Supplementary Material

Supplement to "Identification and frequency domain quasi-maximum likelihood estimation of linearized dynamic stochastic general equilibrium models": Appendix

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The spectral density matrix $f_\theta(\omega)$ is a Hermitian matrix satisfying $f_\theta(\omega)^* = f_\theta(\omega)$. It is in general not symmetric. The following correspondence is useful for understanding and proving the identification results:

$$f_\theta(\omega) \longleftrightarrow f_\theta(\omega)^R \quad \text{with} \quad f_\theta(\omega)^R = \begin{bmatrix} \operatorname{Re}(f_\theta(\omega)) & \operatorname{Im}(f_\theta(\omega)) \\ -\operatorname{Im}(f_\theta(\omega)) & \operatorname{Re}(f_\theta(\omega)) \end{bmatrix}, \tag{A.1}$$

where Re( ) and Im( ) denote the real and the imaginary parts of a complex matrix, that is, if $C = A + Bi$, then $\operatorname{Re}(C) = A$ and $\operatorname{Im}(C) = B$. Because $f_\theta(\omega)$ is Hermitian, $f_\theta(\omega)^R$ is real and symmetric (see Lemma 3.7.1(v) in Brillinger (2001)). To simplify notation, let

$$R(\omega; \theta) = \operatorname{vec}(f_\theta(\omega)^R).$$

The following lemma is crucial for proving the subsequent results.

Lemma A.1. *We have the identity*

$$\left(\frac{\partial \operatorname{vec}(f_\theta(\omega)')}{\partial \theta'}\right)' \left(\frac{\partial \operatorname{vec}(f_\theta(\omega))}{\partial \theta'}\right) = \frac{1}{2}\left(\frac{\partial R(\omega;\theta)}{\partial \theta'}\right)' \left(\frac{\partial R(\omega;\theta)}{\partial \theta'}\right). \tag{A.2}$$

Proof. The (j, k)th element of the term on the left hand side is equal to

$$\begin{aligned}&\left(\frac{\partial \operatorname{vec}(f_\theta(\omega)')}{\partial \theta_j}\right)' \left(\frac{\partial \operatorname{vec}(f_\theta(\omega))}{\partial \theta_k}\right) \\ &\quad = \operatorname{tr}\left\{\frac{\partial f_\theta(\omega)}{\partial \theta_j} \frac{\partial f_\theta(\omega)}{\partial \theta_k}\right\} = \operatorname{tr}\left\{\operatorname{Re}\left(\frac{\partial f_\theta(\omega)}{\partial \theta_j} \frac{\partial f_\theta(\omega)}{\partial \theta_k}\right)\right\}\end{aligned}$$

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$$= \frac{1}{2}\,\mathrm{tr}\left\{\left(\frac{\partial f_\theta(\omega)}{\partial \theta_j}\frac{\partial f_\theta(\omega)}{\partial \theta_k}\right)^R\right\} = \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial (f_\theta(\omega)^R)}{\partial \theta_j}\frac{\partial (f_\theta(\omega)^R)}{\partial \theta_k}\right\}$$

$$= \frac{1}{2}\left(\frac{\partial\,\mathrm{vec}(f_\theta(\omega)^R)}{\partial \theta_j}\right)'\left\{\frac{\partial\,\mathrm{vec}(f_\theta(\omega)^R)}{\partial \theta_k}\right\},$$

where the first equality is because of the identity $\mathrm{vec}(A')'\,\mathrm{vec}(B) = \mathrm{tr}\,(AB)$ for generic matrices A and B, the second is because $f_\theta(\omega)$ is Hermitian, thus this term is real valued, the third equality is because of the definition (A.1), the fourth is because, for generic complex matrices, if $Z = XY$, then $Z^R = X^R Y^R$ (see Lemma 3.7.1(ii) in Brillinger (2001)), and the fifth is because $f_\theta(\omega)^R$ is real and symmetric. The last term in the display is simply the (j,k)th element of the right hand side term in (A.2). This completes the proof. □

Proof of Theorem 1. Lemma A.1 implies that $G(\theta)$ defined by (9) is real, symmetric, positive semidefinite, and equal to

$$\frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{\partial R(\omega;\theta_0)}{\partial \theta'}\right)'\left(\frac{\partial R(\omega;\theta_0)}{\partial \theta'}\right)d\omega.$$

This allows us to adopt the arguments in Theorem 1 in Rothenberg (1971) to prove the result.

Suppose θ_0 is *not* locally identified. Then there exists an infinite sequence of vectors $\{\theta_k\}_{k=1}^{\infty}$ approaching θ_0 such that, for each k,

$$R(\omega;\theta_0) = R(\omega;\theta_k) \quad \text{for all } \omega \in [-\pi,\pi].$$

For an arbitrary $\omega \in [-\pi,\pi]$, by the mean value theorem and the differentiability of $f_\theta(\omega)$ in θ,

$$0 = R_j(\omega;\theta_k) - R_j(\omega;\theta_0) = \frac{\partial R_j(\omega;\tilde{\theta}(j,\omega))}{\partial \theta'}(\theta_k - \theta_0),$$

where the subscript j denotes the jth element of the vector and $\tilde{\theta}(j,\omega)$ lies between θ_k and θ_0 and in general depends on both ω and j. Let

$$d_k = \frac{\theta_k - \theta_0}{\|\theta_k - \theta_0\|}.$$

Then

$$\frac{\partial R_j(\omega;\tilde{\theta}(j,\omega))}{\partial \theta'}d_k = 0 \quad \text{for every } k.$$

