 1 - \phi_n'(\theta) = \frac{n}{n+1}\left(1 - \phi'(\theta)\right)\ .$$

With these we obtain

$$\frac{2 - {\phi_n}'(\theta)}{\theta - \phi_n(\theta)} = \frac{1}{\theta - \phi_n(\theta)} + \frac{1 - {\phi_n}'(\theta)}{\theta - \phi_n(\theta)} = \frac{n+1}{n}\frac{1}{\theta - \phi(\theta)} + \frac{1 - \phi'(\theta)}{\theta - \phi(\theta)} = \frac{1}{n}\frac{1}{\theta - \phi(\theta)} + \frac{2 - \phi'(\theta)}{\theta - \phi(\theta)}$$

for all $n > 0$. Using the above equation and (A.7), we obtain

$$\frac{2 - {\phi_n}'(\theta)}{\theta - \phi_n(\theta)} + \frac{\hat{g}_n'(\theta;q)}{\hat{g}_n(\theta;q)} = \frac{2 - \phi'(\theta)}{\theta - \phi(\theta)} + \frac{g'(\theta)}{g(\theta)} + \left(\frac{1}{n}\frac{1}{\theta - \phi(\theta)} + n\frac{g(\theta)}{G(\theta)}\frac{q - G(\theta)}{1 - G(\theta)}\right) \tag{C.12}$$

for all $n > 0$. Since g is log-concave, we have $\phi'(\theta) < 1$ so that $\theta - \phi(\theta)$ is strictly increasing. Let z be the unique solution to $\theta - \phi(\theta) = 0$. Then $z_n = z$ for all $n > 0$ whenever $z \in [-1,1]$ and $\frac{1}{\theta - \phi(\theta)}$ is strictly decreasing on $(z,1]$. Moreover, as is already shown in the proof of Lemma 1 in Appendix A, $\frac{g(\theta)}{G(\theta)}\frac{q-G(\theta)}{1-G(\theta)}$ is strictly decreasing in θ for all $q \in (0,1)$. Therefore, all terms involving n in (C.12) are all strictly decreasing in θ. Hence, for $\frac{2-{\phi_n}'(\theta)}{\theta-\phi_n(\theta)} + \frac{\hat{g}_n'(\theta;q)}{\hat{g}_n(\theta;q)}$ to be strictly decreasing for all $\theta \in (z,1]$ and $n > 0$, we only need $\frac{2-\phi'(\theta)}{\theta-\phi(\theta)} + \frac{g'(\theta)}{g(\theta)}$ to be decreasing on $(z,1]$. This shows that condition (2) is also sufficient to ensure $N = 0$ and hence complete the proof of Proposition 2.

D Omitted Proofs for Section 6.1: Asymptotic Results

D.1 Proof of Lemma 6

To show part (1), recall that $\phi^* = \phi(v_q^*)$. Together with $\phi^* \in (-1,1)$ and $v_q^* = \phi^*$, this implies $v_q^* = \phi(v_q^*) \in (-1,1)$. (13) then implies $z^* = \phi^* = v_q^*$. To show part (2), note that $\phi(\cdot)$ is a constant and equal to χ when $\rho = 0$. Hence, by (13), $z^* = \phi^* = \chi$ in this case. If $\rho > 0$ then $\phi(\cdot)$ is a strictly increasing function. Together with $\phi^* = \phi(v_q^*)$, we obtain that $v^* > (<)\phi^*$ implies $z^* < (>)\phi^*$.

D.2 Proof of Theorem 2

To prove Theorem 2 we introduce two Lemmas D.1 and D.2, which respectively characterize the asymptotically optimal sender payoff and the set of interval revelation policies that generate this payoff.

Lemma D.1. *Suppose $v_q^*, \phi^* \in [-1,1]$ and let W_n be the value of problem (MP) for any $n \geq 0$. Then*

$$W^* := \lim_{n\to\infty} W_n = \begin{cases} \int_{\underline{\phi}}^1 (k-\phi^*)dF(k), & \text{if } \phi^* < \underline{\phi} \\ \int_{\phi^*}^1 (k-\phi^*)dF(k), & \text{if } \underline{\phi} \leq \phi^* \leq \overline{\phi} \\ \int_{\overline{\phi}}^1 (k-\phi^*)dF(k), & \text{if } \phi^* > \overline{\phi} \end{cases} . \tag{D.1}$$

Proof of Lemma D.1. We start by presenting the sender's asymptotic persuasion problem. Recall that $W_n(\cdot)$ is the sender's indirect utility function and let $W(\theta) := \lim_{n\to\infty} W_n(\theta)$. Then

$$W(\theta) = \begin{cases} \theta - \phi^*, & \text{if } \theta > v_q^* \\ 0, & \text{if } \theta < v_q^* \end{cases} .$$

This is because $v^{(nq+1)} \xrightarrow{p} v_q^*$ and $\varphi_n(v) \xrightarrow{a.s.} \phi^*$. If $\theta < v_q^*$, then $v^{(nq+1)} > \theta$ almost surely so that the status quo is maintained with probability one as $n \to \infty$. The sender's payoff thus converges to 0. Conversely, if $\theta > v_q^*$, then the reform is passed with probability one as $n \to \infty$ so that the sender's payoff converges to $\lim_{n\to\infty} (\theta - \varphi_n(v)) = \theta - \phi^*$. Let

$$\widetilde{W}(\theta) := \begin{cases} W(\theta), & \text{if } \theta \neq v_q^* \\ \max\{v_q^* - \phi^*, 0\}, & \text{if } \theta = v_q^* \end{cases} .$$

The sender's asymptotic persuasion problem is then

$$\max_{H\in\Delta([-1,1])} \int_{-1}^1 \widetilde{W}(\theta)dH(\theta), \text{ s.t. } F \succeq_{MPS} H . \tag{D.2}$$

Because $\widetilde{W}(\cdot)$ is upper semi-continuous, problem (D.2) always admits a solution. We denote the value to (D.2) by W^* and characterize it using Theorem 1 of Dworczak and Martini (2019). For ease of reference we restate their theorem applied to our problem in the following observation.

Observation D.1. *(Theorem 1 of Dworczak and Martini (2019)) If there exists some $H \in \Delta([-1,1])$*

and a convex function $p(\cdot)$ *on* $[-1,1]$ *with* $p(\cdot) \geq \widetilde{W}(\cdot)$ *that satisfy*

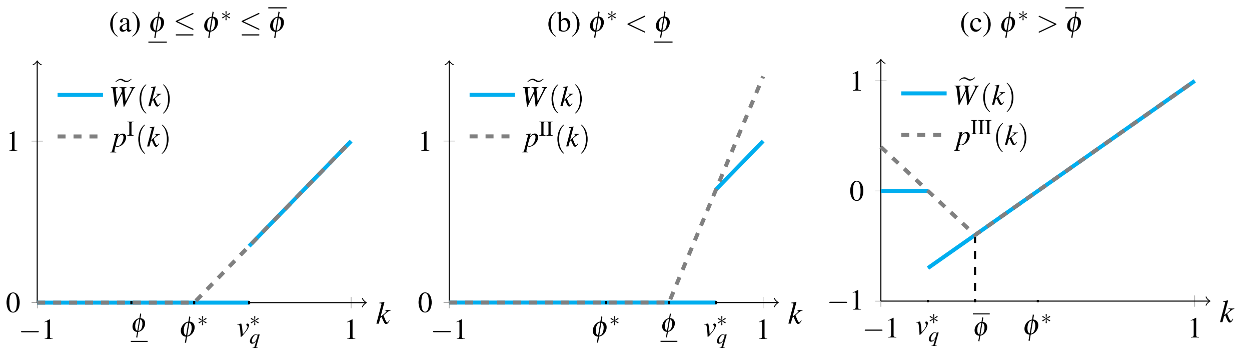
$$supp(H) \subseteq \left\{\theta : p(\theta) = \widetilde{W}(\theta)\right\}, \text{ and} \tag{D.3}$$

$$\int_{-1}^{1} p(\theta)dH(\theta) = \int_{-1}^{1} p(\theta)dH(\theta), \text{ and} \tag{D.4}$$

$$F \succeq_{MPS} H, \tag{D.5}$$

then H *is a solution to* (D.2) *and* $W^* = \int_{-1}^{1} p(\theta)dH(\theta)$.

Figure D.1: Illustration for the proof of Lemma D.1



Note: In all three panels, the blue solid lines denote $\widetilde{W}(\cdot)$ and the gray dashed lines denote the auxiliary functions $p(\cdot)$.

We distinguish between three cases.

Case 1: $\underline{\phi} \leq \phi^* \leq \overline{\phi}$. In this case we have $\mathbb{E}_F[k|k \geq \phi^*] \geq v_q^*$ and $\mathbb{E}_F[k|k \leq \phi^*] \leq v_q^*$. Let $p^{\mathrm{I}}(\theta) := \max\{\theta - \phi^*, 0\}$ for $\theta \in [-1,1]$. $p^{\mathrm{I}}(\cdot)$ is illustrated in Figure D.1a. Consider the cutoff policy $\mathscr{P}(\phi^*)$ and let $\underline{H}^{\mathrm{I}} = H_{\mathscr{P}(\phi^*)}$. The following conditions are easy to verify: (i) $p^{\mathrm{I}}(\cdot)$ is convex and $p^{\mathrm{I}}(\cdot) \geq \widetilde{W}(\cdot)$ on $[-1,1]$; (ii) $F \succeq_{MPS} \underline{H}^{\mathrm{I}}$ and $\int_{-1}^{1} p^{\mathrm{I}}(\theta)d\underline{H}^{\mathrm{I}}(\theta) = \int_{-1}^{1} p^{\mathrm{I}}(\theta)dF(\theta)$, and (iii)

$$\begin{aligned}\operatorname{supp}(\underline{H}^{\mathrm{I}}) &= \{\mathbb{E}_F[k|k \leq \phi^*], \mathbb{E}_F[k|k \geq \phi^*]\} \\ &\subset \left\{\theta \middle| p^{\mathrm{I}}(\theta) = \widetilde{W}(\theta)\right\} = \begin{cases}[-1,1], & \text{if } v_q^* = \phi^* \\ [-1,v_q^*] \cup [\phi^*,1], & \text{if } v_q^* < \phi^* \\ [-1,\phi^*] \cup [v_q^*,1], & \text{if } v_q^* > \phi^*\end{cases}.\end{aligned}$$

