Econometrica Online Article Corrigendum

COMMENT ON "EFFICIENT RESOURCE ALLOCATION ON THE BASIS OF PRIORITIES"

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This note points out that the proof of Theorem 1, the main theorem, in Ergin (2002) needs two corrections. We provide two counterexamples to Ergin's (2002) proof and show that the theorem holds as it is by providing an alternative proof.

KEYWORDS: Matching, stability, efficiency, acyclicity.

1. INTRODUCTION

FOR ALLOCATING RESOURCES TO AGENTS, many institutions use matching mechanisms involving no monetary transfer. For instance, consider a one-to-many matching problem of school choice where each school has a quota on the number of students and a priority ranking of students, and each student has a preference order over the set of schools and remaining unmatched. Schools' quotas and priority rankings are exogenously fixed and common knowledge among all agents, including a planner, while students' preferences are not known to the planner.[1] In such an environment, the planner matches each student with at most one school, so that the number of students matched with each school is within that school's quota.

For such problems, Ergin (2002) presents a necessary and sufficient condition on priorities and quotas under which there exists a *stable* matching rule which associates a stable matching to each preference profile that is (i) Pareto efficient for students, (ii) group strategyproof, or (iii) consistent across outcomes for problems involving different groups of students and levels of quotas.[2] According to his Theorem 1, a stable matching rule satisfying each of the conditions (i), (ii), and (iii) exists if and only if a profile of priority rankings and quotas of schools satisfies a condition called *acyclicity*.[3] Following this result, many papers use acyclicity as a crucial condition. (See, e.g., Kesten (2006), Haeringer and Klijn (2009), and Kojima (2011).)

However, Ergin's (2002) proof of the theorem needs two corrections. In this note, we provide two counterexamples to the proof and show that the theorem holds as it is by providing an alternative proof that accommodates these counterexamples.

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Date of Original Version: February 2009. This work was done when I was an undergraduate student at the University of Tokyo, under supervision of Michihiro Kandori and Fuhito Kojima. I also thank the referees, Haluk Ergin, Akihiko Matsui, and particularly Yusuke Kasuya for help.

[1] This assumption means that schools are *not* strategic in this problem.

[2] Strictly, a matching rule naturally extended to all smaller matching problems involving a subset of students and fewer quotas of schools is said to be *consistent* if the following condition holds: Assume that a matching is determined and a subgroup of students fill their matched schools' positions before the others. Then, the matching of the remaining students coincides with the one given by the matching rule in the smaller problem involving the remaining students and quotas.

[3] Acyclicity requires that for any preference profile and associated student-optimal stable matching, there is no room for a mutually beneficial exchange of matched schools between any two students that is blocked by another student and either one of the two schools to be exchanged.

 https://doi.org/10.3982/ECTA8740

2. NOTATION AND DEFINITIONS

Let A be a set of types of goods (schools in the above example) and N be a set of agents (students in the above example). $q := (q_a)_{a\in A}$ denotes a vector of quotas where for each $a \in A$, $q_a \geq 1$ represents the number of available goods of type a. A *priority structure* is a profile of linear order over agents $\succeq := (\succeq_a)_{a\in A}$, where for each $a \in A$, $\succeq_a$ ranks agents with respect to their priority for a. For each $a \in A$ and $i \in N$, let $U_a(i) := \{j \in N | j \succeq_a i\}$. Distinct $a, b \in A$ and $i, j, k \in N$ constitute a *cycle* if

(C) Cycle condition: $i \succ_a j \succ_a k \succ_b i$.

(S) Scarcity condition: There are (possibly empty) disjoint sets of agents $N_a, N_b \subset N \setminus \{i, j, k\}$ such that $N_a \subset U_a(j)$, $N_b \subset U_b(i)$, $|N_a| = q_a - 1$, and $|N_b| = q_b - 1$.

If $\succeq$ and q have no cycle, they are called *acyclical*. More generally, distinct $a_0, a_1, \dots, a_{n-1} \in A$ and $j, i_0, i_1, \dots, i_{n-1} \in N$ with $n \geq 2$ constitute a *generalized cycle* if

(C) $i_0 \succ_{a_0} j \succ_{a_0} i_{n-1} \succ_{a_{n-1}} i_{n-2} \succ_{a_{n-2}} \cdots i_2 \succ_{a_2} i_1 \succ_{a_1} i_0$.

(S) There are disjoint sets of agents $N_{a_0}, N_{a_1}, \dots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_1, \dots, i_{n-1}\}$ such that $N_{a_0} \subset U_{a_0}(j)$, $N_{a_1} \subset U_{a_1}(i_0)$, $N_{a_2} \subset U_{a_2}(i_1)$, $\dots$, $N_{a_{n-2}} \subset U_{a_{n-2}}(i_{n-3})$, $N_{a_{n-1}} \subset U_{a_{n-1}}(i_{n-2})$, and for all $l = 0, 1, \dots, n-1$, $|N_{a_l}| = q_l - 1$.

