

SUPPLEMENT TO “FIRST-PRICE AUCTIONS WITH GENERAL INFORMATION STRUCTURES: IMPLICATIONS FOR BIDDING AND REVENUE”
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APPENDIX B

B.1. *Pure Common Values*

LET US FIRST CONSIDER A MORE GENERAL VERSION of the pure-common-values model that we studied in Section 3, in which the bidders have the same value which is distributed according to $P(v)$. Recall the information structure of Engelbrecht-Wiggans, Milgrom, and Weber (1983), in which one bidder knows the true value and the remaining bidders are uninformed. The corresponding equilibrium has the informed player bid

$$\sigma(v) = \frac{1}{P(v)} \int_{x=v}^v x P(dx),$$

that is, the expected value of the good conditional on it being below its true value. This bidding function ensures that the uninformed bidders must get zero rents in equilibrium, because no matter what they bid, they must pay the expected value conditional on winning. In equilibrium, the uninformed bidders bid independently of one another and independently of the true value so that the marginal distribution of the highest of the $N - 1$ uninformed bids is equal to the marginal distribution of the informed bid.

Let us compare the welfare properties of the equilibrium under the proprietary information structure with our bounds for the family of power distributions with support equal to $[0, 1]$ and the cumulative distribution

$$P(v) = v^\alpha,$$

where $\alpha \geq 0$. For this family of distributions, the informed bidder’s strategy reduces to a deterministic bid of

$$\sigma(v) = \frac{\alpha}{\alpha + 1} v.$$

Given the interpretation of the informed bid, we can immediately conclude that the expected value of the object is

$$\bar{T} = \frac{\alpha}{\alpha + 1}.$$

We can think of the highest of the $N - 1$ uninformed bids as also being of the same form $\sigma(v)$, but for an independent draw of v from the same prior. Thus, the surplus obtained by the informed bidder is

$$U^{\text{EMW}} = \int_{v=0}^1 \left(v - \frac{\alpha}{1 + \alpha} v \right) v^\alpha \alpha v^{\alpha-1} dv = \frac{\alpha}{(\alpha + 1)(2\alpha + 1)}.$$

Given our calculation of total surplus, revenue must be

$$R^{\text{EMW}} = \frac{2\alpha^2}{(\alpha + 1)(2\alpha + 1)}.$$

On the other hand, when $N = 2$, the revenue-minimizing winning-bid function we obtained earlier is

$$\underline{\beta}(v) = \frac{\alpha}{\alpha + 2}v.$$

Minimum revenue is therefore

$$\underline{R} = \frac{\alpha^2}{(\alpha + 2)(\alpha + 1)}$$

and maximum bidder surplus is

$$\overline{U} = \frac{2\alpha}{(\alpha + 2)(\alpha + 1)}.$$

We can now compare the welfare outcome in the equilibrium with the informed bidder with our bounds for the parameterized family of distributions. Note that the ratio of the bidder surplus between these two information structures is

$$\frac{\overline{U}}{U^{\text{EMW}}} = 2 \left(\frac{2\alpha + 1}{\alpha + 2} \right).$$

This quantity is 2 when $\alpha = 1$, which corresponds to our earlier observation in the uniform example that the two bidders collectively earn twice as many rents in the bidder-surplus-maximizing equilibrium as does the informed bidder. As $\alpha \rightarrow 0$, the ratio converges to 1 so that the informed bidder asymptotically attains the lower bound on bidder surplus (which is zero). As $\alpha \rightarrow \infty$, the bidder-surplus ratio converges to 4, meaning that as the distribution of the common value converges weakly to a point mass on $v = 1$, and each bidder individually receives twice as much surplus as the informed bidder in Engelbrecht-Wiggans, Milgrom, and Weber (1983).