Therefore, by Observation D.1, $\underline{H}^{\mathrm{I}}$ solves problem (D.2) and the value of (D.2) equals

$$\int_{-1}^{1} p^{\mathrm{I}}(k)dF(k) = \int_{-1}^{1} \max\{k - \phi^*, 0\}dF(k) = \int_{\phi^*}^{1} (k - \phi^*)dF(k)\,. \tag{D.6}$$

Case 2: $\phi^* < \underline{\phi}$. For this case we use the definition of $\underline{\phi}$ (cf. (15)) and obtain $\underline{\phi} \in (-1,1)$ and

$\mathbb{E}_F[k|k \geq \underline{\phi}] = v_q^*$. Let

$$p^{\mathrm{II}}(\theta) := \begin{cases} \frac{v_q^*-\phi^*}{v_q^*-\underline{\phi}}\left(\theta - \underline{\phi}\right), & \text{if } \underline{\phi} \leq \theta \leq 1 \\ 0, & \text{if } -1 \leq \theta < \underline{\phi} \end{cases} .$$

$p^{\mathrm{II}}(\cdot)$ is illustrated in Figure D.1b. Consider cutoff policy $\mathscr{P}(\underline{\phi})$ and let $\underline{H}^{\mathrm{II}} = H_{\mathscr{P}(\underline{\phi})}$. The following conditions are easy to verify: (i) $p^{\mathrm{II}}(\cdot)$ is convex and $p^{\mathrm{II}}(\cdot) \geq \widetilde{W}(\cdot)$ on $[-1,1]$; (ii) $F \succeq_{MPS} \underline{H}^{\mathrm{II}}$ and $\int_{-1}^{1} p^{\mathrm{II}}(\theta) d\underline{H}^{\mathrm{II}}(\theta) = \int_{-1}^{1} p^{\mathrm{II}}(\theta) dF(\theta)$; and (iii)

$$\mathrm{supp}(\underline{H}^{\mathrm{II}}) = \left\{\mathbb{E}_F[k|k < \underline{\phi}], v_q^*\right\} \subset \left\{\theta \middle| p^{\mathrm{II}}(\theta) = \widetilde{W}(\theta)\right\} = [-1, \underline{\phi}] \cup \{v_q^*\}.$$

Hence, $\underline{H}^{\mathrm{II}}$ solves problem (D.2) and the value of (D.2) equals

$$\begin{aligned} \int_{-1}^{1} p^{\mathrm{II}}(k) dF(k) &= \left(1 - F(\underline{\phi})\right) \frac{v_q^* - \phi^*}{v_q^* - \underline{\phi}} \left(\mathbb{E}_F[k|k > \underline{\phi}] - \underline{\phi}\right) \\ &= \left(1 - F(\underline{\phi})\right) \left(v_q^* - \phi^*\right) = \int_{\underline{\phi}}^{1} (k - \phi^*) \, dF(k) . \end{aligned} \tag{D.7}$$

Case 3: $\phi^* > \overline{\phi}$. For this case we use the definition of $\overline{\phi}$ (cf. (14)) and obtain $\overline{\phi} \in (-1,1)$ and $\mathbb{E}_F[k|k \leq \overline{\phi}] = v_q^*$. Let

$$p^{\mathrm{III}}(\theta) = \begin{cases} \theta - \phi^*, & \text{if } \overline{\phi} \leq 1 \\ \frac{\overline{\phi}-\phi^*}{\overline{\phi}-v_q^*}\left(\theta - v_q^*\right), & \text{if } -1 \leq \theta < \overline{\phi} \end{cases} .$$

$p^{\mathrm{III}}(\cdot)$ is illustrated in Figure D.1c. Consider cutoff policy $\mathscr{P}(\overline{\phi})$ and let $\underline{H}^{\mathrm{III}} = H_{\mathscr{P}(\overline{\phi})}$. The following conditions are again easy to verify: (i) $p^{\mathrm{III}}(\cdot)$ is convex and $p^{\mathrm{III}}(\cdot) \geq \widetilde{W}(\cdot)$; (ii) $F \succeq_{MPS} \underline{H}^{\mathrm{III}}$ and $\int_{-1}^{1} p^{\mathrm{III}}(\theta) d\underline{H}^{\mathrm{III}}(\theta) = \int_{-1}^{1} p^{\mathrm{III}}(\theta) dF(\theta)$; and (iii)

$$\mathrm{supp}(\underline{H}^{\mathrm{III}}) = \left\{\mathbb{E}_F\left[k|k > \overline{\phi}\right], v_q^*\right\} \subset \left\{\theta \middle| p^{\mathrm{III}}(\theta) = \widetilde{W}(\theta)\right\} = \{v_q^*\} \cup [\overline{\phi}, 1] .$$

Following analogous arguments as in previous cases, we can establish that $\underline{H}^{\mathrm{III}}$ solves problem (D.2) and the value of (D.2) equals

$$\begin{aligned} \int_{-1}^{1} p^{\mathrm{III}}(k) dF(k) &= F\left(\overline{\phi}\right) \frac{\overline{\phi} - \phi^*}{\overline{\phi} - v_q^*} \left(\mathbb{E}[k|k < \overline{\phi}] - v_q^*\right) + \int_{\overline{\phi}}^{1} (k - \phi^*) \, dF(k) \\ &= \int_{\overline{\phi}}^{1} (k - \phi^*) dF(\theta) . \end{aligned} \tag{D.8}$$

Taken together, (D.6) to (D.8) and Observation D.1 imply (D.1). □

Lemma D.2. *Suppose $v_q^*, \phi^* \in [-1,1]$ and let $\mathscr{P}^*$ denote the set of interval revelation policies that generate the asymptotically optimal payoff W^* given by (D.1). Then $\mathscr{P}^*$ is characterized as follows.*

1. *If $\phi^* = v_q^*$ then $\mathscr{P}^* = \{\mathscr{P}(a,b) : -1 \le a \le \phi^* \le b \le 1\}$.*
2. *If $\phi^* < v_q^*$ then $\mathscr{P}^* = \left\{\mathscr{P}(a,b) : -1 \le a \le b = \min\{\phi^*, \overline{\phi}\}\right\}$.*
3. *If $\phi^* > v_q^*$ then $\mathscr{P}^* = \left\{\mathscr{P}(a,b) : \max\{\phi^*, \underline{\phi}\} = a \le b \le 1\right\}$.*

Proof of Lemma D.2. Observe that (D.1) can be rewritten as $W^* = \int_{t^*}^{1} (k-\phi^*)dF(k)$ where $t^* = \text{median}\left\{\overline{\phi}, \phi^*, \underline{\phi}\right\}$. We distinguish between three cases:

Case 1. $\overline{\phi} \ge \phi^* \ge \underline{\phi}$ and therefore $W^* = \int_{\phi^*}^{1} (k-\phi^*)dF(k)$. If $v_q^* = \phi^*$, then all $\mathscr{P}(a,b)$ with $a \le \phi^* \le b$ yield the same optimal asymptotic payoff W^*. If $\phi^* < v_q^*$, then $a \le b = \phi^*$ is necessary. If $\phi^* > v_q^*$ then $\phi^* = a \le b$ is necessary.

Case 2: $\phi^* > \overline{\phi}$ and therefore $W^* = \int_{\overline{\phi}}^{1} (k-\phi^*)dF(k)$. In this case it must hold that $\phi^* > v_q^*$. Policy $\mathscr{P}(a,b)$ can implement the same outcome if and only if $\overline{\phi} = a \le b$.

Case 3: $\phi^* < \underline{\phi}$ and therefore $W^* = \int_{\underline{\phi}}^{1} (k-\phi^*)dF(k)$. In this case it must hold that $\phi^* < v_q^*$. Policy $\mathscr{P}(a,b)$ can implement the same outcome if and only if $a \ge b = \underline{\phi}$.

These together complete the proof of Lemma D.2. □

Now we prove Theorem 2.

Proof of Theorem 2. Consider any pair of sequences $\{b_n\}_{n\ge N}$ and $\{a_n\}_{n\ge N}$. Because both sequences are bounded on a closed interval, by Bolzano–Weierstrass Theorem they must contain at least one convergent subsequence each. Let b^* and a^* be the limits of these convergent subsequences. In what follows we shall characterize b^* and a^*, and then show that all sub-sequences of $\{a_n\}_{n\ge N}$ and $\{b_n\}_{n\ge N}$ converge to them so that $\{a_n\}_{n\ge N}$ and $\{b_n\}_{n\ge N}$ indeed converge.

On the one hand, asymptotic optimality requires that $\mathscr{P}(a^*,b^*) \in \mathscr{P}^*$ must hold, where $\mathscr{P}^*$ is characterized in Lemma D.2. On the other hand, by Lemma 5 (and its proof in Appendix C.2), the single-crossing property implies for sufficiently large n that there exists $z_n \in [-1,1]$ and $-1 \le \ell_n \le z_n \le r_n \le 1$ such that the following conditions must hold:

$$\ell_n \le a_n \le z_n \le b_n \le r_n$$

and $z_n \to z^*$, $\ell_n \to \min\left\{z^*, v_q^*\right\}$ and $r_n \to \max\left\{z^*, v_q^*\right\}$ for all $v_q^* \in (-1,1)$. We now distinguish between three cases. For all these cases recall that $\phi(v_q^*) = \phi^*$ and $\phi(\cdot)$ is non-decreasing.

Case 1. $\phi^* = v_q^*$. In this case $z^* = v_q^*$ because $\phi(v_q^*) = \phi^* = v_q^*$. Therefore, both ℓ_n and r_n converge to ϕ^*. By the squeeze theorem both a_n and b_n must also converge to ϕ^* and thus $a^* = b^* = \phi^*$.

Case 2. $\phi^* < v_q^*$. In this case $z^* \leq \phi^* < v_q^*$ so that $\ell_n \to z^*$ and hence $a^* = z^* < v_q^*$. Moreover, by part (2) of Lemma D.2, $b^* = \phi^*$if $\phi^* \in [\underline{\phi}, v_q^*)$ and $b^* = \underline{\phi}$ if $\phi^* < \underline{\phi}$.