The sequence $\{d_k\}$ is an infinite sequence on the unit sphere and therefore there exists a limit point d (note that d does not depend on j or ω). As $\theta_k \to \theta_0$, d_k approaches d and we have

$$\lim_{k\to\infty}\frac{\partial R_j(\omega;\tilde{\theta}(j,\omega))}{\partial \theta'}d_k = \frac{\partial R_j(\omega;\theta_0)}{\partial \theta'}d = 0,$$

where the convergence result holds because $f_\theta(\omega)$ is continuously differentiable in θ (Assumption 3). Because this holds for an arbitrary j, it holds for the full vector $R(\omega;\theta_0)$. Therefore,

$$\frac{\partial R(\omega;\theta_0)}{\partial \theta'} d = 0,$$

which implies

$$d'\left(\frac{\partial R(\omega;\theta_0)}{\partial \theta'}\right)'\left(\frac{\partial R(\omega;\theta_0)}{\partial \theta'}\right) d = 0.$$

Because the above result holds for an arbitrary $\omega \in [-\pi,\pi]$, it also holds when integrating over $[-\pi,\pi]$. Thus

$$d'\left\{\int_{-\pi}^{\pi}\left(\frac{\partial R(\omega;\theta_0)}{\partial \theta'}\right)'\left(\frac{\partial R(\omega;\theta_0)}{\partial \theta'}\right) d\omega\right\} d = 0.$$

Applying Lemma A.1, because $d \neq 0$, $G(\theta_0)$ is singular.

To show the converse, suppose that $G(\theta)$ has constant rank $\rho < q$ in a neighborhood of θ_0 denoted by $\delta(\theta_0)$. Then consider the characteristic vector $c(\theta)$ associated with one of the zero roots of $G(\theta)$. Because

$$\int_{-\pi}^{\pi}\left(\frac{\partial R(\omega;\theta)}{\partial \theta'}\right)'\left(\frac{\partial R(\omega;\theta)}{\partial \theta'}\right) d\omega \times c(\theta) = 0,$$

we have

$$\int_{-\pi}^{\pi}\left(\frac{\partial R(\omega;\theta)}{\partial \theta'}c(\theta)\right)'\left(\frac{\partial R(\omega;\theta)}{\partial \theta'}c(\theta)\right) d\omega = 0.$$

Since the integrand is continuous in ω and always nonnegative, we must have

$$\left(\frac{\partial R(\omega;\theta)}{\partial \theta'}c(\theta)\right)'\left(\frac{\partial R(\omega;\theta)}{\partial \theta'}c(\theta)\right) = 0$$

for all $\omega \in [-\pi,\pi]$ and all $\theta \in \delta(\theta_0)$. Furthermore, this implies

$$\frac{\partial R(\omega;\theta)}{\partial \theta'}c(\theta) = 0 \tag{A.3}$$

for all $\omega \in [-\pi,\pi]$ and all $\theta \in \delta(\theta_0)$. Because $G(\theta)$ is continuous and has constant rank in $\delta(\theta_0)$, the vector $c(\theta)$ is continuous in $\delta(\theta_0)$. Consider the curve χ defined by the function $\theta(v)$ which solves, for $0 \le v \le \bar{v}$, the differential equation

$$\frac{\partial \theta(v)}{\partial v} = c(\theta),$$
$$\theta(0) = \theta_0.$$

Then

$$\frac{\partial R(\omega;\theta(v))}{\partial v} = \frac{\partial R(\omega;\theta(v))}{\partial \theta(v)'}\frac{\partial \theta(v)}{\partial v} = \frac{\partial R(\omega;\theta(v))}{\partial \theta(v)'}c(\theta) = 0$$

for all $\omega \in [-\pi, \pi]$ and $0 \le v \le \bar{v}$, where the last equality uses (A.3). Thus, $R(\omega;\theta)$ is constant on the curve χ. This implies that $f_\theta(\omega)$ is constant on the same curve and that θ_0 is unidentifiable. This completes the proof. □

Proof of Corollary 1. The statement in the subsequent proof applies to all $\omega \in [-\pi, \pi]$. Using the same argument as in the proof of Lemma A.1, $I(\theta_0)$ can be rewritten as

$$I(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}\right)'\left(\left[f_{\theta_0}(\omega)^R\right]^{-1} \otimes \left[f_{\theta_0}(\omega)^R\right]^{-1}\right)\frac{\partial R(\omega;\theta_0)}{\partial\theta'}\,d\omega. \quad \text{(A.4)}$$

Because spectral density matrices are Hermitian and positive semidefinite, $f_{\theta_0}(\omega)^R$ is real, symmetric, and positive semidefinite (cf. Lemma 3.7.1(vii) in Brillinger (2001)). Furthermore, because here $f_{\theta_0}(\omega)$ has full rank, $f_{\theta_0}(\omega)^R$ is in fact positive definite. Thus, $([f_{\theta_0}(\omega)^R]^{-1} \otimes [f_{\theta_0}(\omega)^R]^{-1})$ is positive definite (cf. Theorem 1 in Magnus and Neudecker (1999, p. 28)).

We now prove $G(\theta_0)$ and $I(\theta_0)$ have the same null space. Since they are both $q \times q$ matrices, the result then implies they have the same rank. First, suppose $c \in \mathbb{R}^q$ and $I(\theta_0)c = 0$. Then $c'I(\theta_0)c = 0$ or, explicitly,

$$\int_{-\pi}^{\pi}\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c\right)'\left(\left[f_{\theta_0}(\omega)^R\right]^{-1} \otimes \left[f_{\theta_0}(\omega)^R\right]^{-1}\right)\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c\right)d\omega = 0.$$

Because the integrand is continuous in ω and always nonnegative, we must have

$$\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c\right)'\left(\left[f_{\theta_0}(\omega)^R\right]^{-1} \otimes \left[f_{\theta_0}(\omega)^R\right]^{-1}\right)\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c\right) = 0.$$

