Supplement to "Frequentist inference in weakly identified dynamic stochastic general equilibrium models"

(*Quantitative Economics*, Vol. 4, No. 2, July 2013, 197–229)

Pablo Guerron-Quintana
Federal Reserve Bank of Philadelphia

Atsushi Inoue
North Carolina State University

Lutz Kilian
University of Michigan and CEPR

Appendix A: Proofs

Proof of Proposition 1. The derivation of the posterior distribution follows from Proposition 12.2 in Hamilton (1994, p. 354). □

Proof of Proposition 2. The proof below is straightforward but is included for completeness. By assumption (a), the constrained MLE $\tilde{\gamma}_T(\theta)$ satisfies the first-order conditions

$$\nabla \ell_T\big(\tilde{\gamma}_T(\theta_0)\big) + D_\gamma f\big(\tilde{\gamma}_T(\theta_0), \theta_0\big)' \tilde{\lambda}_T(\theta_0) = 0_{\dim(\gamma)\times 1}, \tag{A.1}$$

$$f\big(\tilde{\gamma}_T(\theta_0), \theta_0\big) = 0_{r\times 1}, \tag{A.2}$$

where $\tilde{\lambda}_T(\theta_0)$ is the $r \times 1$ vector of Lagrange multipliers. A Taylor series expansion of (A.1) and (A.2) around $(\gamma_{0,T}, 0_{r\times 1})$ yields

$$\begin{aligned} &\begin{bmatrix} \nabla_{\gamma\gamma}\ell_T(\gamma_{0,T}) & D_\gamma f(\gamma_{0,T}, \theta_0)' \\ D_\gamma f(\gamma_{0,T}, \theta_0) & 0_{r\times r} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_T(\theta_0) - \gamma_{0,T} \\ \tilde{\lambda}_T(\theta_0) \end{bmatrix} \\ &\quad = \begin{bmatrix} \nabla \ell_T(\gamma_{0,T}) \\ 0_{r\times 1} \end{bmatrix} + o_p\big(T^{-1/2}\big), \end{aligned} \tag{A.3}$$

where the $o_p(T^{-1/2})$ follows from assumption (b). Solving these equations for $\tilde{\gamma}_T(\theta_0) - \gamma_{0,T}$ produces

$$\begin{aligned} &\tilde{\gamma}_T(\theta_0) - \gamma_{0,T} \\ &\quad = -\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1} \nabla_\gamma \ell_T(\gamma_{0,T}) \end{aligned} \tag{A.4}$$

Pablo Guerron-Quintana: pguerron@gmail.com
Atsushi Inoue: atsushi@ncsu.edu
Lutz Kilian: lkilian@umich.edu

DOI: 10.3982/QE306

$$
\begin{aligned}
&+ \left[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\right]^{-1} R_T' \left\{R_T\left[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\right]^{-1} R_T'\right\}^{-1} \\
&\times R_T\left[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\right]^{-1}\nabla_\gamma \ell_T(\gamma_0) \\
&+ o_p\left(T^{-1/2}\right),
\end{aligned}
$$

where $R_T = D_\gamma f(\gamma_{0,T}, \theta_0)$. Because $\hat{\gamma}_T - \gamma_{0,T} = -[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})]^{-1}\nabla_\gamma \ell_T(\gamma_{0,T}) + o_p(T^{-1/2})$, under assumption (b), it follows from (A.4) that

$$
\begin{aligned}
&\left[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\right]^{1/2}\left(\tilde{\gamma}_T(\theta_0) - \hat{\gamma}_T\right) \\
&\quad = \left[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\right]^{-1/2} R' \left\{R\left[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\right]^{-1} R'\right\}^{-1} \\
&\qquad \times R\left[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\right]^{-1}\nabla_\gamma \ell_T(\gamma_{0,T}) + o_p(1) \\
&\quad = P_T z_T + o_p(1),
\end{aligned}
\tag{A.5}
$$

where $P_T = [\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})]^{-1/2} R_T' \{R_T[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})]^{-1} R_T'\}^{-1} R_T[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})]^{-1/2}$ and $z_T = [\ell_T(\gamma_{0,T})]^{-1/2}\nabla_\gamma \ell_T(\gamma_{0,T})$. Because P_T is idempotent and has rank r, it follows from (A.5), a second-order Taylor series expansion of the LR test statistic around $\hat{\gamma}_T$, and assumptions (a), (b), and (c) that the asymptotic distribution of the LR test is the chi-squared distribution with r degrees of freedom. □

Proof of Theorem 1. It follows from assumption (a) in Proposition 2, assumptions (a), (b), and (c) of Theorem 1, the Taylor theorem, the first-order condition for the unconstrained MLE, and (A.5) that

$$
\begin{aligned}
I_{3,T} &\equiv \int_{B_{\delta_T}(\theta_0)} \pi(\theta) \exp\left(\ell_T\left(\tilde{\gamma}_T(\theta)\right) - \ell_T(\hat{\gamma}_T)\right) d\theta \\
&= \int_{B_{\delta_T}(\theta_0)} \pi(\theta) \exp\left(\frac{1}{2}\left(\tilde{\gamma}_T(\theta) - \hat{\gamma}_T\right)' \nabla_{\gamma\gamma}\ell_T\left(\bar{\gamma}_T(\theta)\right)\left(\tilde{\gamma}_T(\theta) - \hat{\gamma}_T\right)\right) d\theta \\
&= \int_{B_{\delta_T}(\theta_0)} \pi(\theta)\, d\theta \exp\left(-\frac{1}{2} z_T' z_T\right) + o_p(1),
\end{aligned}
\tag{A.6}
$$

where $\bar{\gamma}_T(\theta)$ is a point between $\tilde{\gamma}_T(\theta)$ and $\hat{\gamma}_T$.

Let

$$
I_{4,T} = \int_{\Theta \setminus B_{\delta_T}(\theta_0)} \pi(\theta) \exp\left(\ell_T\left(\tilde{\gamma}_T(\theta)\right) - \ell_T(\hat{\gamma}_T)\right) d\theta,
\tag{A.7}
$$

where $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$. Since $\ell_T(\tilde{\gamma}_T(\theta)) \leq \ell_T(\hat{\gamma}_T)$ by the definition of MLE, it follows from (A.7) that

$$
I_{4,T} \leq \int_{\Theta \setminus B_{\delta_T}(\theta_0)} \pi(\theta)\, d\theta.
\tag{A.8}
$$

Combining these results, the Bayes factor in favor of H_1 can be written as

$$\text{Bayes factor}(\theta_0) = \frac{\displaystyle\int_{B_{\delta_T}(\theta_0)} \pi(\theta)\,d\theta}{\displaystyle\int_{\Theta\setminus B_{\delta_T}(\theta_0)} \pi(\theta)\,d\theta} \frac{I_{4,T}}{I_{3,T}} \le \exp\left(\frac{1}{2} z_T' z_T\right) + o_p(1), \tag{A.9}$$

where the inequality follows from (A.8). □

Proof of Proposition 3a. Because the proof of (7) is analogous to that of Proposition 2, we only provide a sketch of the proof. By assumption (a) in Proposition 2, the constrained MLE $\tilde{\gamma}_T(\alpha_0)$ satisfies the first-order conditions

$$\nabla \ell_T\big(\tilde{\gamma}_T(\theta_0)\big) + D_\gamma f\big(\tilde{\gamma}_T(\alpha_0), \alpha_0, \tilde{\beta}(\alpha_0)\big)' \tilde{\lambda}_T(\alpha_0) = 0_{\dim(\gamma)\times 1}, \tag{A.10}$$

$$D_\beta f\big(\tilde{\gamma}_T(\alpha_0), \alpha_0, \tilde{\beta}_T(\alpha_0)\big) = 0_{k_2\times 1}, \tag{A.11}$$