Case 3. $\phi^* > v_q^*$. In this case $z^* \geq \phi^* > v_q^*$ so that $r_n \to z^*$ and hence $b^* = z^* > v_q^*$. Moreover, by part (3) of Lemma D.2 we have $a^* = \phi^*$if $\phi^* \in (v_q^*, \overline{\phi}]$ and $a^* = \overline{\phi}$ if $\phi^* > \overline{\phi}$.

These complete the characterizations of $a^*$and b^* and these apply for any convergent subsequences of a_n and b_n. Therefore, the limits of $\{a_n\}$ and $\{b_n\}$ exist and are equal to a^* and b^*, respectively. $\square$

D.3 Proof of Proposition 3

It is straightforward from the definitions of $\overline{W}$ and W^{Full} that

$$\overline{W} = \int_{\phi^*}^{1} (k - \phi^*) f(k) dk \quad \text{and} \quad W^{\text{Full}} = \int_{v_q^*}^{1} (k - \phi^*) f(k) dk \,. \tag{D.9}$$

This is because, as $n \to \infty$, in the omniscient scenario the sender prefers reform (status quo) if $k > (<) \phi^*$, while under full information the pivotal voter prefers reform (status quo) if $k > (<) v_q^*$.

Let $\gamma(x) := \int_x^1 (k - \phi^*) dF(k)$ for $x \in [-1, 1]$. We can then rewrite (D.9) as $\overline{W} = \gamma(\phi^*)$ and $W^{\text{Full}} = \gamma(v_q^*)$. Note that $\gamma'(x) = (\phi^* - x) f(x) > (<) 0$ for $x < (>) \phi^*$. This implies that $\gamma(x)$ is single-peaked at ϕ^*. Therefore, $\overline{W} \geq W^{\text{Full}}$ and equality holds if and only if $\phi^* = v_q^*$ whenever $v_q^* \in (-1, 1)$. Moreover, by Lemma D.1, $W^* = \gamma(\phi^*) = \overline{W}$ if $\underline{\phi} \leq \phi^* \leq \overline{\phi}$. These together establish statements (1) and (2). To show statement (3), consider $\phi^* > \overline{\phi}$ first. In this case Lemma D.1 implies that $W^* = \gamma(\overline{\phi})$ with $\overline{\phi} \in (v_q^*, \phi^*)$. Since $\gamma(x)$ is strictly increasing for $x < \phi^*$, we have $\gamma(\phi^*) > \gamma(\overline{\phi}) > \gamma(v_q^*)$, that is $\overline{W} > W^* > W^{\text{Full}}$. The proof for the case $\phi^* < \underline{\phi}$ is analogous.

E Omitted Proofs for Section 6.2: Comparative Statics

In this appendix we prove the comparative static results presented in Section 6.2. We use two different approaches to establish these results.

E.1 First-order approach and proofs of Propositions 4 and 5

In this subsection we use the first-order approach to prove Propositions 4 and 5. We only prove the statements concerning b_n; the proofs for claims concerning a_n are similar and thus omitted.

Recall from (FOC: b_n) in Section 5.2 that the optimality condition for b_n is given by

$$\left(\tilde{b}_n - b_n\right) W_n'(\tilde{b}_n) \leq W_n(\tilde{b}_n) - W_n(b_n) \tag{E.1}$$

where $\tilde{b}_n = \mathbb{E}_F[k|k \geq b_n]$ and this condition is binding whenever $b_n \in (-1,1)$. Recall from (9) that $W_n(\theta) = \int_{\underline{v}}^{\theta} (\theta - \phi_n(x))\, \hat{g}_n(x;q)dx$ and its derivative is given by

$$W_n{}'(\theta) = \hat{G}_n(\theta;q) + (\theta - \phi_n(\theta))\, \hat{g}_n(\theta;q) \ .$$

With these we have

$$\begin{aligned} W_n(\tilde{b}_n) - W_n(b_n) &= \int_{\underline{v}}^{\tilde{b}_n} \left(\tilde{b}_n - \phi_n(x)\right) \hat{g}_n(x;q)dx - \int_{\underline{v}}^{b_n} \left(b_n - \phi_n(x)\right) \hat{g}_n(x;q)dx \\ &= \int_{b_n}^{\tilde{b}_n} \left(\tilde{b}_n - \phi_n(x)\right) \hat{g}_n(x;q)dx + (\tilde{b}_n - b_n) \int_{\underline{v}}^{b_n} \hat{g}_n(x;q)dx \end{aligned}$$

and

$$\left(\tilde{b}_n - b_n\right) W_n'(\tilde{b}_n) = \left(\tilde{b}_n - b_n\right) \left[\left(\tilde{b}_n - \phi_n(\tilde{b}_n)\right) \hat{g}_n(\tilde{b}_n;q) + \int_{\underline{v}}^{\tilde{b}_n} \hat{g}_n(x;q)dx \right] \ .$$

Plugging these into (E.1) and rearranging terms, we verify that (E.1) is equivalent to

$$\tilde{b}_n - b_n \leq \int_{b_n}^{\tilde{b}_n} \frac{b_n - \phi_n(x)}{\tilde{b}_n - \phi_n(\tilde{b}_n)} \frac{\hat{g}_n(x;q)}{\hat{g}_n(\tilde{b}_n;q)} dx \ . \tag{E.2}$$

By Lemma 5, for sufficiently large n it holds that $W_n(\cdot)$ is strictly S-shaped on $[z_n,1]$ with some inflection point $r_n \in [z_n,1]$. This implies that

$$\left(\mathbb{E}_F\left[k|k \geq x\right] - x\right) W_n'\left(\mathbb{E}_F\left[k|k \geq x\right]\right) - \left[W_n\left(\mathbb{E}_F\left[k|k \geq x\right]\right) - W_n\left(x\right)\right]$$

can cross zero at most once and from above as x increases from z_n to 1. In particular, suppose b_n satisfies (E.2) with equality and hold it fixed, then if any parameter change increases the value of the right-hand side of (E.2), then $\left(\tilde{b}_n - b_n\right) W_n'(\tilde{b}_n) - \left[W_n(\tilde{b}_n) - W_n(b_n)\right]$ will be negative following this parameter change. b_n must therefore decrease to restore equality. Comparative static analyses thus can done with the right-hand side of (E.2) alone. With this we can prove Propositions 4 and 5.

Proof of Proposition 4. Let $\gamma_n(x) := \sum_{j=1}^{n+1} w_j \varphi_j(x;q,n)$. If $\rho < 1$, it follows from (3) that

$$\phi_n(x) = \rho\gamma_n(x) + (1-\rho)\chi \ .$$

As is explained in the proof of Lemma 3 in Appendix B, under either condition (i) or (ii) of Lemma 3 it holds that $1 - \phi'(x) > 0$ for all $x \in [\underline{v}, \overline{v}]$. Since $\phi_n(\cdot)$ converges uniformly to $\phi'(\cdot)$ (cf. Lemma 2), $1 - \phi_n'(\cdot) > 0$ on $[\underline{v}, \overline{v}]$ must hold for sufficiently large n. This implies that $x - \phi_n(x)$ is strictly increasing. Moreover, because $\tilde{b}_n = \mathbb{E}_F\left[k|k \geq b_n\right] > b_n$ and $\phi_n(x)$ is non-decreasing (cf. Proposition

1), for all $x \in (b_n, \tilde{b}_n)$ we have

$$1 > \frac{b_n - \phi_n(b_n)}{\tilde{b}_n - \phi_n(\tilde{b}_n)} \geq \frac{b_n - \phi_n(x)}{\tilde{b}_n - \phi_n(\tilde{b}_n)} = \frac{b_n - \rho\gamma_n(x) - (1-\rho)\chi}{\tilde{b}_n - \rho\gamma_n(\tilde{b}_n) - (1-\rho)\chi} .$$

Consider any $\chi_{\mathrm{I}} > \chi_{\mathrm{II}}$. Observe that a decrease of χ from χ_{I} to χ_{II} induces a common increase on both the nominator and the denominator of $\frac{b_n - \phi_n(x)}{\tilde{b}_n - \phi_n(\tilde{b}_n)}$, which is smaller than one for all $x \in \left(b_n, \tilde{b}_n\right)$. This shift of χ therefore strictly increases the value of $\frac{b_n - \phi_n(x)}{\tilde{b}_n - \phi_n(\tilde{b}_n)}$ for all $x \in \left(b_n, \tilde{b}_n\right)$.[10] On the other hand, the term $\frac{\hat{g}_n(x;q)}{\hat{g}_n(\tilde{b}_n;q)}$ is independent of χ for all $x \in \left(b_n, \tilde{b}_n\right)$. These together implies that a shift of χ from χ_{I} to χ_{II} strictly increases the right-hand side of (E.2). Therefore, if $b_n \in (-1, 1)$ under χ_{I} so that (E.2) is binding, such a shift of χ will make the right-hand side of (E.2) strictly higher than the left-hand side. b_n must strictly decrease to make (E.2) binding again or drop to -1. If $b_n = -1$ under χ_{I} so that (E.2) holds with '$\leq$', then this must remain to be the case after the shift of χ so that the optimal b_n remains to be -1. These together show that b_n is non-increasing as χ decreases and thus prove the claim for b_n in Proposition 4. □

Next we prove Proposition 5. To do so we introduce an auxiliary result.

Lemma E.1. *For any* $0 < y < z < 1$, $\int_y^z \frac{\tau_n(x;q)}{\tau_n(z;q)} dx$ *is strictly decreasing in* q*, where* $\tau_n(x;q)$ *is given by* (A.1) *in Appendix A.*[11]

Proof of Lemma E.1. For any pair of $(x,y) \in (0,1)^2$ and $q \in (0,1)$, define

$$\Delta\psi(x,y;q) := q \ln \frac{x}{y} + (1-q) \ln \frac{1-x}{1-y} = \ln \frac{1-x}{1-y} + q \left(\ln \frac{x}{1-x} - \ln \frac{y}{1-y} \right) . \tag{E.3}$$

It then follows from the definition of $\tau_n(\cdot;q)$ that

$$\ln \frac{\tau_n(x;q)}{\tau_n(y;q)} = n \left(q \ln \frac{x}{y} + (1-q) \ln \frac{1-x}{1-y} \right) = n\Delta\psi(x,y;q) .$$

We can thus rewrite $\int_y^z \frac{\tau_n(x;q)}{\tau_n(z;q)} dx$ as

$$\int_y^z \frac{\tau_n(x;q)}{\tau_n(z;q)} dx = \int_y^z e^{n\Delta\psi(x,z;q)} dx .$$

[10] This follows from the fact that $\frac{a+c}{b+c} > \frac{a}{b}$ for all $b, c > 0$ and $b > a$.