Because $([f_{\theta_0}(\omega)^R]^{-1} \otimes [f_{\theta_0}(\omega)^R]^{-1})$ is positive definite, this implies

$$\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c = 0.$$

Therefore,

$$\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}\right)'\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c\right) = 0$$

and, consequently, $G(\theta_0)c = 0$. Next suppose $c \in \mathbb{R}^q$ and $G(\theta_0)c = 0$. Applying the same argument that leads to (A.3), we have

$$\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c\right) = 0.$$

Then, trivially,

$$\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}\right)'\left(\left[f_{\theta_0}(\omega)^R\right]^{-1} \otimes \left[f_{\theta_0}(\omega)^R\right]^{-1}\right)\left(\frac{\partial R(\omega;\theta_0)}{\partial\theta'}c\right) = 0.$$

Upon integration, we have $I(\theta_0)c = 0$. □

Proof of Theorem 2. Using Lemma A.1 again, $\bar{G}(\bar{\theta})$ can be equivalently represented as

$$\bar{G}(\bar{\theta}) = \frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{\partial R(\omega;\theta)}{\partial\bar{\theta}'}\right)'\left(\frac{\partial R(\omega;\theta)}{\partial\bar{\theta}'}\right)d\omega + \left(\frac{\partial\mu(\bar{\theta})}{\partial\bar{\theta}'}\right)'\frac{\partial\mu(\bar{\theta})}{\partial\bar{\theta}'}$$

with both terms on the right hand side being real, symmetric, and positive semidefinite. Let

$$\bar{R}(\omega;\bar{\theta}) = \begin{bmatrix} R(\omega;\theta) \\ \frac{1}{\sqrt{\pi}}\mu(\bar{\theta}) \end{bmatrix}.$$

Then

$$\bar{G}(\bar{\theta}) = \frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{\partial \bar{R}(\omega;\bar{\theta})}{\partial\bar{\theta}'}\right)'\left(\frac{\partial \bar{R}(\omega;\bar{\theta})}{\partial\bar{\theta}'}\right)d\omega.$$

Using this representation, the proof proceeds in the same way as in Theorem 1, with θ replaced by $\bar{\theta}$ and $R(\omega;\theta)$ replaced by $\bar{R}(\omega;\bar{\theta})$. The detail is omitted. □

Proof of Corollary 3. We only prove the first result, as the second can be proven analogously using the formulation in the proof of Theorem 2.

Suppose the subvector θ_0^s is *not* locally identified. Write $\theta = (\theta^{s\prime}, \theta^{r\prime})'$. There exists an infinite sequence of vectors $\{\theta_k\}_{k=1}^{\infty}$ approaching θ_0 such that

$$R(\omega;\theta_0) = R(\omega;\theta_k) \quad \text{for all } \omega\in[-\pi,\pi] \text{ and each } k.$$

By the definition of partial identification, $\{\theta_k^s\}$ can be chosen such that $\|\theta_k^s - \theta_0^s\|/\|\theta_k - \theta_0\| > \varepsilon$, with ε being some arbitrarily small positive number. The values of θ_k^r can either change or stay fixed in this sequence; no restriction is imposed on them besides those in the preceding display. As in the proof of Theorem 1, in the limit, we have

$$\frac{\partial R(\omega;\theta_0)}{\partial\theta'}d = 0,$$

with $d^s \neq 0$ (where d^s comprises the elements in d that correspond to θ^s). Therefore, on one hand,

$$G(\theta_0)d = 0;$$

on the other hand, because $d^s \neq 0$ and by definition $\partial\theta_0^s/\partial\theta' = [I_{\dim(\theta^s)}, 0_{\dim(\theta^r)}]$, we have

$$\frac{\partial\theta_0^s}{\partial\theta'}d = d^s \neq 0,$$

which implies

$$G^a(\theta_0)d \neq 0.$$

Thus, we have identified a vector that falls into the orthogonal column space of $G(\theta_0)$ but not of $G^a(\theta_0)$. Because the former orthogonal space always includes the latter as a subspace, this result implies $G^a(\theta_0)$ has a higher column rank than $G(\theta_0)$.

To show the converse, suppose that $G(\theta)$ and $G^a(\theta)$ have constant ranks in a neighborhood of θ_0 denoted by $\delta(\theta_0)$. Because the rank of $G(\theta)$ is lower than that of $G^a(\theta)$, there exists a vector $c(\theta)$ such that

$$G(\theta)c(\theta) = 0 \quad \text{but} \quad G^a(\theta)c(\theta) \neq 0,$$

which implies for all $\omega \in [-\pi, \pi]$ and all $\theta \in \delta(\theta_0)$ (cf. arguments leading to (A.3)),

$$\frac{\partial R(\omega;\theta)}{\partial \theta'} c(\theta) = 0,$$

but

$$\begin{bmatrix} \partial R(\omega;\theta)/\partial\theta' \\ \partial\theta^s/\partial\theta' \end{bmatrix} c(\theta) = \begin{bmatrix} 0 \\ c^s(\theta) \end{bmatrix} \neq 0,$$

where $c^s(\theta)$ denotes the elements in $c(\theta)$ that correspond to θ^s. Because $G(\theta)$ is continuous and has constant rank in $\delta(\theta_0)$, the vector $c(\theta)$ is continuous in $\delta(\theta_0)$. As in Theorem 1, consider the curve χ defined by the function $\theta(v)$ which solves, for $0 \le v \le \bar{v}$, the differential equation

$$\frac{\partial\theta(v)}{\partial v} = c(\theta), \qquad \theta(0) = \theta_0.$$

On one hand, because $c^s(\theta) \neq 0$ and $c^s(\theta)$ is continuous in θ, points on this curve correspond to different θ^s. On the other hand,

$$\frac{\partial R(\omega;\theta(v))}{\partial v} = \frac{\partial R(\omega;\theta(v))}{\partial\theta(v)'}\frac{\partial\theta(v)}{\partial v} = \frac{\partial R(\omega;\theta(v))}{\partial\theta(v)'} c(\theta) = 0$$

for all $\omega \in [-\pi, \pi]$ and $0 \le v \le \bar{v}$, implying $f_\theta(\omega)$ is constant on the same curve. Therefore, θ_0^s is not locally identifiable. □