$$f\big(\tilde{\gamma}_T(\alpha_0), \alpha_0, \tilde{\beta}_T(\alpha_0)\big) = 0_{r\times 1}, \tag{A.12}$$

where $\tilde{\lambda}_T(\alpha_0)$ is the $r \times 1$ vector of Lagrange multipliers. A Taylor series expansion of these first-order conditions about $[\gamma_{0,T}' \;\; \beta_0' \;\; 0_{1\times r}]'$ yields

$$\begin{aligned} &\begin{bmatrix} \nabla_{\gamma\gamma}\ell_T(\gamma_{0,T}) & 0_{\dim(\gamma)\times k_2} & D_\gamma f(\gamma_{0,T}, \theta_0)' \\ 0_{k_2\times \dim(\gamma)} & 0_{k_2\times k_2} & D_\beta f(\gamma_{0,T}, \theta_0)' \\ D_\gamma f(\gamma_{0,T}, \theta_0) & D_\beta f(\gamma_{0,T}, \theta_{0,T}) & 0_{r\times r} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_T(\alpha_0) - \gamma_{0,T} \\ \tilde{\beta}_T(\alpha_0) - \beta_0 \\ \tilde{\lambda}_T(\alpha_0) \end{bmatrix} \\ &\quad = \begin{bmatrix} -\nabla \ell_T(\gamma_{0,T}) \\ 0_{k_2\times 1} \\ 0_{r\times 1} \end{bmatrix} + o_p\big(T^{-1/2}\big). \end{aligned} \tag{A.13}$$

After solving these equations and some further manipulations, we obtain

$$\begin{aligned} &\tilde{\gamma}_T(\alpha_0) - \hat{\gamma}_T \\ &\quad = H_T^{-1} R' \big(R H_T^{-1} R'\big)^{-1} R H_T^{-1} \nabla \ell_T(\gamma_{0,T}) \\ &\qquad - H_T^{-1} R' \big(R H_T^{-1} R'\big)^{-1} R_\beta \big[R_\beta' \big(R H_T^{-1} R'\big)^{-1} R_\beta\big]^{-1} \\ &\qquad \times R_\beta' \big(R H_T^{-1} R'\big)^{-1} R H_T^{-1} \nabla_\gamma \ell_T(\gamma_{0,T}) + o_p(1), \end{aligned} \tag{A.14}$$

where $H_T = \nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})$ and $R_\beta = D_\beta f_T(\gamma_{0,T}, \theta_0)$. Hence, we can write

$$\begin{aligned} LR_T(\alpha_0) &= \big(\tilde{\gamma}_T(\alpha_0) - \hat{\gamma}_T\big)' \big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big] \big(\tilde{\gamma}_T(\alpha_0) - \hat{\gamma}_T\big) + o_p(1) \\ &= z_T' Q_T z_T + o_p(1), \end{aligned} \tag{A.15}$$

where Q_T is defined in (13). Because Q_T is idempotent and has rank $r - k_2$, we obtain the desired result. □

Proof of Proposition 3b. Note that (A.14) holds not only at $\alpha = \alpha_0$, but also in a neighborhood of α_0 specified by assumption (c) of Theorem 1. The proof of Proposition 3b is analogous to the proof of Theorem 1 except that (A.4) is replaced by (A.14). □

Proof of Theorem 2. First we prove (14). An application of the implicit function theorem to $f_T(\gamma,\alpha)=0$ yields

$$\begin{aligned}\frac{\partial\gamma}{\partial\alpha'}&=-\big[D_\gamma f_T(\gamma,\theta)\big]^{-1}D_\alpha f_T(\gamma,\theta)\\&=-T^{-1/2}\big[D_\gamma f_1(\gamma,\beta)\big]^{-1}D_\alpha f_2(\gamma,\theta)+o\big(T^{-1/2}\big),\end{aligned}\tag{A.16}$$

$$\frac{\partial\gamma}{\partial\beta'}=-\big[D_\gamma f_T(\gamma,\theta)\big]^{-1}D_\beta f_T(\gamma,\theta).\tag{A.17}$$

Thus, the mean value theorem implies

$$\begin{aligned}\gamma_{1,T}-\gamma_{0,T}=&-T^{-1/2}\big[D_\gamma f_1(\bar\gamma_T,\bar\beta_T)\big]^{-1}D_\beta f_1(\bar\gamma_T,\bar\beta_T)c\\&-T^{-1/2}\big[D_\gamma f_1(\bar\gamma_T,\beta_1)\big]^{-1}D_\alpha f_2(\bar\gamma_T,\bar\alpha)(\alpha_1-\alpha_0)\\&+o\big(T^{-1/2}\big),\end{aligned}\tag{A.18}$$

where $[\bar\gamma_T'\ \ \bar\theta_T']'=[\bar\gamma_T'\ \ \bar\alpha_T'\ \ \bar\beta_T']'$ is a point between $[\gamma_{1,T}'\ \ \alpha_1'\ \ \beta_0'+T^{-1/2}c']'$ and $[\gamma_{0,T}'\ \ \alpha_0'\ \ \beta_0']'$. Because $\gamma_{1,T}-\gamma_{0,T}=O(T^{-1/2})$, and f_1 and f_2 are continuously differentiable, we can write (A.18) as

$$\begin{aligned}\gamma_{1,T}-\gamma_{0,T}=&-T^{-1/2}\big[D_\gamma f_1(\gamma_{1,T},\beta_0)\big]^{-1}D_\beta f_1(\gamma_{1,T},\beta_0)c\\&-T^{-1/2}\big[D_\gamma f_1(\gamma_{1,T},\beta_0)\big]^{-1}D_\alpha f_2(\gamma_{1,T},\alpha_1,\beta_0)(\alpha_1-\alpha_0)\\&+o\big(T^{-1/2}\big)\\=&\,T^{-1/2}G(\alpha_0,\alpha_1,\beta_0)\big[c'\quad(\alpha_1-\alpha_0)'\big]'+o\big(T^{-1/2}\big),\end{aligned}\tag{A.19}$$

where $G(\alpha_0,\alpha_1,\beta_1)=\lim_{T\to\infty}\{[D_\gamma f_1(\gamma_{1,T},\beta_0)]^{-1}[D_\beta f_1(\gamma_{1,T},\beta_0)\ \ D_\alpha f_2(\gamma_{1,T},\bar\alpha_T)]\}$.

Using (A.19) and arguments analogous to those in the proof of Theorem 1, we can show $\hat\gamma_T-\gamma_{1,T}=-[\nabla_{\gamma\gamma}\ell_T(\gamma_{1,T})]^{-1}\nabla_\gamma\ell_T(\gamma_{1,T})+o_p(T^{-1/2})$ and

$$\begin{aligned}&\tilde\gamma_T(\theta_0)-\gamma_{0,T}\\&\quad=-\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1}\nabla_\gamma\ell_T(\gamma_{0,T})\\&\qquad+\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1}R_T'\big\{R_T\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1}R_T'\big\}^{-1}\\&\qquad\times R_T\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1}\nabla_\gamma\ell_T(\gamma_{0,T})\\&\qquad+o_p\big(T^{-1/2}\big)\\&\quad=\big\{-\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1}\\&\qquad+\big[\nabla_{\gamma\gamma}\ell_T(\gamma_0)\big]^{-1}R_T'\big\{R_T\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1}R_T'\big\}^{-1}R_T\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1}\big\}\\&\qquad\times\big\{\nabla_\gamma\ell_T(\gamma_{1,T})+T^{-1/2}\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})G(\alpha_0,\alpha_1,\beta_0)\big[c'\quad(\alpha_1-\alpha_0)'\big]'\big\}\\&\qquad+o_p\big(T^{-1/2}\big).\end{aligned}\tag{A.20}$$