[11] Here we implicitly assume nq is an integer for ease of exposition. If this is not the case, then just replace q with $\hat{q} = \lfloor nq \rfloor / n$ and the all arguments hold for $\hat{q}$.

Using (E.3) and the fact that $\ln \frac{x}{1-x}$ is strictly increasing in x, we obtain for all $y < z$ that

$$\frac{\partial}{\partial q} \int_y^z \frac{\tau_n(x;q)}{\tau_n(z;q)} dx = n \int_y^z e^{n\Delta\psi(x,z;q)} \left(\ln \frac{x}{1-x} - \ln \frac{z}{1-z} \right) dx < 0 \, .$$

This implies the strict decreasing property stated in this lemma. □

Proof of Proposition 5. Recall from (A.4) that $\hat{g}_n(x;q) = \tau_n\left(G(x);q\right) g(x)$ for all $x \in [\underline{v}, \overline{v}]$. Plugging this into (E.2), we obtain

$$\begin{aligned} \int_{b_n}^{\tilde{b}_n} \frac{b_n - \phi_n(x)}{\tilde{b}_n - \phi_n(\tilde{b}_n)} \frac{\hat{g}_n(x;q)}{\hat{g}_n(\tilde{b}_n;q)} dx &= \int_{b_n}^{\tilde{b}_n} \frac{b_n - \phi_n(x)}{\tilde{b}_n - \phi_n(\tilde{b}_n)} \frac{\tau_n\left(G(x);q\right) g(x)}{\tau_n\left(G(\tilde{b}_n);q\right) g(\tilde{b}_n)} dx \\ &= \frac{1}{g(\tilde{b}_n)} \int_{G(b_n)}^{G(\tilde{b}_n)} \frac{b_n - \phi_n\left(G^{-1}(y)\right)}{\tilde{b}_n - \phi_n(\tilde{b}_n)} \frac{\tau_n\left(y;q\right)}{\tau_n\left(G(\tilde{b}_n);q\right)} dy \, . \end{aligned} \tag{E.4}$$

Recall that for $\rho = 0$ we have $\phi_n(x) = \chi$ for all $x \in [\underline{v}, \overline{v}]$. Plugging this into (E.4), we get

$$\int_{b_n}^{\tilde{b}_n} \frac{b_n - \phi_n(x)}{\tilde{b}_n - \phi_n(\tilde{b}_n)} \frac{\hat{g}_n(x;q)}{\hat{g}_n(\tilde{b}_n;q)} dx = \frac{1}{g(\tilde{b}_n)} \left(\frac{b_n - \chi}{\tilde{b}_n - \chi} \right) \int_{G(b_n)}^{G(\tilde{b}_n)} \frac{\tau_n(y;q)}{\tau_n\left(G(\tilde{b}_n);q\right)} dy \, . \tag{E.5}$$

Because $\tilde{b}_n > b_n > \chi$ and $G(\tilde{b}_n) > G(b_n)$, Lemma E.1 implies that (E.5) is strictly decreasing in q. Therefore, the right-hand side of (E.2) strictly increases as q rises from q_{I} to q_{II} for all $q_{\mathrm{I}} < q_{\mathrm{II}}$. If $b_n \in (-1, 1)$ under q_{I} so that (E.2) is binding, then such a shift of q will make (E.2) hold with '$>$' so that b_n must strictly increase to regain equality or up to 1. If $b_n = -1$ under q_{I}, then (E.2) holds with '$\leq$'. The shift of q either (i) retains (E.2) with '$\leq$' so that the optimal b_n is still -1, or it shifts '$\leq$' to '$>$' so that the optimal $b_n > -1$. These together show that b_n is non-decreasing as q increases and thus prove the claim for b_n in Proposition 5. □

E.2 Limiting approach for comparative statics

In this subsection we use the limiting approach to prove Proposition 6 and establish Proposition E.1 below, which is an analog of Proposition 4 for the welfare weighting function $w(\cdot)$ when $\rho > 0$.

Suppose the single-crossing property holds and let $a^* := \lim_{n\to\infty} a_n$ and $b^* := \lim_{n\to\infty} b_n$ be the limits of optimal thresholds characterized in Theorem 2. Our comparative statics primarily concern how a^* and b^* vary with voting rule q and a prosocial sender's welfare weighting function $w(\cdot)$. By Theorem 2, this boils down to understanding how these factors affect ϕ^*, v_q^* and z^*. For ϕ^* these are already summarized in Lemma B.6 in Appendix B. From the definition of v_q^* it is obvious that it is strictly increasing in q and independent of $w(\cdot)$. As for z^*, Lemma E.2 below shows that when both G and $1-G$ are strictly log-concave it inherits the comparative statics of ϕ^*.

Lemma E.2. *Suppose both G and $1-G$ are strictly log-concave. Then (i) z^* weakly decreases as $w(\cdot)$ shifts from $w^I(\cdot)$ to $w^{II}(\cdot)$, where $w^I(\cdot) \succeq_{FOSD} w^{II}(\cdot)$; (ii) z^* is weakly decreasing in q.*

Proof of Lemma E.2. By Lemma 3, strict log-concavities of G and $1-G$ ensure the single-crossing property and hence the existence of a unique z^* for all $w(\cdot)$ and q. The definition of z^* (cf. (13)) implies that it must decrease if function $\phi(\cdot)$ systematically shifts downward – i.e., $\phi(x)$ strictly decreases for all $x \in (\underline{v}, \overline{v})$ – after some shift of $w(\cdot)$ or q. The decreasing properties (i) and (ii) then follow from Lemma B.5, which claims that $\phi(\cdot)$ shifts downwards if q increases or if $w(\cdot)$ varies from some $w^{\mathrm{I}}(\cdot)$ to $w^{\mathrm{II}}(\cdot)$ with $w^{\mathrm{I}}(\cdot) \succeq_{FOSD} w^{\mathrm{II}}(\cdot)$, when $\rho > 0$. □

With these we are ready to prove Proposition 6 and establish Proposition E.1. For the proof of Proposition 6 we will frequently use the fact that the two boundaries $\overline{\phi}$ and $\underline{\phi}$ defined by equations (14) and (15) in the main text are both non-decreasing functions of v_q^*. We therefore write them as $\overline{\phi}(v_q^*)$ and $\underline{\phi}(v_q^*)$ in the proof to make this dependence explicit.

Proof of Proposition 6. We prove Proposition 6 by construction. For any pair of q_{I} and q_{II}, we use a_i^* and b_i^* to denote the boundaries of the asymptotically optimal revelation interval under $q = q_i$ for $i \in \{\mathrm{I}, \mathrm{II}\}$. We assume $\phi^* \in (-1,1)$ and let $\widehat{q} := G(\phi^*)$; under $q = \widehat{q}$ we have $\phi^* = G^{-1}(q) = v_q^*$.

First, suppose q_{I} and q_{II} satisfy (i) $\widehat{q} \le q_{\mathrm{I}} < q_{\mathrm{II}}$, (ii) $\phi^* \in \left[\underline{\phi}(v_{q_{\mathrm{I}}}^*), \overline{\phi}(v_{q_{\mathrm{I}}}^*)\right]$, and (iii) $\phi^* < \underline{\phi}(v_{q_{\mathrm{II}}}^*)$. Then, by Theorem 2, $(a_{\mathrm{I}}^*, b_{\mathrm{I}}^*) = (z_{\mathrm{I}}^*, \phi^*)$ and $(a_{\mathrm{II}}^*, b_{\mathrm{II}}^*) = \left(z_{\mathrm{II}}^*, \underline{\phi}(v_{q_{\mathrm{II}}}^*)\right)$. Since $\underline{\phi}(v_{q_{\mathrm{II}}}^*) > \phi^*$ and $z_{\mathrm{II}}^* < z_{\mathrm{I}}^*$ (by Lemma E.2), we get $a_{\mathrm{II}}^* < a_{\mathrm{I}}^* \le b_{\mathrm{I}}^* < b_{\mathrm{II}}^*$. This implies for sufficiently large n that a_n decreases and b_n increases as q shifts from q_{I} to q_{II}. This shows case 1 of Proposition 6 is possible.

To show that case 2 is also possible, consider any q_{I} and q_{II} that satisfy (i) $q_{\mathrm{I}} < q_{\mathrm{II}} \le \widehat{q}$, (ii) $\phi^* > \overline{\phi}(v_{q_{\mathrm{I}}}^*)$, and (iii) $\phi^* \in \left[\underline{\phi}(v_{q_{\mathrm{II}}}^*), \overline{\phi}(v_{q_{\mathrm{II}}}^*)\right]$. By Theorem 2, $(a_{\mathrm{I}}^*, b_{\mathrm{I}}^*) = \left(\overline{\phi}(v_{q_{\mathrm{I}}}^*), z_{\mathrm{I}}^*\right)$ and $(a_{\mathrm{II}}^*, b_{\mathrm{II}}^*) = (\phi^*, z_{\mathrm{II}}^*)$. In this case we have $a_{\mathrm{I}}^* < a_{\mathrm{II}}^* \le b_{\mathrm{II}}^* < b_{\mathrm{I}}^*$. So, for sufficiently large n, a_n increases while b_n decreases as q varies from q_{I} to q_{II}.