B.2. Proof of Proposition 3

PROOF OF PROPOSITION 3: Let us construct the symmetrized winning-bid distributions. Let

$$K : V^N \rightarrow \Delta(B \times \mathcal{N})$$

denote the probability transition kernel associated with the $H_i(b|v)$, that is,

$$K([0, b] \times \{i\}|v) = H_i(b|v).$$

Then we can define a product measure $\phi \in \Delta(V^N \times B \times \mathcal{N})$ according to

$$\phi(X) = \int_{(v,b,i) \in X} K(db, di|v)\mu(dv)$$

for measurable $X \subseteq V^N \times B \times \mathcal{N}$. Now, let us define the mapping

$$f_\xi : V^N \times B \times \mathcal{N} \rightarrow V^N \times B \times \mathcal{N}$$

according to

$$f_{\xi}(v, b, i) = (\xi(v), b, \xi(i)).$$

Then we can define the symmetrized distribution

$$\tilde{\phi} = \frac{1}{N!} \sum_{\xi \in \Xi} \phi \circ f_{\xi}^{-1}.$$

Let us briefly verify that $\tilde{\phi}$ has the symmetrized prior

$$\tilde{\mu} = \frac{1}{N!} \sum_{\xi \in \Xi} \mu \circ \xi^{-1}$$

as a marginal over V^N (which, we remark, is symmetric; cf. the proof of Lemma 3), and that it has the original distribution of winning bids $H(b)$ as its marginal over B . This follows from the observations that for any measurable $X \subseteq V^N$,

$$\phi \circ f_{\xi}^{-1}(X \times B \times \mathcal{N}) = \mu \circ \xi^{-1}(X),$$

and that for any b ,

$$\begin{aligned} \phi \circ f_{\xi}^{-1}(V^N \times [0, b] \times \mathcal{N}) &= \sum_{i=1}^N \int_{v \in V^N} H_{\xi^{-1}(i)}(b | \xi^{-1}(v)) \mu \circ \xi^{-1}(dv) \\ &= \int_{v \in V^N} H(b|v) \mu(dv) = H(b). \end{aligned}$$

We next observe that

$$\xi \circ \alpha^{-1} = \alpha^{-1}$$

for every permutation ξ , where we recall that α denotes the average of the $N - 1$ lowest values. This is because the average and the maximum are invariant to permutations. As a result,

$$\tilde{\mu} \circ \alpha^{-1} = \frac{1}{N!} \sum_{\xi \in \Xi} \mu \circ \xi^{-1} \circ \alpha^{-1} = \frac{1}{N!} \sum_{\xi \in \Xi} \mu \circ \alpha^{-1} = \mu \circ \alpha^{-1}.$$

Thus, the distribution of the average of the $N - 1$ lowest values associated with μ , that is, Q , is exactly the same as that associated with $\tilde{\mu}$.

The final step of the proof is to use $\tilde{\phi}$ to construct winning-bid distributions that are feasible for the relaxed program for the symmetric prior $\tilde{\mu}$. We can disintegrate $\tilde{\phi}$ (Çınlar, (2011, Theorem IV.2.18)) to obtain a probability transition kernel $\tilde{K} : V^N \rightarrow \Delta(B \times \mathcal{N})$ such that

$$\tilde{\phi}(X) = \int_{(v, b, i) \in X} \tilde{K}(db, di|v) \tilde{\mu}(dv).$$

This kernel induces winning-bid distributions

$$\tilde{H}_i(b|v) = \tilde{K}([0, b] \times \{i\}|v).$$

Now, consider the left-hand side of (11):

$$\begin{aligned} & \int_{v \in V^N} (v_i - b) \tilde{H}(b|v) \tilde{\mu}(dv) \\ &= \int_{(v, x, j) \in V^N \times [0, b] \times \mathcal{N}} (v_i - b) \tilde{\phi}(dv, dx, dj) \\ &= \frac{1}{N!} \sum_{\xi \in \Xi} \int_{(v, x, j) \in V^N \times [0, b] \times \mathcal{N}} (v_i - b) (\phi \circ f_\xi^{-1})(dv, dx, dj) \\ &= \frac{1}{N!} \sum_{\xi \in \Xi} \int_{v \in V^N} (v_{\xi(i)} - b) H(b|v) \mu(dv) \\ &= \frac{1}{N} \sum_{j=1}^N \int_{v \in V^N} (v_j - b) H(b|v) \mu(dv). \end{aligned}$$