Let $R=\lim_{T\to\infty} R_T$, $d(\alpha,\alpha_1,\beta_0,c)=V_\gamma^{-1/2}G(\alpha,\alpha_1,\beta_0)[c' \quad (\alpha_1-\alpha)']'$, and $P=V_\gamma^{1/2}\times R'(RV_\gamma R')^{-1}RV_\gamma^{1/2}$, where the dependence of P, R, and V_γ on α_1 and β_0 is omitted for notational simplicity. It follows from (A.19), (A.20), the twice continuous differentiability of $\ell_T(\cdot)$, and assumption (a) in Theorem 2 that

$$\begin{aligned}
&\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{1,T})\big]^{1/2}\big(\tilde{\gamma}_T(\theta_0)-\hat{\gamma}_T\big)\\
&\quad=\big\{\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{1,T})\big]^{-1/2}R_T'\big\{R_T\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{1,T})\big]^{-1}R_T'\big\}^{-1}\\
&\qquad\times R_T\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{1,T})\big]^{-1/2}\big\}\\
&\qquad\times\big\{T^{-1/2}\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{1,T})\big]^{1/2}G(\alpha_0,\alpha_1,\beta_0)\big[c' \quad (\alpha_1-\alpha_0)'\big]'\\
&\qquad+\big[\nabla_{\gamma\gamma}\ell_T(\gamma_{0,T})\big]^{-1/2}\nabla_\gamma\ell_T(\gamma_{1,T})\big\}+o_p(1)\\
&\quad=P\big[d(\alpha_0,\alpha_1,\beta_0,c)+z_T\big]+o_p(1).
\end{aligned}\tag{A.21}$$

Because P is idempotent and has rank r, (A.1) and a second-order Taylor series expansion of the LR test statistic around $\widehat{\gamma}_T$ yield (14).

Next we prove (15). Define

$$I_{j,T}=T^{k_2/2}\int_{\Theta_{j,T}}\pi(\theta)\exp\big(\ell_T\big(\tilde{\gamma}_T(\theta)\big)-\ell_T(\hat{\gamma}_T)\big)\,d\theta\tag{A.22}$$

for $j=5,6,7$, where $G_T(\gamma_{1,T},\theta_{1,T})=-[D_\gamma f_T(\gamma_{1,T},\theta_{1,T})]^{-1}D_\theta(\gamma_{1,T},\theta_{1,T})$,

$$\begin{aligned}
\Theta_{5,T}&=\big\{\theta\in\Theta:|\theta_j-\theta_{0,j}|<\delta_{T,j}\text{ for }j=1,\dots,k\big\},\\
\Theta_{6,T}&=\big\{\theta\in\Theta:|\theta_j-\theta_{0,j}|\ge\delta_{T,j},\ \beta=\beta_0+\bar{c}T^{-1/2}\text{ for some }\bar{c}\in\bar{C}_T\big\},\\
\Theta_{7,T}&=\big\{\theta\in\Theta:|\theta_j-\theta_{0,j}|\ge\delta_{T,j},\ \neg\exists\bar{c}\in\bar{C}_T\text{ such that }\beta=\beta_0+\bar{c}_TT^{-1/2}\big\},
\end{aligned}$$

$\bar{C}_T=\{\bar{c}\in\Re^{k_2}:-c_{\min}T^\eta\le c\le c_{\max}T^\eta\}$, $c_{\max}>0$, $c_{\min}>0$, and $\eta\in(0,1/2)$. Define

$$\begin{aligned}
\bar{\Theta}_{5,T}&=\big\{\big[\alpha' \quad c'\big]\in\mathcal{A}\times\Re^{k_2}:\\
&\qquad|\alpha_j-\alpha_{0,j}|<\delta_{T,j}\text{ for }j=1,\dots,k_1,\ |c_j|<\delta_{T,j+k_1}T^{1/2}\text{ for }j=1,\dots,k_2\big\},\\
\bar{\Theta}_{6,T}&=\big\{\theta\in\Theta:|\theta_j-\theta_{0,j}|\ge\delta_{T,j},\ \beta=\beta_0+\bar{c}T^{-1/2}\text{ for some }\bar{c}\in\bar{C}_T\big\},\\
\bar{\Theta}_{7,T}&=\big\{\theta\in\Theta:|\theta_j-\theta_{0,j}|\ge\delta_{T,j},\ \neg\exists\bar{c}\in\bar{C}_T\text{ such that }\beta=\beta_0+\bar{c}_TT^{-1/2}\big\}.
\end{aligned}$$

It follows that $I_{5,T}$ equals

$$\begin{aligned}
&T^{k_2/2}\int_{\Theta_{5,T}}\pi(\theta)\exp\bigg(\frac{1}{2}\big(\tilde{\gamma}_T(\theta)-\hat{\gamma}_T\big)'\nabla_{\gamma\gamma}\ell_T(\bar{\gamma}_T)\big(\tilde{\gamma}_T(\theta)-\hat{\gamma}_T\big)\bigg)\,d\theta\\
&\quad=\int_{\bar{\Theta}_{5,T}}\pi\big(\big[\alpha',\beta_0'+c'T^{-1/2}\big]'\big)\\
&\qquad\times\exp\bigg\{\frac{1}{2}\big[\tilde{\gamma}_T\big(\big[\alpha' \quad c'T^{-1/2}\big]'\big)-\hat{\gamma}_T\big]'
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \nabla_{\gamma\gamma}\ell_T(\bar{\gamma}_T)\bigl[\tilde{\gamma}_T\bigl(\bigl[\alpha' \quad c'T^{-1/2}\bigr]'\bigr) - \hat{\gamma}_T\bigr]\Bigr\} d\bigl[\alpha' \quad c'\bigr]' \\
&\quad + o_p(1) \\
&= \int_{\bar{\Theta}_{5,T}} \pi\bigl(\bigl[\alpha' \quad \beta_0'\bigr]'\bigr) \\
&\quad \times \exp\Bigl(-\frac{1}{2}\bigl(d\bigl(\alpha, \alpha_1, \beta_0, \bar{c}_T(\beta)\bigr) + z_T\bigr)' \\
&\quad \times P'P\bigl(d\bigl(\alpha, \alpha_1, \beta_0, \bar{c}_T(\beta)\bigr) + z_T\bigr)\Bigr) d\bigl[\alpha' \quad c'\bigr] \\
&\quad + o_p\bigl(T^{-k_2/2}\bigr) + o_p(1) \\
&= \int_{\bar{\Theta}_{5,T}} \pi\bigl(\bigl[\alpha' \quad c'\bigr]'\bigr) d\bigl[\alpha' \quad c'\bigr]' \\
&\quad \times \exp\Bigl\{-\frac{1}{2}\bigl[d(\alpha_0, \alpha_1, \beta_0, 0) + z_T\bigr]' P\bigl[d(\alpha_0, \alpha_1, \beta_0, 0) + z_T\bigr]\Bigr\} \\
&\quad + o_p(1),
\end{aligned}
\tag{A.23}
$$

where the first expression follows from assumption (a) in Proposition 2 and Taylor's theorem, the first equality follows from a change of variables, the second equality follows from (A.21) and assumptions (a) and (b) in Proposition 2, $\bar{\gamma}_T$ is a point between $\hat{\gamma}_T$ and $\tilde{\gamma}_T(\theta)$, $\check{\gamma}_T$ is a point between $\tilde{\gamma}_T(\theta)$ and $\tilde{\gamma}_T(\theta_{1,T})$, and $\check{\theta}_T$ is a point between θ and $\theta_{1,T}$.