To show that case 3 is also possible, consider any q_{I} and q_{II} that satisfy (i) $q_{\mathrm{I}} < \widehat{q} < q_{\mathrm{II}}$, (ii) $\phi^* \in \left[\underline{\phi}(v_{q_{\mathrm{I}}}^*), \overline{\phi}(v_{q_{\mathrm{I}}}^*)\right]$, and (iii) $\phi^* \in \left[\underline{\phi}(v_{q_{\mathrm{II}}}^*), \overline{\phi}(v_{q_{\mathrm{II}}}^*)\right]$. By Theorem 2, $(a_{\mathrm{I}}^*, b_{\mathrm{I}}^*) = (\phi^*, z_{\mathrm{I}}^*)$ and $(a_{\mathrm{II}}^*, b_{\mathrm{II}}^*) = (z_{\mathrm{II}}^*, \phi^*)$. In this case we have $a_{\mathrm{I}}^* > a_{\mathrm{II}}^*$ and $b_{\mathrm{I}}^* > b_{\mathrm{II}}^*$. So, for sufficiently large n, both a_n and b_n decrease as q varies from q_{I} to q_{II}.

Finally, to show that case 4 is also possible, consider any q_{I} and q_{II} that satisfy (i) $q_{\mathrm{I}} < \widehat{q} < q_{\mathrm{II}}$, (ii) $\overline{\phi}(v_{q_{\mathrm{I}}}^*) < \phi^* < \underline{\phi}(v_{q_{\mathrm{II}}}^*)$. Then, by Theorem 2, we have $(a_{\mathrm{I}}^*, b_{\mathrm{I}}^*) = \left(\overline{\phi}(v_{q_{\mathrm{I}}}^*), z_{\mathrm{I}}^*\right)$ and $(a_{\mathrm{II}}^*, b_{\mathrm{II}}^*) = \left(z_{\mathrm{II}}^*, \underline{\phi}(v_{q_{\mathrm{II}}}^*)\right)$. By Lemma E.2, $z_{q_I}^* < \phi^* < z_{q_{II}}^*$ must hold for all $\rho > 0$. If, however, $\rho \to 0$, then $\phi(\cdot)$ must be close to a flat line so that both $z_{q_I}^*$ and $z_{q_{II}}^*$ shall be arbitrarily close to ϕ^*. This implies that $\overline{\phi}\left(v_{q_I}^*\right) < z_{q_{II}}^*$ and $z_{q_I}^* < \underline{\phi}\left(v_{q_{II}}^*\right)$ must hold for ρ sufficiently close to 0. In such case we indeed have $a_I^* < a_{II}^*$ and $b_I^* < b_{II}^*$. □

Proposition E.1. *Suppose $\rho > 0$ and $v_q^* \in (-1,1)$. Let $w^I(\cdot)$ and $w^{II}(\cdot)$ be two absolutely continuous weighting functions that satisfy (i) $w^I(\cdot) \succeq_{FOSD} w^{II}(\cdot)$, and (ii) $\phi_i^* \in \left(\underline{\phi}(v_q^*), \overline{\phi}(v_q^*)\right)$ holds under $w^i(\cdot)$ for at least one $i \in \{I, II\}$. Then, for sufficiently large n, both a_n and b_n decrease as the weighting function $w(\cdot)$ shifts from $w^I(\cdot)$ to $w^{II}(\cdot)$.*

Proof of Proposition E.1. We use a_i^* and b_i^* to denote the boundaries of the asymptotically optimal revelation interval under $w(\cdot) = w^i(\cdot)$ for $i \in \{\mathrm{I}, \mathrm{II}\}$. For ease of exposure we focus on the case where $\phi_i^* \in \left(\underline{\phi}(v_q^*), \overline{\phi}(v_q^*)\right)$ holds under $w^i(\cdot)$ for both $i \in \{\mathrm{I}, \mathrm{II}\}$.[12] In this case, Theorem 2 implies $a_i^* = \min\{z_i^*, \phi_i^*\}$ and $b_i^* = \max\{z_i^*, \phi_i^*\}$ for both $i \in \{\mathrm{I}, \mathrm{II}\}$. By Lemma B.6, we have $\phi_{\mathrm{I}}^* > \phi_{\mathrm{II}}^*$ and $z_{\mathrm{I}}^* > z_{\mathrm{II}}^*$ (equality holds only if both values are -1). These together imply (i) $b_{\mathrm{I}}^* > b_{\mathrm{II}}^*$ and (ii) $a_{\mathrm{I}}^* \geq a_{\mathrm{II}}^*$ (equality holds only if both values are -1). Hence, for sufficiently large n, both a_n and b_n decrease as q shifts from $w^{\mathrm{I}}(\cdot)$ to $w^{\mathrm{II}}(\cdot)$. □

F Equilibrium Derivations and Omitted Proofs for Section 7

In this appendix we solve for the equilibria of the competitive persuasion model in Section 7 and prove Theorems 3 and 4.

Equilibrium. When there are $|M| \geq 2$ senders, let $\pi = \left\langle \{\pi_m\}_{m \in M} \right\rangle$ be any joint information policy induced by all senders and let H_π denote the distribution of the posterior means induced by π. We say that H_π is *unimprovable for sender* $m \in M$ if he has no incentive to unilaterally reveal more information. For $m \in M$, let $\mathscr{H}_m$ denote the set of all unimprovable distributions. The set of distributions H that are *unimprovable for all senders* is then $\mathscr{H} = \cap_{m \in M} \mathscr{H}_m$. By Proposition 2 of Gentzkow and Kamenica (2017*b*), π can be sustained in equilibrium if and only if $H_\pi \in \mathscr{H}$.

To further solve for the unimprovable H's, we introduce some useful observations about properties of solutions to a general linear persuasion problem of the following kind:

$$\max_{H \in \Delta([\underline{\kappa}, \overline{\kappa}])} \int_{\underline{\kappa}}^{\overline{\kappa}} U(\theta) dF(\theta), \quad \text{s.t. } F\big|_{[\underline{\kappa}, \overline{\kappa}]} \succeq_{MPS} H \tag{MP'}$$

where $U(\cdot)$ is a sender's utility function defined on some closed interval $[\underline{\kappa}, \overline{\kappa}] \subseteq [-1,1]$ and $F\big|_{[\underline{\kappa}, \overline{\kappa}]}$ is the cdf of prior F truncated on interval $[\underline{\kappa}, \overline{\kappa}]$.[13] We assume throughout this appendix that $U(\cdot)$ is twice continuously differentiable on $[-1,1]$.[14]

The first observation, due to Theorem 4 of Dworczak and Martini (2019), provides a convenient way to verify whether an induced distribution of posterior means H is unimprovable for a sender

[12] The proof is almost identical for the case where only one weighting function satisfies this condition.

[13] That is, $F\big|_{[\underline{\kappa}, \overline{\kappa}]}(k) = \frac{F(k) - F(\underline{\kappa})}{F(\overline{\kappa}) - F(\underline{\kappa})}$ for $k \in [\underline{\kappa}, \overline{\kappa}]$ and it equals 1 (resp. 0) for $k > \overline{\kappa}$ (resp. $k < \underline{\kappa}$).

[14] This ensures that $U(\cdot)$ is *regular*: $[-1,1]$ can be partitioned into finitely many intervals on which $U(\cdot)$ is either strictly concave, strictly convex, or affine. This regularity property is needed for Observations F.1 and F.4 below.

with utility function $U(\cdot)$ by solving a monopolistic persuasion problem with his utility function modified by its convex translations.

Observation F.1. *(Theorem 4 of Dworczak and Martini (2019)) If $H \in \Delta([\underline{\kappa}, \overline{\kappa}])$ is unimprovable for a sender with utility function $U(\cdot)$ on $[\underline{\kappa}, \overline{\kappa}]$,*[15] *then there exists a convex function $\omega(\cdot)$ on $[\underline{\kappa}, \overline{\kappa}]$ such that H is a solution to (MP') with $U(\cdot)$ therein replaced by $\widehat{U}(\cdot) = U(\cdot) + \omega(\cdot)$.*

Using Observation F.1, we establish Lemma F.1, which implies that any unimprovable H must induce cutoff partitions at z_n^m for all $m \in M$. As a result, any $H \in \mathscr{H}$ must be unimprovable separately on each segment of interval $[-1, 1]$ partitioned by the z_n^m's.

Lemma F.1. *Suppose that the single-crossing property holds for a sender $m \in M$ and $\phi_n^m(x) - x$ crosses zero only once and from above at $z_n^m \in (-1, 1)$. Then $H \succeq_{MPS} H_{\mathscr{P}(z_n^m)}$ must hold for any H that is unimprovable for sender m on $[-1, 1]$.*

Proof of Lemma F.1. By Observation F.1, H is unimprovable for sender m if and only if H is a solution to (MP) with utility function $W_n^m(\cdot)$ replaced by $\widehat{W}_n^m(\cdot) = W_n^m(\cdot) + \omega_m(\cdot)$ for some convex $\omega_m(\cdot)$. By Lemma 4, $W_n^m(\cdot)$ satisfies the increasing slope property at z_n^m. We show that $\widehat{W}_n^m(\cdot)$ must also satisfy this property and hence $H \succeq_{MPS} H_{\mathscr{P}(z_n^m)}$ follows Lemma 4. To see why, observe that

$$\frac{\widehat{W}_n^m(x) - \widehat{W}_n^m(z_n^m)}{x - z_n^m} = \frac{W_n^m(x) - W_n^m(z_n^m)}{x - z_n^m} + \frac{\omega_m(x) - \omega_m(z_n^m)}{x - z_n^m}$$

and the latter term is non-decreasing in x because $\omega_m(\cdot)$ is convex. $\widehat{W}_n^m(\cdot)$ thus satisfies the increasing slope property whenever $W_n^m(\cdot)$ does. □

The next two observations establish, for the strictly S-shaped and inverse-S-shaped utility functions, the unimprovability and best-response properties of interval revelation policies.