Proof of Corollary 5. The proof is essentially the same as in Rothenberg (1971, Theorem 2) and is included for the sake of completeness. Suppose $\Psi(\theta)$ has rank s for all θ in a neighborhood of θ_0. Then, by the implicit function theorem, there exists a partition of θ into $\theta^1 \in \mathbb{R}^s$ and $\theta^2 \in \mathbb{R}^{q-s}$ such that

$$\theta^1 = q(\theta^2)$$

for all solutions of $\psi(\theta) = 0$ in a neighborhood of θ_0 with θ_0^2 being an interior point of that neighborhood. Consequently, the spectral density can be rewritten as

$$f_\theta(\omega) = f_{q(\theta^2),\theta^2}(\omega),$$

which involves only $q - s$ parameters. Let

$$Q(\theta^2) = \frac{\partial q(\theta^2)}{\partial\theta^{2\prime}} \quad \text{and} \quad \tilde{G}(\theta) = [\,Q(\theta^2)' \quad I\,]\,G(\theta)\begin{bmatrix} Q(\theta^2) \\ I \end{bmatrix}.$$

Then, by Theorem 1, θ_0 is identified if and only if $\tilde{G}(\theta_0)$ has full rank.

Suppose there exists a vector $d \in \mathbb{R}^{q-s}$ such that

$$\tilde{G}(\theta_0)d = 0. \tag{A.5}$$

Then the structure of $G(\theta)$ (cf. Lemma A.1) implies that (A.5) holds if and only if

$$G(\theta_0)\begin{bmatrix} Q(\theta_0^2) \\ I \end{bmatrix} d = 0.$$

Let

$$c = \begin{bmatrix} Q(\theta_0^2) \\ I \end{bmatrix} d.$$

Then we have (a) $c \neq 0$ if and only $d \neq 0$, and (b)

$$\begin{bmatrix} G(\theta_0) \\ \Psi(\theta_0) \end{bmatrix} c = 0$$

if and only if (A.5) holds, where $\Psi(\theta_0)c = 0$ always holds because θ_0 satisfies the constraint $\psi(\theta) = 0$. Thus, the preceding matrix has full rank if and only if θ_0 is identified under the constraints. This completes the proof. □

Proof of Corollary 6. Without loss of generality, assume $n_Y = 1$. Otherwise, the proof can be carried out by analyzing $R(\omega; \theta)$. The map $\theta \longmapsto f_\theta$ is infinite dimensional. The proof therefore involves two steps. The first is to reduce it to a finite dimensional problem. The second is to apply a constant rank theorem (a generalization of the implicit function theorem).

Consider a positive integer N and a partition of the interval $[-\pi, \pi]$ by $\omega_j = (2\pi j/2^N) - \pi$, with $j = 0, 1, \dots, 2^N$. Then the map

$$\theta \longmapsto (f_\theta(\omega_0), \dots, f_\theta(\omega_{2^N})) \tag{A.6}$$

is finite dimensional. To simplify notation, let $f_{\theta,N} = (f_\theta(\omega_0), \dots, f_\theta(\omega_{2^N}))'$. Conventionally, the rank of the above map is defined as the rank of the Jacobian matrix $\partial f_{\theta,N}/\partial\theta'$, which is of dimension $(2^N + 1) \times q$ with rank no greater than $q - 1$ at θ_0, because if the rank equals q, then θ_0 becomes locally identified, contradicting the assumption in the corollary. Note that, for a given N, its rank can be strictly less than $q - 1$.

We now show that there exists a finite N such that $\partial f_{\theta,N}/\partial\theta'$ has rank $q - 1$ at θ_0. Suppose such an N does not exist. Then the rank of $\partial f_{\theta,N}/\partial\theta'$ is at most $q - 2$ for arbitrarily large N. This implies that the rank of

$$G_N(\theta_0) = \frac{2\pi}{2^N + 1}\sum_{j=0}^{2^N}\left(\frac{\partial f_{\theta_0}(\omega_j)}{\partial\theta'}\right)'\left(\frac{\partial f_{\theta_0}(\omega_j)}{\partial\theta'}\right)$$

is at most $q - 2$ for arbitrarily large N, because vectors orthogonal to $\partial f_{\theta,N}/\partial\theta'$ are also orthogonal to $G_N(\theta)$ by construction. Let $\lambda_{N,j}$ $(j = 1, \dots, q)$ be the eigenvalues of $G_N(\theta_0)$ sorted in an increasing order. Then, for any finite N,

$$\lambda_{N,1} = \lambda_{N,2} = 0.$$

On the other hand, because $G_N(\theta_0) \to G(\theta_0)$, so do its eigenvalues. Thus, for any $\varepsilon > 0$, there exists a finite N such that $|\lambda_2 - \lambda_{N,2}| < \varepsilon$, where λ_2 is the second smallest eigenvalue of $G(\theta_0)$. Choosing $\varepsilon = \lambda_2/2$ leads to

$$|\lambda_{N,2}| > \lambda_2/2.$$

Since $\mathrm{rank}(G(\theta_0)) = q-1$ by assumption, λ_2 is strictly positive. Thus, we reach a contradiction. Because the convergence of $G_N(\theta) \to G(\theta)$ is uniform in an open neighborhood of θ_0, say $\delta(\theta_0)$, the above analysis also implies there exists an N such that $\partial f_{\theta,N}/\partial\theta'$ has constant rank $q-1$ in that neighborhood.