By a similar sequence of steps, we conclude that the right-hand side of (11) is

$$\int_{v \in V^N} \int_{x=0}^b (v_i - x) \tilde{H}_i(dx|v) \tilde{\mu}(dv) = \frac{1}{N} \sum_{j=1}^N \int_{v \in V^N} \int_{x=0}^b (v_j - x) H_j(dx|v) \mu(dv).$$

Since (11) is satisfied for every $j = 1, \dots, N$ for the measure μ and winning-bid distributions H_i , we conclude that (11) will also be satisfied for the symmetrized prior $\tilde{\mu}$ and winning-bid distributions \tilde{H}_i . Since both induce the same distribution of winning bids, we conclude that the solution to the relaxed program for $\tilde{\mu}$ must be weakly lower than the solution for μ , and we have argued that the solution for $\tilde{\mu}$ is the \underline{H} defined relative to the distribution of the average of the $N - 1$ lowest values for μ . Finally, since any information structure \mathcal{S} and equilibrium σ (under μ) induce a winning-bid distribution $H(\mathcal{S}, \sigma)$ that is feasible for the relaxed program for μ , we conclude that $H(\mathcal{S}, \sigma) \leq \underline{H}$. *Q.E.D.*

B.3. Proof of Theorem 2

PROOF OF THEOREM 2: We will show that our proposed strategies are an equilibrium. This is trivially true when the information structure reveals the entire profile of values, so let us focus on the case where the highest value $v^{(1)}$ is strictly larger than the second-highest value $v^{(2)}$. Suppose that bidder i has the highest value. Then he receives a signal, which we can think of as a recommendation to bid

$$(23) \quad b_i = xv^{(1)} + (1-x)v^{(2)}$$

for some $x \in (0, 1)$. The other bidders are given recommendations:

$$(24) \quad b_j = y_j v^{(1)} + (1 - y_j) v^{(2)},$$

where the y_j are independent random variables in $[0, x]$ drawn from the cumulative distribution

$$(25) \quad F(y|x) = \left(\frac{y}{1-y} \frac{1-x}{x} \right)^{1/(N-1)}.$$

We claim that the truthful bidding strategies in which bidders follow their recommendations b_i are an equilibrium for this information structure, even conditional on $x \in (0, 1)$ and conditional on the realized profile of values, v_1, \dots, v_N . Now, if the bid b is a recommendation for a low-value bidder with value v_i , then it is never profitable to deviate to a higher bid since, by construction, $b \geq v_i$. Similarly, lowering the bid below b is not profitable either, as it will not change the outcome of the auction. Next, if the bid b is a recommendation for the high-value bidder i , then $b < v_i = v^{(1)}$ and a bid increase is not profitable, as it does not change the outcome but rather leads to higher sale price. It remains to verify that the winning bidder has no incentive to lower his bid. Given the equilibrium bid, the payoff for the winning bidder is

$$v^{(1)} - b = (v^{(1)} - v^{(2)})(1 - x).$$

By deviating to a lower bid b' , the deviator will win whenever the realized y_j 's are all below a critical level defined by $b' = yv^{(1)} + (1-y)v^{(2)}$. Given the distribution of y as defined by (25), the payoff from such a deviation is

$$(v^{(1)} - (yv^{(1)} + (1-y)v^{(2)})) \left(\frac{y}{1-y} \frac{1-x}{x} \right) = (v^{(1)} - v^{(2)})(1-x) \frac{y}{x},$$

which is increasing in y . Thus there is no profitable deviation for the winning bidder either.

While our argument that these strategies are an equilibrium has presumed that x is known, in fact x is not directly observed and it is drawn from a non-atomic and full-support distribution on $[0, 1]$. As a result, conditional on the highest and second-highest values, the distributions of both winning bids and losing bids have support equal to $[v^{(2)}, v^{(1)}]$. Moreover, each bidder has a $1/N$ chance of being the high-value bidder, so that any set of bid recommendations $X \subseteq [v^{(2)}, v^{(1)}]$ for bidder i that has positive probability conditional on $\{v^{(1)}, v^{(2)}\}$ also has positive probability when $v_i = v^{(1)}$, so that the proposed equilibrium strategy is not weakly dominated.