Note that the arguments used to derive (A.21) are valid not only for a particular value of α_0 and c, but for all α and c. The compactness of α, the continuous differentiability of f_T, and the continuity of $\nabla_{\gamma\gamma}\ell_T$ imply that (A.20) holds uniformly in $\alpha \in \mathcal{A}$, which in turn implies that (A.21) holds uniformly in $\alpha \in \mathcal{A}$. Thus, $I_{6,T}$ equals

$$
\begin{aligned}
&T^{k_2/2} \int_{\Theta_{6,T}} \pi(\theta) \exp\Bigl(\frac{1}{2}\bigl(\tilde{\gamma}_T(\theta) - \hat{\gamma}_T\bigr)' \nabla_{\gamma\gamma}\ell_T(\bar{\gamma}_T)\bigl(\tilde{\gamma}_T(\theta) - \hat{\gamma}_T\bigr)\Bigr) d\theta \\
&= \int_{\bar{\Theta}_{6,T}} \pi\bigl(\bigl[\alpha', \beta_0' + c'T^{-1/2}\bigr]'\bigr) \\
&\quad \times \exp\Bigl\{\frac{1}{2}\bigl[\tilde{\gamma}_T\bigl(\bigl[\alpha' \quad c'T^{-1/2}\bigr]'\bigr) - \hat{\gamma}_T\bigr]' \\
&\quad \times \nabla_{\gamma\gamma}\ell_T(\bar{\gamma}_T)\bigl[\tilde{\gamma}_T\bigl(\bigl[\alpha' \quad c'T^{-1/2}\bigr]'\bigr) - \hat{\gamma}_T\bigr]\Bigr\} d\bigl[\alpha' \quad c'\bigr]' \\
&\quad + o_p(1) \\
&= \int_{\bar{\Theta}_{6,T}} \pi\bigl(\bigl[\alpha', \beta_0'\bigr]'\bigr) \\
&\quad \times \exp\Bigl\{-\frac{1}{2}\bigl[d(\alpha, \alpha_1, \beta_0, c) + z_T\bigr]' P\bigl[d(\alpha, \alpha_1, \beta_0, c) + z_T\bigr]\Bigr\} d\alpha + o_p(1)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{|\alpha_j - \alpha_{1,j}| \geq \delta_{j,T}} \pi\big([\alpha', \beta_0']'\big) \\
&\quad \times \exp\bigg\{ \frac{1}{2} z_T' \big[P - P V_\gamma^{-1/2} G(\alpha, \alpha_1, \beta_0) \big(G(\alpha, \alpha_1, \beta_0)' V_\gamma^{-1/2} \\
&\quad \times P V_\gamma^{-1/2} G(\alpha, \alpha_1, \beta_0)\big)^{-1} G(\alpha, \alpha_1, \beta_0)' V_\gamma^{-1/2} P\big] z_T \bigg\} \, d\alpha \\
&\quad + o_p(1) \\
&= \int_{\mathcal{A} \times \bar{C}_T} \pi\big([\alpha', \beta_0']'\big) \\
&\quad \times \exp\bigg\{ \frac{1}{2} z_T' \big[P - P V_\gamma^{-1/2} G(\alpha, \alpha_1, \beta_0) \big(G(\alpha, \alpha_1, \beta_0)' V_\gamma^{-1/2} \\
&\quad \times P V_\gamma^{-1/2} G(\alpha, \alpha_1, \beta_0)\big)^{-1} G(\alpha, \alpha_1, \beta_0)' V_\gamma^{-1/2} P\big] z_T \bigg\} \, d\alpha \\
&\quad + o_p(1).
\end{aligned}
\tag{A.24}
$$

Moreover,

$$
I_{7,T} = T^{k_2/2} \int_{\Theta_{7,T}} \pi(\theta) \exp\big(\ell_T\big(\tilde{\gamma}_T(\theta)\big) - \ell_T(\hat{\gamma}_T)\big) \, d\theta = o_p(1), \tag{A.25}
$$

where the last equality follows from assumption (c) in Theorem 2.

Because $\int_{\bar{\Theta}_{5,T}} \pi([\alpha', \beta_0' + T^{-1/2} c']') \, d[\alpha' \;\; c']' = \int_{\Theta_{5,T}} \pi([\alpha', \beta']') \, d[\alpha', \beta']'$, it follows from (A.23), (A.24), and (A.25) that the Bayes factor in favor of H_1 can be written as

$$
\begin{aligned}
&T^{k_2/2} \, \text{Bayes factor}(\theta_0) \\
&\quad = T^{k_2/2} \frac{\displaystyle\int_{B_{\delta_T}(\theta_0)} \pi(\theta) \, d\theta}{\displaystyle\int_{\Theta \setminus B_{\delta_T}(\theta_0)} \pi(\theta) \, d\theta} \frac{I_{6,T} + I_{7,T}}{I_{5,T}} \\
&\quad = \int_{\mathcal{A} \times \bar{C}_T} \pi(\alpha, \beta_0) \\
&\qquad \times \exp\bigg(-\frac{1}{2} \big(d(\alpha, \alpha_1, \beta_0, c) + P z_T\big)' \\
&\qquad \times \big(d(\alpha, \alpha_1, \beta_0, c) + P z_T\big) \bigg) \, d\big[\alpha' \quad c'\big]' \\
&\qquad \Big/ \bigg(\exp\bigg(-\frac{1}{2} \big(d(\alpha_0, \alpha_1, \beta_0, 0) + P z_T\big)' \big(d(\alpha_0, \alpha_1, \beta_0, 0) + P z_T\big) \bigg) \bigg) \\
&\qquad + o_p(1),
\end{aligned}
\tag{A.26}
$$

which completes the proof of (14). □

Proof of Proposition 4. Define

$$I_{j,T}=\int_{\Theta_{j,T}}\pi(\theta)\exp\bigl(\ell_T\bigl(\tilde{\gamma}_T(\theta)\bigr)-\ell_T(\hat{\gamma}_T)\bigr)\,d\theta \tag{A.27}$$

for $j=8,9,10$, where

$$\begin{aligned}
\Theta_{8,T}&=\bigl\{\theta\in\Theta:|\theta_j-\theta_{0,j}|<\delta_{T,j}\text{ for }j=1,\dots,k\bigr\},\\
\Theta_{9,T}&=\bigl\{\theta\in\Theta:|\theta_j-\theta_{0,j}|\geq\delta_{T,j}\bigr\},\\
\Theta_{10,T}&=\Theta\setminus(\Theta_{8,T}\cup\Theta_{9,T}).
\end{aligned}$$

Because $\hat{\gamma}_T$ is the MLE, we have

$$\begin{aligned}
I_{8,T}&=\int_{A_{8,T}}\pi(\theta)\exp\bigl(\ell_T\bigl(\tilde{\gamma}_T(\theta)\bigr)-\ell_T(\hat{\gamma}_T)\bigr)\,d\theta=o_p(1),\\
I_{9,T}&\equiv\int_{B_{\delta_T}(\theta_0)}\pi(\theta)\exp\bigl(\ell_T\bigl(\tilde{\gamma}_T(\theta)\bigr)-\ell_T(\hat{\gamma}_T)\bigr)\,d\theta\\
&=\int_{B_{\delta_T}(\theta_0)}\pi(\theta)\exp\biggl(\frac{1}{2}\bigl(\tilde{\gamma}_T(\theta)-\hat{\gamma}_T\bigr)'\nabla_{\gamma\gamma}\ell_T\bigl(\bar{\gamma}_T(\theta)\bigr)\bigl(\tilde{\gamma}_T(\theta)-\hat{\gamma}_T\bigr)\biggr)\,d\theta\\
&=\int_{B_{\delta_T}(\theta_0)}\pi(\theta)\,d\theta\exp\biggl(-\frac{1}{2}z_T'z_T\biggr)+o_p(1).
\end{aligned} \tag{A.28}$$

Thus, $P(H_0|X)=I_{8,T}/(I_{9,T}+I_{10,T})$ decays exponentially and faster than $\pi(H_0)$. Therefore, the Bayes factor diverges to positive infinity. □

Appendix B: Deriving the degrees of freedom for the LR and BF test statistics

Deriving the degrees of freedom for DGP 1

DGP 1 in the Monte Carlo experiment has a state-space representation

$$x_{t+1}=Ax_t+Bu_t, \tag{B.1}$$

$$y_t=Cx_t, \tag{B.2}$$

where x_t is a 3×1 vector of state variables, y_t is a 2×1 vector of observed variables, $u_t\sim N(0_{2\times1},I_2)$, A, B, and C are 3×3, 3×2, and 2×3 matrices such that

$$A=\begin{bmatrix}\times&\times&\times\\0&\times&0\\0&0&0\end{bmatrix},\qquad B=\begin{bmatrix}0&0\\\times&0\\0&\times\end{bmatrix},\qquad C=\begin{bmatrix}\times&\times&\times\\\times&\times&\times\end{bmatrix},$$

where $\times$ denotes an element that is not identical to zero.