Use such an N and consider again the map $\theta \longmapsto f_{\theta,N}$, which is finite dimensional, is continuously differentiable, and has constant rank $q-1$ in $\delta(\theta_0)$. Define the level set

$$\left\{\theta \in \delta(\theta_0) : f_{\theta,N} = f_{\theta_0,N}\right\}.$$

Then the rank theorem (Krantz and Parks (2002, Theorem 3.5.1 and the discussion on p. 56)) implies that the level set constitutes a smooth, parameterized one dimensional manifold. Thus, there exists a unique level curve passing through θ_0 satisfying $f_{\theta,N} = f_{\theta_0,N}$.

Therefore, we have established the result for a particular finite N. Further increasing N leads to finer partitions of $[-\pi, \pi]$. This cannot decrease the rank of the map (A.6) and therefore cannot increase the number of level curves passing through θ_0. Thus, in the limit, there is at most one level curve passing through θ_0. The existence of such a curve for the infinite dimensional case has already been shown in the main text, given by (10). This completes the proof. □

Proof of Lemma 1. Applying Lemma A.3.3 (1) in Hosoya and Taniguchi (1982), for a given $\theta \in \Theta$, we have

$$\operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_{j=1}^{T-1} \mathrm{tr}\{W(\omega_j) f_\theta^{-1}(\omega_j) I_T(\omega_j)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left\{W(\omega) f_\theta^{-1}(\omega) f_{\theta_0}(\omega)\right\} dw.$$

To prove stochastic equicontinuity, consider for any $\theta_1, \theta_2 \in \Theta$,

$$\frac{1}{T}\sum_{j=1}^{T-1} \mathrm{tr}\left\{W(\omega_j)\left(f_{\theta_1}^{-1}(\omega_j) - f_{\theta_2}^{-1}(\omega_j)\right) I_T(\omega_j)\right\}.$$

Apply a first order Taylor expansion

$$\begin{aligned} &\frac{1}{T}\sum_{j=1}^{T-1} \mathrm{tr}\left\{W(\omega_j)\left(f_{\theta_1}^{-1}(\omega_j) - f_{\theta_2}^{-1}(\omega_j)\right) I_T(\omega_j)\right\} \\ &= \frac{1}{T}\sum_{j=1}^{T-1} \frac{\partial\, \mathrm{tr}\{W(\omega_j) f_{\bar\theta}^{-1}(\omega_j) I_T(\omega_j)\}}{\partial\theta'}(\theta_1 - \theta_2) \end{aligned} \tag{A.7}$$

$$= -\frac{1}{T}\sum_{j=1}^{T-1} W(\omega_j)\operatorname{vec}(I_T(\omega_j)')'\{f_{\tilde\theta}^{-1}(\omega_j)'\otimes f_{\tilde\theta}^{-1}(\omega_j)\}$$
$$\times \frac{\partial \operatorname{vec}(f_{\tilde\theta}(\omega_j))}{\partial\theta'}(\theta_1-\theta_2),$$

where $\tilde\theta$ lies between θ_1 and θ_2. The norm of (A.7) is bounded by

$$\frac{1}{T}\sum_{j=1}^{T-1}\|\operatorname{vec}(I_T(\omega_j)')\|\left\|\{f_{\tilde\theta}^{-1}(\omega_j)'\otimes f_{\tilde\theta}^{-1}(\omega_j)\}\frac{\partial \operatorname{vec}(f_{\tilde\theta}(\omega_j))}{\partial\theta'}\right\|\|\theta_1-\theta_2\|.$$

The quantity

$$\left\|(f_{\tilde\theta}^{-1}(\omega_j)'\otimes f_{\tilde\theta}^{-1}(\omega_j))\frac{\partial \operatorname{vec}(f_{\tilde\theta}(\omega_j))}{\partial\theta'}\right\|$$

is uniformly bounded by Assumption 5(ii). The term $T^{-1}\sum_{j=1}^{T-1}\|\operatorname{vec}(I_T(\omega_j)')\|$ only depends on θ_0 and is $O_p(1)$ because the diagonal elements of $T^{-1}\sum_{j=1}^{T-1} I_T(\omega_j)$ are positive and satisfy a law of large numbers (Hosoya and Taniguchi (1982, Lemma A.3.3 (1))), and the norm of the off-diagonal elements can be bounded by the diagonal elements using the Cauchy–Schwarz inequality. Therefore, the term (A.7) can be made uniformly small by choosing a small $\|\theta_1-\theta_2\|$. Meanwhile,

$$\frac{1}{T}\sum_{j=1}^{T-1} W(\omega_j)\log\det f_\theta(\omega_j)\to\frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega)\log\det f_\theta(\omega)\,dw$$

uniformly in $\theta\in\Theta$. Thus, the first result holds.

For the second result, we first show that θ_0 maximizes $L_\infty(\theta)$. Apply the same argument as in Hosoya and Taniguchi (1982, p. 149). For every $\omega\in[-\pi,\pi]$,

$$W(\omega)\big[\log\det f_\theta(\omega)+\operatorname{tr}\{f_\theta^{-1}(\omega)f_{\theta_0}(\omega)\}\big]$$
$$= W(\omega)\log\det f_{\theta_0}(\omega)+W(\omega)\big[\operatorname{tr}\{f_\theta^{-1}(\omega)f_{\theta_0}(\omega)\}-\log\det\{f_\theta^{-1}(\omega)f_{\theta_0}(\omega)\}\big]$$
$$= W(\omega)\log\det f_{\theta_0}(\omega)+W(\omega)\left[\sum_{j=1}^{n_Y}\lambda_j(\omega)-\log\lambda_j(\omega)-1\right]+W(\omega)n_Y,$$

where $\lambda_j(\omega)$ is the jth eigenvalue of $f_\theta^{-1}(\omega)f_{\theta_0}(\omega)$. Because $\lambda_j(\omega)-\log\lambda_j(\omega)-1\ge 0$ and the equality holds if and only if $\lambda_j(\omega)=1$, $j=1,\dots,n_Y$, this implies

$$L_\infty(\theta)\le-\frac{1}{2\pi}\int_{-\pi}^{\pi}W(\omega)\big(\log\det f_{\theta_0}(\omega)+n_Y\big)\,d\omega,$$

which holds with equality if and only if $\lambda_j(\omega)=1$ for all $\omega\in[-\pi,\pi]$ $(j=1,\dots,n_Y)$. However, $\lambda_j(\omega)=1$ $(j=1,\dots,n_Y)$ implies $f_{\theta_0}(\omega)=f_\theta(\omega)$ because the latter are positive definite Hermitian matrices. Hence, θ_0 is a global maximizer.