Finally, for each x , the expected winning bid is simply a convex combination of the expected highest and the expected second-highest values, with weights x and $1-x$, respectively. As the distribution of x approaches a Dirac measure on 1, the expected winning bid converges to the expected highest value, and bidder surplus must therefore converge to zero. *Q.E.D.*

B.4. Inefficient Equilibria

We argue that the pure strategies given by

$$(26) \quad \sigma(s) = \frac{1}{P(s)} \int_{x=v}^s x P(dx)$$

constitute an equilibrium. First, consider a bidder i who observes a signal s_i . If bidder i follows the equilibrium strategy and bids $\sigma(s_i)$, then he will win with probability $1/(N-1)$

when he had a high signal, which is when some other bidder had a higher value. But since bidder i 's value is independent of the highest of others' values, the posterior distribution of bidder i 's value on this event is precisely the truncated prior $P(v_i)/P(s_i)$ with support equal to $[\underline{v}, s_i]$. Thus, the expected valuation conditional on winning is precisely $\sigma(s_i)$, and the bidder obtains zero rents in equilibrium.

Now, consider a bidder i who deviates down to some $\sigma(s')$ with $s' < s_i$. We can separately consider the case of $N = 2$ and $N > 2$. In the latter case, there is more than one bidder who sees a signal is equal to the highest value, and therefore in equilibrium there is a tie for the highest bid at $\sigma(s_i)$. Thus, a downward deviator will always lose the auction and obtain zero rents. On the other hand, if $N = 2$, then the bidder wins whenever the other bidder's signal was less than s' . But since the other bidder's signal is equal to v_i , the event where bidder i wins is precisely when v_i is in the range $[\underline{v}, s']$, so that the expectation conditional on winning is $\sigma(s')$, and the deviator still obtains zero rents.

Finally, let us consider a bidder i who deviates up to $\sigma(s')$ with $s' > s_i$. This bidder will now win outright whenever s_i was equal to the maximum valuation. Moreover, when s_i was a losing signal, the bidder will now win whenever others' signals were less than s' . But on this event, others' signals are equal to $v_i = \max v$. Thus, the upward deviator will win whenever $v_i \leq s'$, and again the expected value conditional on winning is precisely $\sigma(s')$, so that the deviator's surplus is still zero.

We observe that the realized value among the winning bidders is exactly given by the average value among the $N - 1$ bidders with the lowest values, or

$$\alpha(v) = \frac{1}{N-1} \left(\sum_{i=1}^N v_i - \max v \right).$$

It follows that the revenue of the seller is given exactly by the expectation over the average value among the $N - 1$ lowest valuations. We have therefore proven the following.

THEOREM 1—Inefficient Equilibrium: *The strategies (26) are an equilibrium for the information structure in which each bidder observes $s_i = \max v_{-i}$. In this equilibrium, revenue and total surplus are both equal to*

$$\int_{w=\underline{v}}^{\bar{v}} w Q(dw),$$

and bidder surplus is zero.

We note that this equilibrium construction can be extended well beyond the independent-values case. In such a generalization, the equilibrium bid would be each bidder's expected value conditional on it being less than the observed maximum of others' values. As long as there is sufficient positive correlation between bidders' values, for example, affiliation, this bidding function will be strictly increasing, and for this more general class, the upward incentive constraints will be satisfied as strict inequalities.