Observation F.2. *Let $U(\cdot)$ be sender m's utility function and suppose that $U(\cdot)$ is strictly S-shaped on $[\underline{\kappa}, \overline{\kappa}]$ with inflection point r. The following properties hold:*

1. *(Kolotilin, Mylovanov and Zapechelnyuk, 2022) The unique solution H to problem (MP') is induced by an upper censorship policy $\mathscr{P}(\underline{\kappa}, b)$ with $b \geq \underline{\kappa}$.*
2. *(Sun, 2022b) Let $\mathscr{H}_m$ denote the set of unimprovable outcomes for sender m, then (i) $H \succeq_{MPS} H_{\mathscr{P}(\underline{\kappa}, b)}$ for all $H \in \mathscr{H}_m$, and (ii) $H_{\mathscr{P}(\underline{\kappa}, d)} \in \mathscr{H}_m$ for all $d \in [b, \overline{\kappa}]$.*
3. *(Sun, 2022b) Given any pure strategy profile of other senders π_{-m}, there exists some $d \in [b, r]$ such that the upper censorship policy $\mathscr{P}(\underline{\kappa}, d)$ is sender m's best response to π_{-m}.*

[15] Formally, H is unimprovable for a sender with utility function $U(\cdot)$ on $[\underline{\kappa}, \overline{\kappa}]$ if $\int_{\underline{\kappa}}^{\overline{\kappa}} U(\theta) dH(\theta) \geq \int_{\underline{\kappa}}^{\overline{\kappa}} U(\theta) d\widetilde{H}(\theta)$ holds for all $\widetilde{H} \in \Delta([\underline{\kappa}, \overline{\kappa}])$ that satisfies the feasibility constraint $F\big|_{[\underline{\kappa}, \overline{\kappa}]} \succeq_{MPS} \widetilde{H} \succeq_{MPS} H$.

Observation F.3. *Let $U(\cdot)$ be sender m's utility function and suppose that $U(\cdot)$ is strictly inverse S-shaped on $[\underline{\kappa},\overline{\kappa}]$ with inflection point ℓ. The following properties hold:*

1. *(Kolotilin, Mylovanov and Zapechelnyuk, 2022) The unique solution H to problem (MP') is induced by an lower censorship policy $\mathscr{P}(a,\overline{\kappa})$ with $a \leq \overline{\kappa}$.*

2. *(Sun, 2022b) Let $\mathscr{H}_m$ denote the set of unimprovable outcomes for sender m, then (i) $H \succeq_{MPS} H_{\mathscr{P}(a,\overline{\kappa})}$ for all $H \in \mathscr{H}_m$, and (ii) $H_{\mathscr{P}(c,\overline{\kappa})} \in \mathscr{H}_m$ for all $c \in [\underline{\kappa},a]$.*

3. *(Sun, 2022b) Given any pure strategy profile of other senders π_{-m}, there exists some $c \in [\ell,a]$ such that the lower censorship policy $\mathscr{P}(c,\overline{\kappa})$ is sender m's best response to π_{-m}.*

The first statements of the two observations above suggest that under monopolistic persuasion *upper* (resp. *lower*) censorship policies are uniquely optimal for a sender whose utility function $U(\cdot)$ is strictly S-shaped (resp. inverse S-shaped). The remaining statements extend this insight to competition in persuasion. For a sender whose utility function is either strictly S-shaped or inverse S-shaped, any interval revelation policy that is no less informative than the monopolistic optimal one is unimprovable. Moreover, given any pure strategy profile of other senders, he can always find a best response from a subset of interval revelation policies. Notice that all information policies in this subset are no less informative than his monopolistically optimal one. Finally, we introduce a sufficient condition for full disclosure to be the unique equilibrium outcome under competition in persuasion.

Observation F.4. *(Sun, 2022a) Let $\{U_m(\cdot)\}_{m\in M}$ be a profile of twice continuously differentiable utility functions defined on $[\underline{\kappa},\overline{\kappa}]$. Then full disclosure on $[\underline{\kappa},\overline{\kappa}]$ is the unique equilibrium outcome if for a.e. $k \in [\underline{\kappa},\overline{\kappa}]$ there exists a sender m for whom $U''_m(k) > 0$ holds.*

With these ingredients we are now ready to prove Theorems 3 and 4.

F.1 Proof of Theorem 3

For each $m \in M$ and $n \geq N_m$, recall from (BR) that

$$\mathscr{P}_n^m := \{\mathscr{P}(c,d) : \ [a_n^m, b_n^m] \subseteq [c,d] \subseteq [\ell_n^m, r_n^m]\}$$

where a_n^m and b_n^m are the thresholds of the monopolistically optimal revelation interval. Clearly, $\mathscr{P}_n^m$ is a subset of interval revelation policies. We show below that for any pure strategy profile π_{-m} of other senders, there exists a $\pi_m \in \mathscr{P}_n^m$ such that π_m is sender m's best response to π_{-m}. We distinguish between three cases depending on the value of z_n^m.

If $z_n^m = -1$, then $\ell_n^m = a_n^m = 0$ and $W_n^m(\cdot)$ is strictly S-shaped on $[-1,1]$ with inflection point $r_n^m \geq b_n^m$. By Observation F.2, for any π_{-m} there exists $d \in [b_n^m, r_n^m]$ such that $\pi_m = \mathscr{P}(-1,d)$ is sender m's best response to π_{-m}. Similarly, if $z_n^m = 1$ then $r_n^m = b_n^m = 1$ and $W_n^m(\cdot)$ is strictly inverse S-shaped on $[-1,1]$ with inflection point $\ell_n^m \leq a_n^m$. By Observation F.3, for any π_{-m} there exists $c \in [\ell_n^m, a_n^m]$ such that $\pi_m = \mathscr{P}(c,1)$ is m's best response to π_{-m}. In both cases $\pi_m \in \mathscr{P}_n^m$ holds.

Finally, consider the case $z_n^m \in (-1,1)$ and let π_m be any best response to π_{-m} for sender m. Because the information environment is Blackwell-connected, the induced joint information policy $\langle \pi_m, \pi_{-m} \rangle$ must be unimprovable for sender m. Recall that $W_n^m(\cdot)$ satisfies the increasing slope property at point z_n^m (cf. Lemma 4), it thus follows from Lemma F.1 that $\langle \pi_m, \pi_{-m} \rangle$ must be Blackwell more informative than the cutoff policy $\mathscr{P}(z_n^m)$. This implies that there always exists a best response π_m that is Blackwell more informative than $\mathscr{P}(z_n^m)$ (i.e., $H_{\pi_m} \succeq_{MPS} H_{\mathscr{P}(z_n^m)}$).[16] For such π_m, it must be a best response to π_{-m} on both $[-1, z_n^m]$ and $[z_n^m, 1]$ separately. By Lemma 5, $W_n^m(\cdot)$ is strictly inverse S-shaped on $[-1, z_n^m]$ with inflection point $\ell_n^m < z_n^m$ and strictly S-shaped on $[z_n^m, 1]$ with inflection point $r_n^m > z_n^m$. It follows that there exists $c \in [\ell_n^m, a_n^m]$ and $d \in [b_n^m, r_n^m]$ such that $\mathscr{P}(c, z_n^m)$ is a best response to π_{-m} on $[-1, z_n^m]$ and $\mathscr{P}(z_n^m, d)$ is a best response on $[z_n^m, 1]$. These together produce an interval revelation policy $\pi_m = \mathscr{P}(c,d)$, which belongs to $\mathscr{P}_n^m$, that is a best response to π_{-m}. This completes the proof of Theorem 3.

F.2 Proof of Theorem 4

We use two lemmas to establish part (1) of the theorem and then a third to prove part (2).

Lemma F.2. *Suppose Assumption 1 holds. Then any unimprovable outcome $H \in \mathscr{H}$ must satisfy $H \succeq_{MPS} H_{\mathscr{P}(z_n^{\min}, z_n^{\max})}$, where $z_n^{\min} = \min_{m \in M}\{z_n^m\}$ and $z_n^{\max} = \max_{m \in M}\{z_n^m\}$.*

Proof of Lemma F.2. By Lemma F.1, any $H \in \mathscr{H}$ must satisfy $H \succeq_{MPS} H_{\mathscr{P}(z_n^{\min})}$, $H \succeq_{MPS} H_{\mathscr{P}(z_n^{\max})}$, and be unimprovable on $[z_n^{\min}, z_n^{\max}]$. It then suffices to show that any unimprovable H must be fully revealing on $[z_n^{\min}, z_n^{\max}]$ to complete the proof. Recall that

$$W_n^{m\prime\prime}(k) = \left(2 - \phi_n^{m\prime}(k)\right) \hat{g}_n(k;q) + \left(k - \phi_n^m(k)\right) \hat{g}_n'(k;q) \ .$$

The first term is strictly positive for all $m \in M$ because part (2) of Assumption 1 ensures that $\phi_n^{m\prime}(k) < 2$ for all $k \in [z_n^{\min}, z_n^{\max}]$. Let I (resp. II) denote the index of the sender for whom $z_n^{\mathrm{I}} = z_n^{\min}$ (resp. $z_n^{\mathrm{II}} = z_n^{\max}$). Then for all $k \in [z_n^{\min}, z_n^{\max}]$ we have $\phi_n^{\mathrm{I}}(k) \leq k \leq \phi_n^{\mathrm{II}}(k)$. So, no matter what the sign of $\hat{g}_n'(k;q)$ is, $(k - \phi_n^m(k))\, \hat{g}_n'(k;q)$ must be non-negative for at least one $m \in \{\mathrm{I}, \mathrm{II}\}$. Hence, for any $k \in [z_n^{\min}, z_n^{\max}]$, $W_n^{m\prime\prime}(k) > 0$ must hold for at least one $m \in \{\mathrm{I}, \mathrm{II}\} \subseteq M$. Therefore, by Observation F.4, any unimprovable H must be fully revealing on $[z_n^{\min}, z_n^{\max}]$ and hence $H \succeq_{MPS} H_{\mathscr{P}(z_n^{\min}, z_n^{\max})}$. □

[16] To see why, let $\pi = \langle \pi_m, \pi_{-m} \rangle$ and $\pi' = \langle \pi_m, \pi_{-m}, \mathscr{P}(z_n^m) \rangle$. The best response property of π_m ensures that $H_\pi \succeq_{MPS} H_{\mathscr{P}(z_n^m)}$. Therefore, $H_\pi = H_{\pi'}$. Consider $\pi_m' = \langle \pi_m, \mathscr{P}(z_n^m) \rangle$ (which is always feasible) and observe that $H_{\pi_m'} \succeq_{MPS} H_{\mathscr{P}(z_n^m)}$ and $\pi' = \langle \pi_m', \pi_{-m} \rangle$. Then π_m' must also be a best response to π_{-m} because $H_\pi = H_{\pi'}$.