The above result implies that any other parameter vector, say θ_1, is a maximizer if and only if $f_{\theta_1}(\omega) = f_{\theta_0}(\omega)$ for all $\omega \in [-\pi, \pi]$. Now suppose the parameters are locally identified. Then there are no parameter values close to θ_0 satisfying this equality. Thus, θ_0 is the locally unique maximizer. To see the converse, suppose θ_0 is the locally unique maximizer. Then there cannot be any parameter close to θ_0 satisfying $f_{\theta_0}(\omega) = f_\theta(\omega)$ for all ω. Thus, by definition, we have local identification. The argument to establish the result for the global identification proceeds in the same way.

The third result follows directly from the uniform weak law of large numbers. $\square$

Proof of Theorem 3. We only prove the second result, which includes the first as a special case. The first order condition (FOC) gives

$$2\pi T^{-1/2}\sum_{j=0}^{T-1} W(\omega_j)\frac{\partial\,\mathrm{vec}(f_{\hat{\bar\theta}_T}(\omega_j)')'}{\partial\bar\theta}\left\{f_{\hat{\bar\theta}_T}^{-1}(\omega_j)'\otimes f_{\hat{\bar\theta}_T}^{-1}(\omega_j)\right\}\mathrm{vec}\big(I_T(\omega_j)-f_{\hat{\bar\theta}_T}(\omega_j)\big)$$
$$+\,2T^{-1/2}\sum_{t=1}^{T}\frac{\partial\mu(\hat{\bar\theta}_T)'}{\partial\bar\theta}f_{\hat{\bar\theta}_T}^{-1}(0)(Y_t-\mu(\hat{\bar\theta}_T))=0.$$

Note that the first summation starts at $j=0$ and $I_T(0)=I_{\hat{\bar\theta}_T,T}(0)$. The above FOC implies

$$2\pi T^{-1/2}\sum_{j=0}^{T-1} W(\omega_j)\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega_j)')'}{\partial\bar\theta}\big(f_{\theta_0}^{-1}(\omega_j)'\otimes f_{\theta_0}^{-1}(\omega_j)\big)$$
$$\times\,\mathrm{vec}\big(I_T(\omega_j)-f_{\hat{\bar\theta}_T}(\omega_j)\big)$$
$$+\,2T^{-1/2}\sum_{t=1}^{T}\frac{\partial\mu(\bar\theta_0)'}{\partial\bar\theta}f_{\theta_0}^{-1}(0)(Y_t-\mu(\hat{\bar\theta}_T))=o_p(1),$$

which holds because $\hat{\bar\theta}_T\to^p\bar\theta_0$, $f_{\theta_0}(\omega_j)$ and $\mu(\bar\theta_0)$ are continuously differentiable, and $f_{\theta_0}^{-1}(\omega_j)$ have bounded eigenvalues. Apply a first order Taylor expansion around $\bar\theta_0$. Then the left hand side of the preceding display is equal to

$$2\pi T^{-1/2}\sum_{j=0}^{T-1} W(\omega_j)\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega_j)')'}{\partial\bar\theta}\big(f_{\theta_0}^{-1}(\omega_j)'\otimes f_{\theta_0}^{-1}(\omega_j)\big)$$
$$\times\,\mathrm{vec}\big(I_T(\omega_j)-f_{\theta_0}(\omega_j)\big) \qquad \text{(I)}$$
$$+\,2T^{-1/2}\sum_{t=1}^{T}\frac{\partial\mu(\bar\theta_0)'}{\partial\bar\theta}f_{\theta_0}^{-1}(0)(Y_t-\mu(\bar\theta_0)) \qquad \text{(II)}$$
$$-\,2\pi T^{-1}\sum_{j=0}^{T-1} W(\omega_j)\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega_j)')'}{\partial\bar\theta}\big(f_{\theta_0}^{-1}(\omega_j)'\otimes f_{\theta_0}^{-1}(\omega_j)\big) \qquad \text{(A.8)}$$
$$\times\,\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega_j))}{\partial\bar\theta'}T^{1/2}(\hat{\bar\theta}-\bar\theta_0) \qquad \text{(III)}$$

$$- 2\frac{\partial \mu(\bar{\theta}_0)'}{\partial \bar{\theta}} f_{\theta_0}^{-1}(0) \frac{\partial \mu(\bar{\theta}_0)}{\partial \bar{\theta}'} T^{1/2}(\hat{\bar{\theta}} - \bar{\theta}_0) \qquad \text{(IV)}$$
$$+ o_p(1).$$

First consider term (III). The quantity in front of $T^{1/2}(\hat{\bar{\theta}} - \bar{\theta}_0)$ converges to

$$\int_{-\pi}^{\pi} W(\omega) \frac{\partial \operatorname{vec}(f_{\theta_0}(\omega)')'}{\partial \bar{\theta}} \big(f_{\theta_0}^{-1}(\omega)' \otimes f_{\theta_0}^{-1}(\omega)\big) \frac{\partial \operatorname{vec}(f_{\theta_0}(\omega))}{\partial \bar{\theta}'} \, d\omega,$$

whose (h, k)th element is given by

$$\int_{-\pi}^{\pi} \operatorname{tr}\left\{ W(\omega) f_{\theta_0}(\omega) \frac{\partial f_{\theta_0}^{-1}(\omega)}{\partial \bar{\theta}_h} f_{\theta_0}(\omega) \frac{\partial f_{\theta_0}^{-1}(\omega)}{\partial \bar{\theta}_k} \right\} d\omega.$$