B.5. Reserve Prices

In this section, we analyze the first-price auction with a minimum bid in the case of pure common values. Let P denote the distribution of the value on $V = [\underline{v}, \bar{v}]$, and let $r \in V$

denote the reserve price. The auction is as described in Section 2, except that

$$q_i(b) = \frac{\mathbb{I}_{i \in W(b)}}{|W(b)|},$$

where

$$W(b) = \{i | b_i \geq b_j \forall j \text{ and } b_i \geq r\}.$$

In other words, a bidder only wins if they bid more than the reserve price and have a high bid, and ties are broken uniformly. Let us construct an equilibrium as follows. Let x_i be i.i.d. draws from $F(s) = (P(s))^{1/N}$, correlated with the value so that $v = \max_i x_i$. Bidder i 's signal is $s_i = \max\{\hat{v}, x_i\}$, where \hat{v} solves

$$(27) \quad \frac{1}{P(\hat{v})} \int_{v=\hat{v}}^{\hat{v}} v P(dv) = r.$$

Bidders follow a pure strategy in equilibrium $\underline{\beta}$, defined by $\underline{\beta}(s_i) = 0$ if $s_i = \hat{v}$, and otherwise bid

$$(28) \quad \underline{\beta}(s_i) = \frac{1}{(P(s_i))^{(N-1)/N}} \left(r(P(\hat{v}))^{(N-1)/N} + \int_{v=\hat{v}}^{s_i} \frac{N-1}{N} v \frac{P(dv)}{(P(v))^{1/N}} \right),$$

which we note for future reference is the solution to the differential equation

$$\beta'(s_i) = \frac{N-1}{N} (s_i - \beta(s_i)) \frac{P(ds_i)}{P(s_i)}$$

with the boundary condition $\beta(\hat{v}) = r$.

To verify that this is an equilibrium, first observe that a bidder with signal $s_i = \hat{v}$ who deviates up to $b_i = r$ will win if and only if $s_j = \hat{v}$ for all $j \neq i$, which is when $v < \hat{v}$. The conditional expectation of the value of the good is no more than r , so that the bidder obtains non-positive surplus conditional upon winning. Now consider the surplus from bidding as a type $w > \hat{v}$ when $s_i = \hat{v}$. In this case, surplus is

$$(r - \underline{\beta}(w))(P(\hat{v}))^{(N-1)/N} + \int_{v=\hat{v}}^w (v - \underline{\beta}(w)) \frac{N-1}{N} \frac{P(dv)}{(P(v))^{1/N}}.$$

The marginal change in surplus from an increase in w is therefore

$$(w - \underline{\beta}(w)) \frac{N-1}{N} \frac{P(dw)}{(P(w))^{1/N}} - \underline{\beta}'(w)(P(w))^{(N-1)/N} = 0$$

by definition of the bidding function. Similarly, now consider a bidder with signal $s_i > \hat{v}$ who bids $\underline{\beta}(w)$ for some $w \geq s_i$. Surplus is

$$(s_i - \underline{\beta}(w))(P(s_i))^{(N-1)/N} + \int_{v=s_i}^w (v - \underline{\beta}(w)) \frac{N-1}{N} \frac{P(dv)}{(P(v))^{1/N}},$$

which also has zero derivative. Finally, let us verify that a bidder with signal $s_i > \hat{v}$ does not want to deviate down to a bid $\hat{\beta}(w)$ for $w \leq s_i$. It is straightforward to see that $\underline{\beta}(s_i) \leq s_i$ for all s_i , so that surplus in equilibrium is non-negative. Thus, it is not attractive to deviate for $w = \hat{v}$. Otherwise, surplus from the downward deviation is

$$(s_i - \underline{\beta}(w))(P(w))^{(N-1)/N},$$

which has derivative

$$(s_i - \underline{\beta}(w)) \frac{N-1}{N} \frac{P(dw)}{(P(w))^{1/N}} - \underline{\beta}'(w)(P(w))^{(N-1)/N},$$

which, using the formula for $\underline{\beta}'$, becomes

$$\begin{aligned} & (s_i - \underline{\beta}(w)) \frac{N-1}{N} \frac{P(dw)}{(P(w))^{1/N}} - \frac{N-1}{N} (w - \underline{\beta}(w)) \frac{P(dw)}{P(w)} (P(w))^{(N-1)/N} \\ &= (s_i - w) \frac{N-1}{N} \frac{P(dw)}{(P(w))^{1/N}}, \end{aligned}$$