When evaluated at 10,000 random draws of the structural parameter vector $\theta\in\Theta$, $[C'\ \ A'C']$ has rank 2 and $[B\ \ AB\ \ A^2B]$ has rank 3 for each of the 10,000 draws. Thus, while the state-space representation above is reachable, it is not observable and hence is not minimal. The rank condition for observability implies that the minimal representation has only two state variables. Inspection reveals that the third state variable is an i.i.d. process, which can be absorbed into another state variable by allowing some elements of B to be nonzero. Thus, the resulting minimal state-space representation can be written as

$$x^*_{t+1} = A^* x^*_t + B^* u_t, \tag{B.3}$$

$$y_t = C^* x^*_t, \tag{B.4}$$

where x^*_t is a 2×1 vector of state variables, and A^*, B^*, and C^* are 2×2 matrices such that

$$A^* = \begin{bmatrix} \times & \times \\ 0 & \times \end{bmatrix}, \qquad B^* = \begin{bmatrix} \times & \times \\ 0 & \times \end{bmatrix}, \qquad C = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}.$$

For the $(2,1)$ elements of TA^*T^{-1} and TB^* to be zero like the corresponding elements in A^* and B^*, the $(2,1)$ element of T must be zero. Because there are 10 free elements in A^*, B^*, and C^*, and there are 3 free elements in T, we conclude that the number of degrees of freedom is 7.

In fact, y_t has a bivariate VAR(1) representation. Because the matrix A has three distinct eigenvalues including zero, it can be written as

$$A = V \Lambda V^{-1},$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and V is the matrix whose first, second, and third columns are eigenvectors associated with λ_1, λ_2, and 0, respectively. Then

$$\begin{aligned} y_t &= C(I - AL)^{-1} B u_t \\ &= CBu_t + CABu_{t-1} + CA^2Bu_{t-2} + \cdots \\ &= \sum_{j=0}^{\infty} CA^j B u_{t-j} \\ &= \sum_{j=0}^{\infty} CV\Lambda^j V^{-1} B u_{t-j} \\ &= \sum_{j=0}^{\infty} M_1 \bar{\Lambda} M_2 u_{t-j}, \end{aligned}$$

where M_1 is the left 2×2 submatrix of CV, M_2 is the upper 2×2 submatrix of $V^{-1}B$, and $\bar{\Lambda}$ is the 2×2 diagonal matrix whose diagonal elements are given by λ_1 and λ_2. This moving average representation has an autoregressive representation

$$M_1^{-1} y_t = \bar{\Lambda} M_1^{-1} y_{t-1} + M_2 u_t$$

or

$$y_t = \Phi y_{t-1} + \Sigma^{1/2} u_t,$$

where $\Phi = M_1 \bar{\Lambda} M_1^{-1}$ and $\Sigma = M_1 M_2 M_2' M_1'$.

Deriving the degrees of freedom for DGP 2

DGP 2 has four state variables and three structural shocks, with A, B, and C being of dimension 4×4, 4×3, and 3×4, respectively. The rank conditions for observability and reachability are both satisfied at any one of 10,000 randomly sampled values of the vector θ. Thus we treat the original state-space representation as minimal. Because A and B take the forms

$$A = \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{bmatrix}$$

and possible similarity transformation matrices must take the form

$$T = \begin{bmatrix} t_{11} & 0 & 0 & 0 \\ 0 & t_{22} & 0 & 0 \\ 0 & 0 & t_{33} & 0 \\ 0 & 0 & 0 & t_{44} \end{bmatrix},$$

where $t_{ii} \neq 0$ for $i = 1, 2, 3, 4$, this leaves us with 18 degrees of freedom in pinning down the 22 reduced-form parameters in the A, B, and C matrices.

Unlike DGP 1, DGP 2 has a VARMA(2, 1) reduced-form representation, which can be written as

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}' \\ 0_{3\times 1} & \Lambda \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} 0_{1\times 3} \\ B_2 \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon_t \end{bmatrix}, \tag{B.5}$$

$$y_t = \begin{bmatrix} c_1 & C_2 \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}, \tag{B.6}$$

where x_{1t} and a_{11} are scalars; x_{2t}, a_{12}, $\varepsilon_t = [\varepsilon_t^{mk} \ \ \varepsilon_t^z \ \ \varepsilon_t^{\xi}]'$, c_1, and $y_t = [\pi_t \ \ x_t \ \ R_t]'$ are 3×1; Λ is the 3×3 diagonal matrix that consists of ρ_{mk}, ρ_z, and ρ_ξ; B_2 is the 3×3 diagonal matrix that consists of σ^{mk}, σ^z, and σ^ξ; and C_2 is 3×3. Because $x_{2t} = (I_3 - \Lambda L)^{-1} B_2 \varepsilon_t$ and $x_{1t} = (1 - a_{11}L)^{-1} a_{12}' x_{2t-1} = (1 - a_{11}L)^{-1} a_{12}' (I_3 - \Lambda L)^{-1} B_2 L \varepsilon_t$, y_t can be expressed as

$$y_t = c_1 (1 - a_{11}L)^{-1} a_{12}' (I_3 - \Lambda L)^{-1} B_2 L \varepsilon_t + C_2 (I_3 - \Lambda L)^{-1} B_2 \varepsilon_t.$$

After some manipulations, y_t can be expressed as the VARMA(2, 1) model

$$(1 - a_{11}L)(I_3 - \Phi L)y_t = (I_3 - \Theta L)\Sigma^{1/2}\varepsilon_t, \tag{B.7}$$

with $D = C_2^{-1}(a_{11}C_2 - c_1 a_{12}')$, $\Phi = C_2(\Lambda - D)\Lambda(\Lambda - D)^{-1}C_2^{-1}$, $\Theta = C_2(\Lambda - D)D(\Lambda - D)^{-1}C_2^{-1}$, and $\Sigma^{1/2} = C_2 B_2$. It turns out that there is a root cancellation. To see this, note that y_t can be also written as

$$(1 - a_{11}L)(I_3 - \Lambda L)(\Lambda - D)^{-1}C_2^{-1}y_t = P(I_3 - JL)P^{-1}(\Lambda - D)^{-1}B_2\varepsilon_t, \tag{B.8}$$

where PJP^{-1} is the Jordan decomposition of D and

$$J = \begin{bmatrix} a_{11} & 1 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the system can also be written as

$$(1 - a_{11})\big(I_3 - \big(P^{-1}\Lambda P\big)L\big)y_t^* = (I_3 - JL)\varepsilon_t^*, \tag{B.9}$$

where $y_t^* = P^{-1}(\Lambda - D)^{-1}C_2^{-1}y_t$ and $\varepsilon_t^* = P^{-1}(\Lambda - D)^{-1}B_2\varepsilon_t$. The root cancellation in the second equation is not a problem for our testing procedure as long as we base inference on the minimal state-space representation (B.5) and (B.6) rather than the unrestricted VARMA(2, 1) model (B.7).