Combining Lemma F.1, Observations F.2 and F.3, and Lemma 5, we obtain

Lemma F.3. *Suppose the single-crossing property holds for each $m \in M$. Then, for any $n \geq N_m$, (i) $H \succeq_{MPS} H_{\mathscr{P}(a_n^m, b_n^m)}$ for all $H \in \mathscr{H}_m$, and (ii) $\mathscr{P}(a,b) \in \mathscr{H}_m$ for all $a \in [-1, a_n^m]$ and $b \in [b_n^m, 1]$.*

In words, Lemma F.3 says that under the single-crossing property and for sufficiently large n all unimprovable outcomes for sender m must be no less informative than his monopolistically optimal interval revelation policy. Moreover, all interval revelation policies that are more informative than the monopolistically optimal one are unimprovable for sender m.

We now use Lemmas F.2 and F.3 to establish part (1) of Theorem 4. Consider any $n \geq N$. By Lemma F.2, $H \succeq_{MPS} H_{\mathscr{P}(z_n^{\min}, z_n^{\max})}$ must hold for all $H \in \mathscr{H} = \cap_{m \in M} \mathscr{H}_m$. By Lemma F.3, $H \succeq_{MPS} H_{\mathscr{P}(a_n^m, b_n^m)}$ must hold for all $H \in \mathscr{H}_m$. Moreover, for each $m \in M$, it holds that $H_{\mathscr{P}(c,d)} \in \mathscr{H}_m$ for all $c \in [-1, a_n^m]$ and $d \in [b_n^m, 1]$. Therefore, $\mathscr{P}(a_n^{\min}, b_n^{\max})$ is unimprovable for all senders and hence $H_{\mathscr{P}(a_n^{\min}, b_n^{\max})} \in \mathscr{H}$. Next we show that any $H \in \mathscr{H}$ must be weakly more informative than $\mathscr{P}(a_n^{\min}, b_n^{\max})$, that is $H \succeq_{MPS} H_{\mathscr{P}(a_n^{\min}, b_n^{\max})}$. Let $\tilde{i}$ (resp. $\tilde{j}$) denote the identity of the sender with $a_n^{\tilde{i}} = a_n^{\min}$ (resp. $b_n^{\tilde{j}} = b_n^{\max}$). Recall that any $H \in \mathscr{H}$ must satisfy $H \succeq_{MPS} H_{\mathscr{P}(a_n^m, b_n^m)}$ with $a_n^m \leq z_n^m \leq b_n^m$ for all $m \in M$. The choices of $\tilde{i}$ and $\tilde{j}$ imply that $\left[a_n^{\tilde{i}}, b_n^{\tilde{i}}\right]$, $[z_n^{\min}, z_n^{\max}]$ and $\left[a_n^{\tilde{j}}, b_n^{\tilde{j}}\right]$ are overlapping and $\left[a_n^{\tilde{i}}, b_n^{\tilde{i}}\right] \cup [z_n^{\min}, z_n^{\max}] \cup \left[a_n^{\tilde{j}}, b_n^{\tilde{j}}\right] = [a_n^{\min}, b_n^{\max}]$. Therefore, $H \succeq_{MPS} H_{\mathscr{P}(a_n^{\min}, b_n^{\max})}$ must hold for all $H \in \mathscr{H}$ and this completes the proof for statement (1) of Theorem 4.

Next we prove statement (2) of Theorem 4. Given any $\mathscr{P}(c,d)$ with revelation interval $[c,d] \subseteq [-1,1]$, each sender m's expected payoff under this policy is

$$\mathbb{E}_{\mathscr{P}(c,d)}\left[W_n^m(\cdot)\right] = F(c) W_n^m\left(\underline{\mu}_F(c)\right) + \int_c^d W_n^m(k) dF(k) + (1 - F(d)) W_n^m\left(\overline{\mu}_F(d)\right) \tag{F.1}$$

where $\underline{\mu}_F(c) := \mathbb{E}_F[k | k < c]$ and $\overline{\mu}_F(d) := \mathbb{E}_F[k | k > d]$. Lemma F.4 shows that $\mathbb{E}_{\mathscr{P}(c,d)}[W_n^m(\cdot)]$ is single-peaked in both thresholds c and d.

Lemma F.4. *The following inequalities hold:*

(i) $\frac{\partial}{\partial d} \mathbb{E}_{\mathscr{P}(c,d)}[W_n^m(\cdot)] > (<) 0$ for $d < (>) b_n^m$, and

(ii) $\frac{\partial}{\partial c} \mathbb{E}_{\mathscr{P}(c,d)}[W_n^m(\cdot)] > (<) 0$ for $c > (<) a_n^m$.

Proof of Lemma F.4. Taking derivatives of (F.1) with respect to c and d yield

$$\begin{aligned}
\frac{\partial}{\partial d} \mathbb{E}_{\mathscr{P}(c,d)}\left[W_n^m(\cdot)\right] &= f(d)\left(W_n^m(d) - W_n^m\left(\overline{\mu}_F(d)\right)\right) + (1 - F(d)) W_n^{m\prime}\left(\overline{\mu}_F(d)\right) \overline{\mu}_F'(d) \\
&= f(d) \cdot \left[W_n^m\left(\overline{\mu}_F(d)\right) - W_n^m(d) - \left(\overline{\mu}_F(d) - d\right) W_n^{m\prime}\left(\overline{\mu}_F(d)\right)\right] \\
\frac{\partial}{\partial c} \mathbb{E}_{\mathscr{P}(c,d)}\left[W_n^m(\cdot)\right] &= f(c)\left(W_n^m\left(\underline{\mu}_F(c)\right) - W_n^m(c)\right) + F(c) \cdot W_n^{m\prime}\left(\underline{\mu}_F(c)\right) \underline{\mu}_F'(c) \\
&= f(c) \cdot \left[\left(c - \underline{\mu}_F(c)\right) W_n^{m\prime}\left(\underline{\mu}_F(c)\right) - \left(W_n^m(c) - W_n^m\left(\underline{\mu}_F(c)\right)\right)\right].
\end{aligned}$$

Because both $\tilde{f}(d)$ and $\overline{\mu}_F(d)-d$ are positive, $\frac{\partial}{\partial d}\mathbb{E}_{\mathscr{P}(c,d)}[W_n^m(\cdot)]$ is sign-equivalent to

$$W_n^m(\overline{\mu}_F(d)) - W_n^m(d) - (\overline{\mu}_F(d)-d)\, W_n^{m\prime}(\overline{\mu}_F(d)) \ . \tag{F.2}$$

By Lemma 5, $W_n^m(\cdot)$ is strictly S-shaped on $[z_n^m, 1]$ and thus (F.2) crosses zero at most once and above at b_n^m, which is determined by (FOC: b_n). This proves part (i). The proof for part (ii) is similar; it exploits the inverse S-shape property of $W_n^m(\cdot)$ on $[-1, z_n^m]$ and the definition of a_n^m. $\square$

We establish below that any equilibrium outcome in pure and weakly undominated strategies must be both no more and no less informative than $\mathscr{P}(a_n^{\min}, b_n^{\max})$. This implies the uniqueness of $\mathscr{P}(a_n^{\min}, b_n^{\max})$ as the induced outcome of any pure strategy equilibrium in weakly undominated strategies. We first show for any $m \in M$ that all $\mathscr{P}(c,d)$ with $d > b_n^{\max}$ are weakly dominated by $\mathscr{P}(c, b_n^{\max})$, when all senders $i \neq m$ choose strategies from $\mathscr{P}_n^i$. Let π_{-m} denote a strategy profile by other senders and under π_{-m} let $\eta \in [b_n^{\max}, 1]$ be the threshold such that $k \in [b_n^{\max}, \eta]$ are revealed, while $k > \eta$ are censored. Replacing $\mathscr{P}(c,d)$ with $\mathscr{P}(c, b_n^{\max})$ can only make a difference for $k \in [b_n^{\max}, 1]$. If $d \leq \eta$, then such replacement has no effect on the joint information policy so sender m is indifferent to it. If $d > \eta$, then it lowers the threshold of the upper censoring interval and thus reduces the informativeness of the joint policy. By Lemma F.4, for each $m \in M$, $\mathbb{E}_{\mathscr{P}(c,d)}[W_n^m(\cdot)]$ is single-peaked in d with a peak at b_n^m. Since $\eta \geq b_n^{\max} > b_n^m$, it follows that any sender m's expected payoff would increase were $\mathscr{P}(c,d)$ replaced by $\mathscr{P}(c, b_n^{\max})$. Hence, any $\mathscr{P}(c,d)$ with $d > b_n^{\max}$ is weakly dominated by $\mathscr{P}(c, b_n^{\max})$. Using an analogous argument we can also show that any $\mathscr{P}(c,d)$ with $c < a_n^{\min}$ is weakly dominated by $\mathscr{P}(a_n^{\min}, d)$. Together these imply that any $\mathscr{P}(c,d)$ with $d > b_n^{\max}$ or $c < a_n^{\min}$ is weakly dominated. This shows that any outcome induced by a pure-strategy equilibrium with undominated strategies must be weakly less informative than $\mathscr{P}(a_n^{\min}, b_n^{\max})$.

Next we show that no equilibrium outcome can be strictly less informative than $\mathscr{P}(a_n^{\min}, b_n^{\max})$.[17] Observe that the structure of $\mathscr{P}_n^m$ implies that any feasible outcome must be weakly more informative than $\mathscr{P}(a_n^m, b_n^m)$ for all $m \in M$. Therefore, if $\cup_{m\in M}[a_n^m, b_n^m] = \left[a_n^{\min}, b_n^{\max}\right]$ the result holds trivially. In what follows we assume that $\cup_{m\in M}[a_n^m, b_n^m]$ is a proper subset of $\left[a_n^{\min}, b_n^{\max}\right]$. Then, there must be at least one sender pair $l, r \in M$ such that (i) $b_n^l < a_n^r$, and (ii) for all $m \in M \backslash \{l, r\}$ there are $[a_n^m, b_n^m] \cap \left(b_n^l, a_n^r\right) = \emptyset$.[18] By the construction of $\mathscr{P}_n^m$ for all $m \in M$, there could be at most one nontrivial pooling interval $(x,y) \subseteq \left(b_n^l, a_n^r\right)$. Fix interval (x,y) and let $\mu(x,y) := \mathbb{E}_F[k | k \in (x,y)]$,

[17] Note that the information environment is no longer Blackwell-connected if each sender m is restricted to choose information policies from $\mathscr{P}_n^m$ only. Therefore, Proposition 2 of Gentzkow and Kamenica (2017b) no longer applies (i.e., a feasible outcome being unimprovable to all senders is no longer necessary for that outcome to be an equilibrium).