Therefore, the above expansion implies (cf. Theorem 3 for the definition of $\bar{M}$)

$$T^{1/2}(\hat{\bar{\theta}} - \bar{\theta}_0) = \bar{M}^{-1} * \text{(I)} + \bar{M}^{-1} * \text{(II)} + o_p(1).$$

Term (I) satisfies a central limit theorem (CLT), whose covariance matrix has the (h, k)th element given by (see Theorem 3.1 and Proposition 3.1 in Hosoya and Taniguchi (1982); in particular, their formula for U_{jl})

$$\begin{aligned}
&4\pi \int_{-\pi}^{\pi} W(\omega) \operatorname{tr}\left\{ f_{\theta_0}(\omega) \frac{\partial f_{\theta_0}^{-1}(\omega)}{\partial \bar{\theta}_h} f_{\theta_0}(\omega) \frac{\partial f_{\theta_0}^{-1}(\omega)}{\partial \bar{\theta}_k} \right\} d\omega \\
&+ \sum_{a,b,c,d=1}^{n_\epsilon} \kappa_{abcd} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} W(\omega) H^*(\omega) \frac{\partial f_{\theta_0}^{-1}(\omega)}{\partial \bar{\theta}_h} H(\omega) \, d\omega \right]_{ab} \\
&\times \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} W(\omega) H^*(\omega) \frac{\partial f_{\theta_0}^{-1}(\omega)}{\partial \bar{\theta}_k} H(\omega) \, d\omega \right]_{cd}.
\end{aligned}$$

Term (II) also satisfies a CLT, with covariance matrix given by

$$8\pi \frac{\partial \mu(\bar{\theta}_0)'}{\partial \bar{\theta}} f_{\theta_0}^{-1}(0) \frac{\partial \mu(\bar{\theta}_0)}{\partial \bar{\theta}'}.$$

To complete the proof, we only need to verify the covariance matrix between (I) and (II). Let

$$A = \operatorname{Cov}(\text{(I)}, \text{(II)})$$

and consider its (h, k)th element

$$\begin{aligned}
A_{hk} = 4\pi \operatorname{Cov}\Bigg\{ &\operatorname{tr}\left(\frac{1}{\sqrt{T}} \sum_{j=0}^{T-1} W(\omega_j) \frac{\partial f_{\theta_0}^{-1}(\omega_j)}{\partial \bar{\theta}_h} \big(I_T(\omega_j) - f_{\theta_0}(\omega)\big) \right), \\
&\left(\frac{1}{\sqrt{T}} \frac{\partial \mu(\bar{\theta}_0)'}{\partial \bar{\theta}_k} f_{\theta_0}^{-1}(0) \sum_{t=1}^{T} (Y_t - \mu(\bar{\theta}_0)) \right) \Bigg\}.
\end{aligned}$$

Define

$$\phi^h(\omega_j) = \frac{\partial f_{\theta_0}^{-1}(\omega_j)}{\partial \bar{\theta}_h} \quad \text{and} \quad \psi^k(0) = \frac{\partial \mu(\bar{\theta}_0)'}{\partial \bar{\theta}_k} f_{\theta_0}^{-1}(0).$$

Then

$$\begin{aligned} A_{hk} &= 4\pi \operatorname{Cov}\Bigg\{ \operatorname{tr}\Bigg(\frac{1}{\sqrt{T}} \sum_{j=0}^{T-1} W(\omega_j)\phi^h(\omega_j)\big(I_T(\omega_j) - f_{\theta_0}(\omega)\big)\Bigg), \\ &\quad \Bigg(\frac{1}{\sqrt{T}} \psi^k(0) \sum_{t=1}^{T} (Y_t - \mu(\bar{\theta}_0))\Bigg)\Bigg\} \\ &= 4\pi \operatorname{Cov}\Bigg\{ \frac{1}{\sqrt{T}} \sum_{j=0}^{T-1} W(\omega_j) \sum_{a,b=1}^{n_Y} \phi^h_{ab}(\omega_j)\big(I_{Tba}(\omega_j) - f_{\theta_0 ba}(\omega)\big), \\ &\quad \frac{1}{\sqrt{T}} \sum_{c=1}^{n_Y} \psi_c^k(0) \sum_{t=1}^{T} (Y_{tc} - \mu_c(\bar{\theta}_0))\Bigg\} \\ &= 4\pi \sum_{a,b,c=1}^{n_Y} \operatorname{Cov}\Bigg\{ \frac{1}{\sqrt{T}} \sum_{j=0}^{T-1} W(\omega_j)\phi^h_{ab}(\omega_j)\big(I_{Tba}(\omega_j) - f_{\theta_0 ba}(\omega)\big), \\ &\quad \frac{1}{\sqrt{T}} \psi_c^k(0) \sum_{t=1}^{T} (Y_{tc} - \mu_c(\bar{\theta}_0))\Bigg\}, \end{aligned}$$

where $I_{Tba}(\omega_j)$ is the (b,a)th element of $I_T(\omega_j)$ and other quantities are defined analogously. Consider the two terms inside the curly brackets separately. Applying the same argument as in Theorem 10.8.5 in Brockwell and Davis (1991), we have

$$\begin{aligned} &\frac{1}{\sqrt{T}} \sum_{j=0}^{T-1} W(\omega_j)\phi^h_{a,b}(\omega_j)\big(I_{Tba}(\omega_j) - f_{\theta_0 ba}(\omega)\big) \\ &= \frac{1}{\sqrt{T}} \sum_{j=0}^{T-1} \sum_{f,g=1}^{n_\epsilon} W(\omega_j)\phi^h_{ab}(\omega_j) H_{bf}(\omega_j)(I^\epsilon_{Tfg}(\omega_j) - EI^\epsilon_{Tfg}(\omega_j)) \\ &\quad \times H^*_{ga}(\omega_j) + o_p(1), \end{aligned}$$