Deriving the degrees of freedom for the medium-scale DSGE model used as the empirical example

Let $\times_{p\times q}$ denote a $p \times q$ matrix whose elements are not identical to zero. The medium-scale model used in the empirical application has a state-space representation with

$$A = \begin{bmatrix} \times_{7\times 8} & & & \times_{7\times 5} & & \\ & 1 & 0 & 0 & 0 & 0 \\ & \times & 0 & 0 & 0 & 0 \\ 0_{6\times 8} & 0 & \times & 0 & 0 & 0 \\ & 0 & 0 & \times & 0 & 0 \\ & 0 & 0 & 0 & \times & 0 \\ & 0 & 0 & 0 & 0 & \times \end{bmatrix},$$

$$B = \begin{bmatrix} & & 0_{8\times 5} & & \\ \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & \times_{1\times 12} \\ 0_{4\times 1} & \times_{4\times 12} \end{bmatrix},$$

and an intercept term for each of the five measurement equations. An evaluation of the two rank conditions at 10,000 randomly chosen structural parameter values indicates that this representation with 13 state variables is not minimal and that the minimal representation has only 11 state variables. Inspection reveals that the 13th state variable can be absorbed into the first 8 state variables by allowing nonzero elements in the B matrix because it is i.i.d. and is independent of the other state variables. Thus, the minimal representation can be written as a state-space model with

$$A^* = \begin{bmatrix} \times_{6\times6} & \times_{6\times5} \\ 0_{5\times6} & \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \end{matrix} \end{bmatrix}, \qquad B^* = \begin{bmatrix} 0_{6\times4} & \times_{6\times1} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \end{bmatrix},$$

$C^* = \times_{5\times11}$, and the five intercept terms in the measurement equations. This leaves us with 140 reduced-form parameters. For TA^*T^{-1} and B^* to have zeros in the same location as A^* and B^*, respectively, the upper-right 6×5 and the lower-left 5×6 submatrices of T must be zero, and the lower-right 5×5 submatrix must be diagonal, so there remain 41 free parameters in T, reducing the effective number of identified reduced-form parameters to 99.

Appendix C: Additional tables

Table C.1. Effective coverage rates of nominal 90% confidence intervals.

	ϕ_π	ϕ_x	α	θ	ρ_z	ρ_r	σ_z	σ_r
(a) Small-Scale New Keynesian Model With Two Observables: LR								
$T=96$								
LR	0.962	0.964	0.962	0.970	0.948	0.951	0.942	0.939
$T=188$								
LR	0.972	0.972	0.973	0.976	0.964	0.968	0.953	0.960
(b) Small-Scale New Keynesian Model With Two Observables: Uniform Priors								
$T=96$								
Median $\pm$ 1.645SD	0.965	0.960	0.834	1.000	0.797	0.900	0.649	0.629
Mean $\pm$ 1.645SD	0.981	0.957	0.906	1.000	0.796	0.889	0.622	0.563
Mode $\pm$ 1.645SD	0.920	0.968	0.823	0.772	0.879	0.886	0.893	0.917
Percentile	0.994	0.952	0.975	1.000	0.795	0.881	0.548	0.416
BF	1.000	1.000	1.000	1.000	0.997	0.999	0.994	0.990
$T=188$								
Median $\pm$ 1.645SD	0.986	0.989	0.927	1.000	0.795	0.906	0.728	0.735
Mean $\pm$ 1.645SD	0.994	0.992	0.965	1.000	0.789	0.905	0.694	0.681
Mode $\pm$ 1.645SD	0.963	0.981	0.878	0.795	0.876	0.893	0.922	0.947
Percentile	0.997	0.989	0.987	1.000	0.789	0.903	0.628	0.581
BF	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.998

TABLE C.1. *Continued.*

	ϕ_π	ϕ_x	α	θ	ρ_z	ρ_r	σ_z	σ_r
(c) Small-Scale New Keynesian Model With Two Observables: Informative Priors								
$T = 96$								
Median $\pm$ 1.645SD	1.000	0.972	0.997	1.000	0.896	0.902	0.895	0.909
Mean $\pm$ 1.645SD	1.000	0.987	0.996	1.000	0.900	0.905	0.916	0.921
Mode $\pm$ 1.645SD	1.000	0.799	0.946	1.000	0.750	0.883	0.636	0.708
Percentile	1.000	0.997	0.996	1.000	0.902	0.910	0.939	0.944
BF	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000
$T = 188$								
Median $\pm$ 1.645SD	1.000	0.944	1.000	1.000	0.906	0.932	0.906	0.907
Mean $\pm$ 1.645SD	1.000	0.966	1.000	1.000	0.905	0.934	0.914	0.935
Mode $\pm$ 1.645SD	1.000	0.773	0.983	1.000	0.758	0.913	0.669	0.711
Percentile	1.000	0.985	1.000	1.000	0.906	0.938	0.932	0.952
BF	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(d) Small-Scale New Keynesian Model With Two Observables: Asymmetric Priors								
$T = 96$								
Median $\pm$ 1.645SD	0.966	0.953	0.103	0.400	0.796	0.886	0.636	0.619
Mean $\pm$ 1.645SD	0.978	0.956	0.109	0.713	0.804	0.884	0.592	0.542
Mode $\pm$ 1.645SD	0.931	0.972	0.521	0.504	0.868	0.888	0.874	0.919
Percentile	0.996	0.950	0.118	0.185	0.803	0.883	0.519	0.384
BF interval	1.000	1.000	0.968	1.000	0.999	0.999	0.994	0.981
$T = 188$								
Median $\pm$ 1.645SD	0.989	0.992	0.163	0.342	0.789	0.909	0.704	0.756
Mean $\pm$ 1.645SD	0.994	0.991	0.170	0.646	0.787	0.907	0.677	0.696
Mode $\pm$ 1.645SD	0.958	0.984	0.550	0.500	0.869	0.892	0.899	0.949
Percentile	1.000	0.986	0.176	0.145	0.784	0.908	0.599	0.566
BF	1.000	1.000	0.978	1.000	1.000	1.000	0.995	0.994

TABLE C.2. Median lengths of nominal 90% confidence intervals.

	ϕ_π	ϕ_x	α	θ	ρ_z	ρ_r	σ_z	σ_r
(a) Small-Scale New Keynesian Model With Two Observables: LR								
$T = 96$								
LR	1.57	2.38	0.25	5.24	0.40	0.25	0.24	0.31
$T = 188$								
LR	1.55	2.38	0.25	5.80	0.40	0.25	0.22	0.30
(b) Small-Scale New Keynesian Model With Two Observables: Uniform Priors								
$T = 96$								
Median/mean/mode	3.34	0.77	0.22	12.80	0.08	0.17	0.49	0.39
Percentile	3.34	0.76	0.21	12.27	0.08	0.17	0.49	0.39
BF	3.99	1.48	0.32	13.99	0.18	0.33	0.76	0.79
$T = 188$								
Median/mean/mode	2.83	0.52	0.21	12.81	0.06	0.12	0.34	0.26
Percentile	2.77	0.50	0.20	12.27	0.06	0.12	0.34	0.26
BF	3.99	1.27	0.30	13.98	0.14	0.26	0.74	0.69

Table C.2. *Continued.*

	ϕ_π	ϕ_x	α	θ	ρ_z	ρ_r	σ_z	σ_r
(c) Small-Scale New Keynesian Model With Two Observables: Informative Priors								
$T = 96$								
Median/mean/mode	0.76	0.22	0.12	6.41	0.07	0.15	0.21	0.14
Percentile	0.76	0.21	0.12	6.44	0.07	0.15	0.21	0.14
BF	1.45	0.34	0.25	12.5	0.12	0.29	0.38	0.18
$T = 188$								
Median/mean/mode	0.75	0.21	0.12	6.39	0.05	0.11	0.18	0.12
Percentile	0.76	0.20	0.12	6.41	0.05	0.11	0.18	0.12
BF	1.34	0.29	0.22	12.3	0.09	0.20	0.32	0.19
(d) Small-Scale New Keynesian Model With Two Observables: Asymmetric Priors								
$T = 96$								
Median/mean/mode	3.42	0.81	0.14	8.96	0.08	0.17	0.49	0.41
Percentile	3.37	0.79	0.14	8.51	0.08	0.17	0.50	0.41
BF	3.99	1.54	0.24	9.49	0.19	0.35	0.78	0.79
$T = 188$								
Median/mean/mode	2.80	0.53	0.13	8.95	0.06	0.12	0.35	0.26
Percentile	2.75	0.51	0.13	8.51	0.06	0.12	0.35	0.26
BF	3.99	1.22	0.23	9.50	0.15	0.28	0.73	0.70

Table C.3. Effective coverage rates of nominal 90% confidence sets.