In the definition, we consider n to be the "size" of the generalized cycle. Then, it is clear that the generalized cycle of size 2 is a cycle. Ergin (2002) argues that the following proposition holds and uses this as a critical lemma for proving Theorem 1.

LEMMA—Ergin (2002, p. 2495): *If $\succeq$ and q have a generalized cycle, then they also have a cycle.*

3. COUNTEREXAMPLE TO ERGIN'S PROOF

To prove the Lemma above, suppose that $a_0, a_1, \dots, a_{n-1} \in A$ and $j, i_0, i_1, \dots, i_{n-1} \in N$ constitute the shortest generalized cycle of size $n > 2$ with $N_{a_0}, N_{a_1}, \dots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_1, \dots, i_{n-1}\}$ as in the above definition. In the original proof of the Lemma, Ergin (2002) utilizes the following two claims.

CLAIM 1: *If $i_0 \succ_{a_2} i_2$, then a_1, a_2 and i_0, i_1, i_2 constitute a cycle with N_{a_1} and N_{a_2}, that is, $i_0 \succ_{a_2} i_2 \succ_{a_2} i_1 \succ_{a_1} i_0$, $N_{a_2} \subset U_{a_2}(i_2)$, and so on.*

CLAIM 2: *If $i_2 \succ_{a_2} i_0$, then $a_0, a_2, a_3, \dots, a_{n-1}$ and $j, i_0, i_2, i_3, \dots, i_{n-1}$ constitute a generalized cycle with $N_{a_0}, N_{a_2}, N_{a_3}, \dots, N_{a_{n-1}}$, that is, $i_0 \succ_{a_0} j \succ_{a_0} i_{n-1} \succ_{a_{n-1}} \cdots i_3 \succ_{a_3} i_2 \succ_{a_2} i_0$, $N_{a_2} \subset U_{a_2}(i_0)$, and so on.*

However, both of these claims turn out to be incorrect. We provide two counterexamples, the former to Claim 1 and the latter to Claim 2. For simplicity, consider a case where for $a_0, a_1, a_2 \in A$, $q_{a_0} = q_{a_1} = q_{a_2} = 2$, and suppose that a_0, a_1, a_2 and $j, i_0, i_1, i_2 \in N$ constitute the shortest generalized cycle of size 3 with N_{a_0}, N_{a_1}, and N_{a_2}, that is, $i_0 \succ_{a_0} j \succ_{a_0} i_2 \succ_{a_2} i_1 \succ_{a_1} i_0$ and $N_{a_0}, N_{a_1}, N_{a_2} \in N \setminus \{j, i_0, i_1, i_2\}$ are disjoint sets satisfying $N_{a_0} \subset U_{a_0}(j)$, $N_{a_1} \subset U_{a_1}(i_0)$, $N_{a_2} \subset U_{a_2}(i_1)$ and $|N_{a_0}| = |N_{a_1}| = |N_{a_2}| = 1$. Let $k \in N$ be an agent such that $N_{a_2} = \{k\}$.

COUNTEREXAMPLE TO CLAIM 1: Suppose $i_0 \succ_{a_2} i_2 \succ_{a_2} k \succ_{a_2} i_1$. (Note that $i_0 \succ_{a_2} i_2$, i.e., the assumption of Claim 1, is satisfied.) Then, N_{a_2} does not satisfy that $N_{a_2} \subset U_{a_2}(i_2)$ because $N_{a_2} = \{k\}$, but $i_2 \succ_{a_2} k$, so $N_{a_2} \cap U_{a_2}(i_2) = \emptyset$. Therefore, a_1, a_2 and i_0, i_1, i_2 cannot constitute a cycle with N_{a_1} and N_{a_2}.

COUNTEREXAMPLE TO CLAIM 2: Suppose $i_2 \succ_{a_2} i_0 \succ_{a_2} k \succ_{a_2} i_1$. (Note that $i_2 \succ_{a_2} i_0$, i.e., the assumption of Claim 2, is satisfied.) Then, N_{a_2} does not satisfy that $N_{a_2} \subset U_{a_2}(i_0)$ because $N_{a_2} = \{k\}$, but $i_0 \succ_{a_2} k$, so $N_{a_2} \cap U_{a_2}(i_0) = \emptyset$. Therefore, a_0, a_2 and j, i_0, i_2 cannot constitute a generalized cycle of size 2, that is, a cycle, with N_{a_0} and N_{a_2}.

Note that we do not exclude a possibility that a cycle or a generalized cycle other than those considered in Claim 1 and Claim 2 may exist. In the last section, we provide an alternative proof of the aforementioned Lemma that accommodates these counterexamples.