[18] In fact, there could be at most $|M|-1$ such pairs. The argument presented below holds for any such pair.

the expected utility of sender $m = \{l, r\}$ conditional on $k \in \left(b_n^l, a_n^r\right)$ is

$$V_m = \int_{b_n^l}^{x} W_n^m(k) d\widetilde{F}(k) + \left(\widetilde{F}(y) - \widetilde{F}(x)\right) W_n^m(\mu(x,y)) + \int_{y}^{a_n^r} W_n^m(k) d\widetilde{F}(k)$$

where $\widetilde{F}(\cdot)$ is the cdf of the distribution of k conditional on $k \in \left[b_n^l, a_n^r\right]$. Taking derivatives yields[19]

$$\begin{aligned}
\frac{\partial V_m}{\partial x} &= \widetilde{f}(x)\left(W_n^m(x) - W_n^m(\mu(x,y))\right) + \left(\widetilde{F}(y) - \widetilde{F}(x)\right) W_n^{m\prime}(\mu(x,y))\, \mu_x(x,y) \\
&= \widetilde{f}(x)\left[(\mu(x,y) - x) W_n^{m\prime}(\mu(x,y)) - \left(W_n^m(\mu(x,y)) - W_n^m(x)\right)\right] \\
\frac{\partial V_m}{\partial y} &= \widetilde{f}(y)\left(W_n^m(\mu(x,y)) - W_n^m(y)\right) + \left(\widetilde{F}(y) - \widetilde{F}(x)\right) W_n^{m\prime}(\mu(x,y))\, \mu_y(x,y) \\
&= \widetilde{f}(x)\left[(y - \mu(x,y)) W_n^{m\prime}(\mu(x,y)) - \left(W_n^m(y) - W_n^m\right)(\mu(x,y))\right] .
\end{aligned}$$

For both senders l and r to have no incentive to reveal any extra information, it is necessary that $\frac{\partial V_l}{\partial x} \leq 0$ and $\frac{\partial V_r}{\partial y} \geq 0$, or equivalently[20]

$$(\mu(x,y) - x) W_n^{l\prime}(\mu(x,y)) \leq W_n^l(\mu(x,y)) - W_n^l(x) \quad \text{and} \tag{F.3}$$

$$(y - \mu(x,y)) W_n^{r\prime}(\mu(x,y)) \geq W_n^r(y) - W_n^l(\mu(x,y)) . \tag{F.4}$$

Because $z_n^l \leq b_n^l \leq x < y \leq a_n^r \leq z_n^r$, both $[z_n^l, 1]$ and $[-1, z_n^r]$ must contain (x,y) in their interior. For (F.3) to hold, $\mu(x,y) > r_n^l$ must be true so that $\mu(x,y)$ is in the concave region of $W_n^l(\cdot)$. Similarly, for (F.4) to hold, $\mu(x,y) < \ell_n^r$ must hold for $\mu(x,y)$ to be in the concave region of $W_n^r(\cdot)$. These jointly imply that both $W_n^l(\cdot)$ and $W_n^r(\cdot)$ are strictly concave on an open neighborhood of $\mu(x,y)$ contained in $[z_n^l, z_n^r]$. This is, however, impossible because for any $k \in [z_n^l, z_n^r]$, $W_n^{m\prime\prime}(k) > 0$ must hold for at least one $m \in \{l, r\}$ (apply $M = \{l, r\}$ in the proof of Lemma F.2. Therefore, the incentive compatibility conditions for senders l and r cannot be jointly satisfied and it is impossible to have any non-trivial pooling interval (x,y) in equilibrium. Hence, no equilibrium outcome can be strictly less informative than $\mathscr{P}(a_n^{\min}, b_n^{\max})$. This completes the proof for statement (2) of Theorem 4.

G Omitted Materials for Section 8

In this appendix we first verify that for the combinations of parameters in Example 5 of Section 8.2, Theorem 1 holds independently of electorate size n.

[19] Here we use $\mu_x(x,y) := \frac{\partial \mu}{\partial x} = \frac{\widetilde{f}(x)(\mu(x,y)-x)}{\widetilde{F}(y)-\widetilde{F}(x)}$ and $\mu_y(x,y) := \frac{\partial \mu}{\partial y} = \frac{\widetilde{f}(y)(y-\mu(x,y))}{\widetilde{F}(y)-\widetilde{F}(x)}$.

[20] This is because each $m \in \{l, r\}$ can only choose policies from $\mathscr{P}_n^m$, whose revelation interval must contain $[a_n^m, b_n^m]$. Therefore, since $(x,y) \subseteq \left(b_n^l, a_n^r\right)$, only sender l can marginally increase x while only sender r can marginally decrease y. These two inequalities ensure that such marginal deviations are not profitable for either sender.

Let G be a normal distribution $\mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma > 0$; that is, $g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ and $G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$ for all $x \in \mathbb{R}$. Then $\frac{g'(x)}{g(x)} = \frac{\mu - x}{\sigma^2}$. It is well known that the expectations of one side truncated normal distributions are given by

$$\mathbb{E}_G[v_i | v_i \geq x] = \mu + \sigma^2 \frac{g(x)}{1 - G(x)} \quad \text{and} \quad \mathbb{E}_G[v_i | v_i \leq x] = \mu - \sigma^2 \frac{g(x)}{G(x)} .$$

Therefore,

$$\phi(x) = q\mathbb{E}_G[v_i | v_i \leq x] + (1-q)\mathbb{E}_G[v_i | v_i \geq x] = \mu - \sigma^2 \left[q\frac{g(x)}{G(x)} - (1-q)\frac{g(x)}{1-G(x)} \right] .$$

This implies

$$x - \phi(x) = x - \mu + \sigma^2 \left[q\frac{g(x)}{G(x)} - (1-q)\frac{g(x)}{1-G(x)} \right]$$

and

$$\begin{aligned} 2 - \phi'(x) &= 2 + \sigma^2 \left[q\left(\frac{g(x)}{G(x)}\right)' - (1-q)\left(\frac{g(x)}{1-G(x)}\right)' \right] \\ &= 2 + \sigma^2 \left[q\frac{g(x)}{G(x)}\left(\frac{\mu - x}{\sigma^2} - \frac{g(x)}{G(x)}\right) - (1-q)\frac{g(x)}{1-G(x)}\left(\frac{\mu - x}{\sigma^2} + \frac{g(x)}{1-G(x)}\right) \right] \end{aligned}$$

where in the second step we use the fact that $\frac{g'(x)}{G(x)} = \frac{\mu - x}{\sigma^2}$. Given μ, σ, and q, we can numerically solve for z, the unique solution to $x - \phi(x) = 0$. Then, by part (2) of Proposition 2, we only need to check that function $\lambda(x) := \frac{2-\phi'(x)}{x-\phi(x)} + \frac{g(x)}{G(x)}$ is decreasing on $[-1, z)$ and $(z, 1]$, respectively. Figure G.1 plots $\lambda(x)$ for all combinations parameters we used in Section 8.3. These confirm that $\lambda(x)$ is indeed decreasing in relevant intervals for all parameter combinations considered, and thus interval revelation policies are indeed optimal for all $n > 0$.

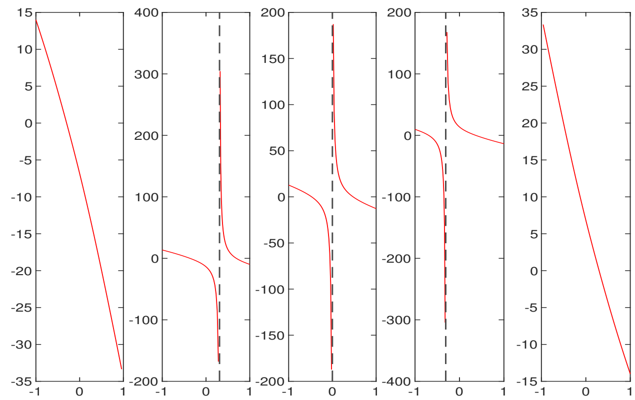
Finally, we prove Proposition 7, which claims that when the intermediary's payoff function is given by (1), all interval revelation policies are credible in the notion of Lin and Liu (2023).

Proof of Proposition 7. We start by writing the sender's expected utility $W_n(k, y)$ as a function of both the state k and the pivotal voter's 'action' y – defined as a cutoff strategy such that the pivotal voter passes the reform if and only if $x \leq y$. Then we have

$$W_n(k, y) = \int_{\underline{v}}^{y} (k - \phi_n(x))\,\hat{g}_n(x; q)dx = k \cdot \hat{G}_n(y; q) - \int_{\underline{v}}^{y} \phi_n(x)\hat{g}_n(x; q)dx . \tag{G.1}$$

Note that $W_n(k, y)$ is supermodular in k and y because $\frac{\partial^2 W_n}{\partial k \partial y} = \hat{g}_n(y; q) > 0$. Given any posterior expected state θ, the pivotal voter's best response is $y = \theta$. Bringing this to (G.1) yields value

Figure G.1: Plots of $\lambda(x)$ (red curves) under different sets of parameters



Note: The model parameters are $\mu = 0$, $\sigma = 0.25$, and $q = 0,\ 1/4,\ 1/2,\ 3/4,\ 1$ (from left to right). In these panels the dashed vertical lines indicate the values of z whenever $z \in (-1,1)$.

function (9). Given any interval revelation policy $\mathscr{P}(a,b)$, the induced optimal action is non-decreasing in state, meaning that joint distribution of state and action is comonotone. Then by Lemma 1 and Remark 1 of Lin and Liu (2023), $\mathscr{P}(a,b)$ is indeed credible. $\square$

References for Appendices

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