		$T = 96$	$T = 188$
Small-Scale New Keynesian Model With Three Observables			
LR set		0.856	0.868
BF set	Uniform prior	0.943	0.911
BF set	Informative prior	0.828	0.787
BF set	Asymmetric prior	0.896	0.870

Table C.4. Effective coverage rates of nominal 90% confidence intervals.

	ϕ_π	ϕ_x	α	θ	ρ_z	ρ_r	ρ_{mk}	ρ_m	σ_z	σ_r	σ_{mk}
(a) Small-Scale New Keynesian Model With Three Observables: LR											
$T = 96$											
LR interval	0.968	0.968	0.960	0.993	0.942	0.991	0.975	0.955	0.943	0.984	0.936
$T = 188$											
LR interval	0.971	0.973	0.967	0.993	0.960	0.991	0.973	0.969	0.944	0.982	0.951

TABLE C.4. *Continued.*

	ϕ_π	ϕ_x	α	θ	ρ_z	ρ_r	ρ_{mk}	ρ_m	σ_z	σ_r	σ_{mk}
(b) Small-Scale New Keynesian Model With Three Observables: Uniform Priors											
$T=96$											
Median ± 1.645SD	0.900	0.907	0.852	1.000	0.846	0.781	0.859	0.829	0.735	0.920	0.878
Mean ± 1.645SD	0.902	0.889	0.874	1.000	0.848	0.794	0.874	0.848	0.737	0.902	0.854
Mode ± 1.645SD	0.952	0.950	0.765	0.809	0.809	0.891	0.837	0.874	0.824	0.941	0.896
Percentile	0.787	0.802	0.891	1.000	0.846	0.828	0.874	0.870	0.733	0.819	0.752
BF interval	0.998	0.996	1.000	1.000	1.000	0.998	1.000	1.000	0.998	1.000	1.000
$T=188$											
Median ± 1.645SD	0.921	0.920	0.875	1.000	0.872	0.846	0.863	0.861	0.806	0.904	0.878
Mean ± 1.645SD	0.926	0.915	0.900	1.000	0.878	0.850	0.865	0.869	0.817	0.894	0.876
Mode ± 1.645SD	0.930	0.937	0.787	0.791	0.811	0.891	0.837	0.874	0.841	0.928	0.889
Percentile	0.837	0.815	0.928	1.000	0.870	0.854	0.872	0.883	0.813	0.833	0.826
BF interval	0.998	0.998	1.000	1.000	1.000	1.000	1.000	0.998	1.000	1.000	1.000
(c) Small-Scale New Keynesian Model With Three Observables: Informative Priors											
$T=96$											
Median ± 1.645SD	0.956	0.948	0.722	1.000	0.830	0.933	0.891	0.909	0.837	0.896	0.893
Mean ± 1.645SD	0.961	0.952	0.765	1.000	0.841	0.933	0.896	0.915	0.850	0.902	0.898
Mode ± 1.645SD	0.870	0.902	0.556	1.000	0.670	0.885	0.857	0.859	0.630	0.863	0.765
Percentile	0.969	0.956	0.820	1.000	0.857	0.930	0.894	0.909	0.867	0.898	0.911
BF interval	1.000	1.000	1.000	1.000	0.998	1.000	0.998	1.000	0.998	1.000	1.000
$T=188$											
Median ± 1.645SD	0.928	0.956	0.772	1.000	0.856	0.909	0.878	0.900	0.830	0.913	0.865
Mean ± 1.645SD	0.939	0.959	0.793	1.000	0.859	0.907	0.878	0.898	0.839	0.920	0.878
Mode ± 1.645SD	0.844	0.896	0.678	1.000	0.813	0.867	0.841	0.848	0.728	0.854	0.800
Percentile	0.948	0.950	0.820	1.000	0.863	0.913	0.881	0.904	0.850	0.928	0.891
BF interval	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.998	0.994	1.000	0.996
(d) Small-Scale New Keynesian Model With Three Observables: Asymmetric Priors											
$T=96$											
Median ± 1.645SD	0.883	0.889	0.737	0.254	0.837	0.794	0.894	0.820	0.724	0.915	0.837
Mean ± 1.645SD	0.874	0.883	0.752	0.367	0.857	0.813	0.893	0.837	0.733	0.894	0.819
Mode ± 1.645SD	0.952	0.939	0.746	0.459	0.785	0.883	0.846	0.865	0.781	0.954	0.856
Percentile	0.752	0.781	0.774	0.126	0.867	0.844	0.891	0.865	0.711	0.837	0.733
BF interval	0.996	0.998	1.000	1.000	1.000	0.996	1.000	1.000	0.996	0.998	0.998
$T=188$											
Median ± 1.645SD	0.944	0.933	0.746	0.196	0.870	0.870	0.893	0.859	0.768	0.913	0.857
Mean ± 1.645SD	0.926	0.920	0.752	0.287	0.867	0.872	0.891	0.863	0.770	0.902	0.835
Mode ± 1.645SD	0.963	0.954	0.785	0.474	0.853	0.935	0.874	0.885	0.826	0.926	0.878
Percentile	0.815	0.824	0.776	0.078	0.872	0.869	0.896	0.861	0.778	0.839	0.763
BF interval	0.998	1.000	0.998	1.000	1.000	1.000	1.000	0.998	1.000	1.000	1.000

TABLE C.5. Median lengths of nominal 90% confidence intervals.

	ϕ_π	ϕ_x	α	θ	ρ_z	ρ_r	ρ_{mk}	ρ_m	σ_z	σ_r	σ_{mk}
(a) Small-Scale New Keynesian Model With Three Observables: LR											
$T=96$											
LR	1.606	2.388	0.311	2.773	0.400	0.256	0.407	0.169	0.202	0.306	0.360
$T=188$											
LR	1.521	2.380	0.286	2.698	0.401	0.256	0.403	0.086	0.201	0.302	0.361
(b) Small-Scale New Keynesian Model With Three Observables: Uniform Priors											
$T=96$											
Median/mean/mode	0.139	0.036	0.091	5.719	0.028	0.031	0.032	0.080	0.060	0.021	0.039
Percentile	0.140	0.035	0.084	5.612	0.028	0.031	0.032	0.079	0.060	0.020	0.038
BF	2.316	0.569	0.582	13.99	0.215	0.213	0.271	0.486	0.366	0.181	0.334
$T=188$											
Median/mean/mode	0.121	0.033	0.104	5.531	0.030	0.029	0.027	0.064	0.043	0.017	0.031
Percentile	0.114	0.032	0.103	5.361	0.029	0.028	0.027	0.066	0.043	0.017	0.031
BF	1.207	0.323	0.475	13.98	0.163	0.159	0.205	0.360	0.250	0.103	0.213
(c) Small-Scale New Keynesian Model With Three Observables: Informative Priors											
$T=96$											
Median/mean/mode	0.158	0.044	0.067	2.959	0.032	0.033	0.026	0.080	0.053	0.020	0.026
Percentile	0.153	0.042	0.061	2.916	0.032	0.033	0.024	0.081	0.053	0.020	0.026
BF	1.046	0.280	0.408	12.96	0.178	0.184	0.253	0.433	0.295	0.115	0.223
$T=188$											
Median/mean/mode	0.130	0.035	0.072	2.907	0.026	0.026	0.032	0.063	0.043	0.016	0.028
Percentile	0.125	0.033	0.067	2.908	0.025	0.026	0.032	0.063	0.042	0.016	0.028
BF	0.894	0.240	0.389	12.86	0.149	0.145	0.210	0.316	0.228	0.086	0.184
(d) Small-Scale New Keynesian Model With Three Observables: Asymmetric Priors											
$T=96$											
Median/mean/mode	0.146	0.046	0.072	4.078	0.031	0.036	0.028	0.070	0.055	0.025	0.038
Percentile	0.136	0.043	0.069	3.926	0.030	0.036	0.026	0.070	0.052	0.024	0.037
BF	2.254	0.545	0.450	9.495	0.214	0.211	0.267	0.493	0.321	0.175	0.332
$T=188$											
Median/mean/mode	0.148	0.033	0.062	3.991	0.026	0.026	0.033	0.068	0.046	0.016	0.030
Percentile	0.145	0.032	0.060	3.876	0.026	0.026	0.033	0.067	0.049	0.017	0.029
BF	1.287	0.322	0.373	9.495	0.156	0.165	0.211	0.352	0.248	0.108	0.224

TABLE C.6. 90% confidence intervals.