4. ALTERNATIVE PROOF

Suppose that $\succeq$ and q have a generalized cycle and let the size of the shortest generalized cycle be $n > 2$, that is, $a_0, a_1, \ldots, a_{n-1} \in A$; $j, i_0, i_1, \ldots, i_{n-1} \in N$ and $N_{a_0}, N_{a_1}, \ldots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_1, \ldots, i_{n-1}\}$ constitute the shortest generalized cycle of size $n > 2$.[4]

Case (1-α): If $i_0 \succ_{a_2} i_2$ and, for all $i \in N_{a_2}$, $i \succ_{a_2} i_2$, then $i_0 \succ_{a_2} i_2 \succ_{a_2} i_1 \succ_{a_1} i_0$ and $N_{a_2}, N_{a_1} \subset N \setminus \{i_2, i_1, i_0\}$ are disjoint sets satisfying $N_{a_2} \subset U_{a_2}(i_2)$, $N_{a_1} \subset U_{a_1}(i_0)$, $|N_{a_2}| = q_{a_2} - 1$, and $|N_{a_1}| = q_{a_1} - 1$. Therefore, a_2, a_1 and i_0, i_2, i_1 constitute a cycle, that is, a generalized cycle of size 2, which is a contradiction.

Case (1-β): If $i_0 \succ_{a_2} i_2$ and there exists $i \in N_{a_2}$ such that $i_2 \succ_{a_2} i \succ_{a_2} i_1$, let i_* be the minimum element in N_{a_2} with respect to $\succ_{a_2}$ and $N'_{a_2} = N_{a_2} \cup \{i_2\} \setminus \{i_*\}$. Then, $i_0 \succ_{a_2} i_* \succ_{a_2} i_1 \succ_{a_1} i_0$ and $N_{a_1}, N'_{a_2} \subset N \setminus \{i_1, i_0, i_*\}$ are disjoint sets satisfying $N'_{a_2} \subset U_{a_2}(i_*)$, $N_{a_1} \subset U_{a_1}(i_0)$, $|N'_{a_2}| = q_{a_2} - 1$, and $|N_{a_1}| = q_{a_1} - 1$. Therefore, a_2, a_1 and i_0, i_*, i_1 constitute a cycle, that is, a generalized cycle of size 2, which is a contradiction.

Case (2-α): If $i_2 \succ_{a_2} i_0$ and, for all $i \in N_{a_2}$, $i \succ_{a_2} i_0$, then $i_0 \succ_{a_0} j \succ_{a_0} i_{n-1} \succ_{a_{n-1}} \cdots i_3 \succ_{a_3} i_2 \succ_{a_2} i_0$, and $N_{a_0}, N_{a_2}, \ldots, N_{a_{n-1}} \subset N \setminus \{j, i_0, i_2, \ldots, i_{n-1}\}$ are disjoint sets satisfying $N_{a_0} \subset U_{a_0}(j)$, $N_{a_2} \subset U_{a_2}(i_0)$, $N_{a_3} \subset U_{a_3}(i_2)$, $\ldots$, $N_{a_{n-2}} \subset U_{a_{n-2}}(i_{n-3})$, $N_{a_{n-1}} \subset U_{a_{n-1}}(i_{n-2})$, and for all $l = 0, 2, 3, \ldots, n-1$, $|N_{a_l}| = q_{a_l} - 1$. Therefore, $a_0, a_2, a_3, \ldots, a_{n-1}$ and $j, i_0, i_2, i_3, \ldots, i_{n-1}$ constitute a generalized cycle of size $n-1$, which is a contradiction.

Case (2-β): If $i_2 \succ_{a_2} i_0$ and there exists $i \in N_{a_2}$ such that $i_0 \succ_{a_2} i \succ_{a_2} i_1$, let i_{**} be the minimum element in N_{a_2} with respect to $\succ_{a_2}$ and $N''_{a_2} = N_{a_2} \cup \{i_2\} \setminus \{i_{**}\}$. Then, $i_0 \succ_{a_2} i_{**} \succ_{a_2} i_1 \succ_{a_1} i_0$ and $N_{a_1}, N''_{a_2} \subset N \setminus \{i_1, i_0, i_{**}\}$ are disjoint sets satisfying $N''_{a_2} \subset U_{a_2}(i_{**})$, $N_{a_1} \subset U_{a_1}(i_0)$, $|N''_{a_2}| = q_{a_2} - 1$, and $|N_{a_1}| = q_{a_1} - 1$. Therefore, a_2, a_1 and i_0, i_{**}, i_1 constitute a cycle, that is, a generalized cycle of size 2, which is a contradiction. *Q.E.D.*

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Co-editor Eddie Dekel handled this manuscript.

Manuscript received 31 July, 2009; final version accepted 17 October, 2009.

[4]Virtually, Ergin (2002) only considers two of the four cases, Case (1-α) and Case (2-α), of the alternative proof.