which is positive. This completes the proof that the construction is an equilibrium, and we note that revenue is simply

$$\underline{R} = \int_{v=\hat{v}}^{\bar{v}} \underline{\beta}(v) P(dv).$$

Now let us prove that this construction attains a lower bound on revenue. As before, we set up a relaxed program:

$$\max H(\hat{b})$$

subject to

$$\int_{v \in V} (v - b) H(b|v) P(dv) \leq \int_{v \in V} \int_{x=r}^b (v - x) H_i(dx|v) P(dv)$$

for all $b \geq r$, where the H_i are conditional distributions over the high bidder and the highest bid given the true value, and $H(b)$ is the aggregate distribution of highest bids. Note that we now distinguish between “highest” and “winning,” since no one wins the good when the winning bid is less than r , and revenue is only

$$R = \int_{b=r}^{\bar{v}} b H(db).$$

The constraint of course represents the uniform upward incentive constraint for deviating up to $b \geq r$. It differs from (11) in that the right-hand side only counts the equilibrium surplus from winning with the highest bid recommendation when it is at least r , although the left-hand side still counts surplus from winning whenever the highest recommendation is less than b .

By similar arguments as those provided in Section 4, we can conclude that it is without loss of generality to look at solutions that are symmetric and monotonic. Thus, there is a deterministic and increasing highest bid $\beta(v)$ as a function of the true value. The incentive constraint thus becomes

$$\int_{v=\underline{v}}^w (v - \beta(w))P(dv) \leq \frac{1}{N} \int_{v=\hat{v}}^w (v - \beta(v))P(dv),$$

where \hat{v} is the critical type at which $\beta(v) > r$ for all $v > \hat{v}$. Thus, uniform deviations up to \hat{v} are not attractive as long as

$$\int_{v=\underline{v}}^{\hat{v}} (v - r)P(dv) \leq 0,$$

and since we wish to minimize β (by making it zero for as many types as possible), it is optimal to set \hat{v} as in the construction, so that (27) holds. Thus, the integral inequality becomes

$$\beta(w) \geq \frac{1}{P(w)} \left(rP(\hat{v}) + \int_{v=\hat{v}}^w \left(\frac{N-1}{N}v + \frac{1}{N}\beta(v) \right) P(dv) \right).$$

Using the same arguments as those employed in the proof of Proposition 1, we can conclude that there is a lowest β that satisfies this integral inequality, which is the unique fixed point given by (28). This completes the proof that the construction attains a lower bound on revenue.

We conclude with a description of minimum bidding in the uniform case, in which $V = [0, 1]$ and $P(v) = v$. In that case, $\hat{v} = 2r$, and

$$\begin{aligned} \underline{\beta}(w) &= \frac{1}{w^{(N-1)/N}} \left(r\hat{v}^{(N-1)/N} + \frac{N-1}{N} \int_{v=\hat{v}}^w v^{(N-1)/N} dv \right) \\ &= \frac{N-1}{2N-1} w + \frac{1}{w^{(N-1)/N}} \left(r(2r)^{(N-1)/N} - \frac{N-1}{2N-1} (2r)^{(2N-1)/N} \right). \end{aligned}$$

Revenue is therefore

$$\underline{R} = \frac{N-1}{2N-1} \frac{1}{2} (1 - 4r^2) + \left(r(2r)^{(N-1)/N} - \frac{N-1}{2N-1} (2r)^{(2N-1)/N} \right) N(1 - (2r)^{1/N}).$$

For the case of $N = 2$, this reduces to

$$\underline{R} = \frac{1}{6} (1 - 4r^2) + \frac{2\sqrt{2}}{3} r^{3/2} (1 - \sqrt{2r}).$$

Differentiating with respect to r , we obtain

$$-\frac{4}{3}r + \sqrt{2}\sqrt{r}(1 - \sqrt{2r}) - \frac{2}{3}r = \sqrt{r}(\sqrt{2} - 4\sqrt{r}),$$

which has zeros at 0 and at $1/8$. We leave it as an exercise for the reader to verify second-order conditions at the positive solution.

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