	Posterior				Percentile Intervals	BF Intervals
	Means	Medians	Modes	SD		
	(a) Medium-Scale New Keynesian Model With Agnostic Priors					
	Rigidity Parameters					
ζ_p	0.692	0.693	0.695	0.047	[0.611, 0.767]	[0.512, 0.833]
ζ_w	0.222	0.217	0.164	0.073	[0.113, 0.350]	[0.035, 0.504]
	Other Endogenous Propagation Parameters					
ν_l	1.776	1.703	1.388	0.568	[0.983, 2.807]	[0.385, 4.862]
ψ_1	2.382	2.375	2.353	0.247	[1.993, 2.813]	[1.555, 3.314]
ψ_2	0.075	0.074	0.074	0.024	[0.038, 0.117]	[0.011, 0.190]
ρ_r	0.723	0.724	0.715	0.036	[0.660, 0.780]	[0.573, 0.828]
ι_p	0.105	0.077	0.014	0.096	[0.008, 0.293]	[0.000, 0.673]
ι_w	0.274	0.264	0.281	0.120	[0.097, 0.484]	[0.000, 0.746]
S''	9.106	8.920	8.986	2.016	[6.193, 12.705]	[3.512, 19.591]
h	0.753	0.756	0.762	0.054	[0.662, 0.837]	[0.530, 0.911]
a''	0.241	0.223	0.168	0.111	[0.094, 0.448]	[0.024, 0.750]
	Exogenous Propagation Parameters					
ρ_z	0.229	0.216	0.185	0.120	[0.056, 0.445]	[0.001, 0.709]
ρ_ϕ	0.957	0.960	0.964	0.019	[0.921, 0.984]	[0.869, 0.998]
ρ_{λ_f}	0.940	0.950	0.960	0.038	[0.861, 0.982]	[0.712, 0.998]
ρ_g	0.915	0.915	0.918	0.026	[0.870, 0.956]	[0.795, 0.995]
	(b) Medium-Scale New Keynesian Model With Low-Rigidity Priors					
	Rigidity Parameters					
ζ_p	0.659	0.661	0.695	0.045	[0.581, 0.729]	[0.470, 0.796]
ζ_w	0.266	0.264	0.269	0.057	[0.177, 0.364]	[0.080, 0.515]
	Other Endogenous Propagation Parameters					
ν_l	1.848	1.771	1.389	0.593	[1.014, 2.917]	[0.330, 5.810]
ψ_1	2.416	2.411	2.241	0.248	[2.019, 2.835]	[1.611, 3.622]
ψ_2	0.074	0.072	0.064	0.023	[0.038, 0.116]	[0.012, 0.183]
ρ_r	0.724	0.726	0.714	0.037	[0.660, 0.780]	[0.548, 0.837]
ι_p	0.152	0.127	0.029	0.116	[0.015, 0.379]	[0.000, 0.742]
ι_w	0.271	0.263	0.277	0.113	[0.103, 0.472]	[0.000, 0.812]
S''	8.939	8.769	8.480	2.003	[5.950, 12.501]	[3.086, 19.161]
h	0.741	0.742	0.773	0.055	[0.648, 0.827]	[0.532, 0.927]
a''	0.243	0.227	0.125	0.103	[0.103, 0.436]	[0.032, 0.809]
	Exogenous Propagation Parameters					
ρ_z	0.218	0.208	0.204	0.117	[0.042, 0.426]	[0.000, 0.697]
ρ_ϕ	0.956	0.958	0.954	0.018	[0.924, 0.982]	[0.864, 0.999]
ρ_{λ_f}	0.950	0.955	0.970	0.027	[0.898, 0.984]	[0.775, 1.000]
ρ_g	0.911	0.912	0.925	0.027	[0.866, 0.954]	[0.790, 0.995]

Table C.6. *Continued.*

	Posterior Means	Posterior Medians	Posterior Modes	Posterior SD	Percentile Intervals	BF Intervals
	(c) Medium-Scale New Keynesian Model With High-Rigidity Priors					
	Rigidity Parameters					
ζ_p	0.772	0.770	0.786	0.058	[0.676, 0.855]	[0.626, 0.887]
ζ_w	0.446	0.428	0.391	0.114	[0.286, 0.647]	[0.239, 0.714]
	Other Endogenous Propagation Parameters					
ν_l	1.507	1.508	1.078	0.614	[0.579, 2.408]	[0.305, 3.431]
ψ_1	2.319	2.325	2.302	0.273	[1.829, 2.790]	[1.568, 2.928]
ψ_2	0.055	0.054	0.056	0.021	[0.018, 0.090]	[0.009, 0.112]
ρ_r	0.746	0.751	0.716	0.039	[0.666, 0.794]	[0.625, 0.805]
ι_p	0.089	0.063	0.015	0.083	[0.008, 0.266]	[0.003, 0.440]
ι_w	0.189	0.185	0.239	0.092	[0.036, 0.345]	[0.004, 0.481]
S''	10.195	9.862	9.065	2.285	[6.612, 14.516]	[5.712, 15.943]
h	0.798	0.811	0.825	0.053	[0.692, 0.865]	[0.620, 0.907]
a''	0.208	0.174	0.160	0.118	[0.074, 0.411]	[0.047, 0.669]
	Exogenous Propagation Parameters					
ρ_z	0.247	0.241	0.236	0.128	[0.049, 0.463]	[0.004, 0.602]
ρ_ϕ	0.905	0.921	0.940	0.052	[0.799, 0.968]	[0.710, 0.974]
ρ_{λ_f}	0.863	0.902	0.867	0.100	[0.646, 0.970]	[0.633, 0.996]
ρ_g	0.926	0.926	0.903	0.028	[0.875, 0.969]	[0.847, 0.994]
	(d) Medium-Scale New Keynesian Model: LR Intervals					
	Rigidity Parameters					
ζ_p				[0.543, 0.854]		
ζ_w				[0.017, 0.470]		
	Other Endogenous Propagation Parameters					
ν_l				[0.233, 6.108]		
ψ_1				[1.665, 4.049]		
ψ_2				[0.002, 0.173]		
ρ_r				[0.646, 0.852]		
ι_p				[0.000, 0.259]		
ι_w				[0.001, 0.542]		
S''				[7.527, 25.23]		
h				[0.417, 0.998]		
a''				[0.020, 0.938]		
	Exogenous Propagation Parameters					
ρ_z				[0.002, 0.771]		
ρ_ϕ				[0.892, 0.996]		
ρ_{λ_f}				[0.815, 0.996]		
ρ_g				[0.856, 0.999]		

Reference

Hamilton, J. D. (1994), *Time Series Analysis*. Princeton University Press, Princeton, NJ.

Submitted September, 2012. Final version accepted October, 2012.