where and $I^\epsilon_{Tfg}(\omega_j)$ denote the (f,g)th element of the periodogram of ϵ_t. Applying Theorem 10.3.1 in Brockwell and Davis (1991), we have

$$\frac{1}{\sqrt{T}} \psi_c^k(0) \sum_{t=1}^{T} (Y_{tc} - \mu_c(\bar{\theta}_0)) = \frac{1}{\sqrt{T}} \sum_{l=1}^{n_\epsilon} \sum_{t=1}^{T} \psi_c^k(0) H_{cl}(0)\epsilon_{tl} + o_p(1),$$

where $H(0)=\sum_{j=0}^{\infty} h_j(\theta_0)$ (cf. (3)). Therefore, their covariance is equal to

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{T-1}\sum_{f,g,l=1}^{n_\epsilon} W(\omega_j)\phi_{ab}^{h}(\omega_j)H_{bf}(\omega_j)H_{ga}^{*}(\omega_j)\psi_c^k(0)H_{cl}(0)\\
&\quad\times E\{(I_{Tfg}^{\epsilon}(\omega_j)-EI_{Tfg}^{\epsilon}(\omega_j))\epsilon_{tl}\}+o_p(1)\\
&=\frac{1}{T}\sum_{t=1}^{T}\sum_{f,g,l=1}^{n_\epsilon} W(\omega_j)\phi_{ab}^{h}(\omega_j)H_{bf}(\omega_j)H_{ga}^{*}(\omega_j)\psi_c^k(0)H_{cl}(0)\xi_{fgl}+o_p(1)\\
&=\frac{1}{2\pi}\sum_{f,g,l=1}^{n_\epsilon}\left\{\int_{-\pi}^{\pi} W(\omega)H^{*}(\omega)_{ga}\phi_{ab}^{h}(\omega)H_{bf}(\omega_j)\,d\omega\right\}\\
&\quad\times \xi_{fgl}\times\{\psi_c^k(0)H_{cl}(0)\}+o_p(1).
\end{aligned}$$

Some algebra shows that

$$\begin{aligned}
A_{hk}&=2\sum_{f,g,l=1}^{n_\epsilon}\left[\int_{-\pi}^{\pi} W(\omega)H(\omega)^{*}\frac{\partial f_{\theta_0}^{-1}(\omega)}{\partial\bar\theta_h}H(\omega)\,d\omega\right]_{gf}\\
&\quad\times \xi_{gfl}\times\left[\frac{\partial\mu(\bar\theta_0)'}{\partial\bar\theta_k}f_{\theta_0}^{-1}(0)H(0)\right]_l.
\end{aligned}$$

□

Proof of Corollary 7. We prove the second result. Because the argument is very similar to Theorem 3 and Taniguchi (1979, Theorem 2), we only provide an outline. The estimate $\widehat{\bar\theta}$ solves

$$\frac{\partial\bar L_T(\widehat{\bar\theta})}{\partial\bar\theta}=0 \tag{A.9}$$

and the pseudo-true value $\bar\theta_0^m$ satisfies

$$\frac{\partial\bar L_\infty^m(\bar\theta_0^m)}{\partial\bar\theta}=0. \tag{A.10}$$

Consider a Taylor expansion of (A.9) around $\bar\theta_0^m$,

$$\frac{\partial\bar L_T(\bar\theta_0^m)}{\partial\bar\theta}+\frac{\partial^2\bar L_T(\widetilde{\bar\theta})}{\partial\bar\theta\,\partial\bar\theta'}(\widehat{\bar\theta}-\bar\theta_0^m)=0,$$

where $\widetilde{\bar\theta}$ lies between $\widehat{\bar\theta}$ and $\bar\theta_0^m$. Rearrange terms and apply (A.10):

$$T^{1/2}(\widehat{\bar\theta}-\bar\theta_0^m)=\left[-\frac{2\pi}{T}\frac{\partial^2\bar L_T(\widetilde{\bar\theta})}{\partial\bar\theta\,\partial\bar\theta'}\right]^{-1}\left(2\pi T^{-1/2}\frac{\partial\bar L_T(\bar\theta_0^m)}{\partial\bar\theta}-2\pi T^{1/2}\frac{\partial\bar L_\infty^m(\bar\theta_0^m)}{\partial\bar\theta}\right).$$

Furthermore,

$$-\frac{2\pi}{T}\frac{\partial^2\bar L_T(\widetilde{\bar\theta})}{\partial\bar\theta\,\partial\bar\theta'}\to\int_{-\pi}^{\pi} W(\omega)\left[\frac{\partial^2}{\partial\bar\theta\,\partial\bar\theta'}\log\det\bigl(f_{\theta_0^m}(\omega)\bigr)+\frac{\partial^2}{\partial\bar\theta\,\partial\bar\theta'}\operatorname{tr}\{f_{\theta_0^m}^{-1}(\omega)f_0(\omega)\}\right]$$

$$+2\frac{\partial\mu(\bar{\theta}_0^m)'}{\partial\bar{\theta}}f_{\bar{\theta}_0^m}^{-1}(0)\frac{\partial\mu(\bar{\theta}_0^m)}{\partial\bar{\theta}'}$$

because $\widetilde{\bar{\theta}} \to^p \bar{\theta}_0^m$ and because of the continuity of the integrand. Also,

$$\begin{aligned}
&2\pi T^{-1/2}\frac{\partial\bar{L}_T(\bar{\theta}_0^m)}{\partial\bar{\theta}} - 2\pi T^{1/2}\frac{\partial\bar{L}_\infty^m(\bar{\theta}_0^m)}{\partial\bar{\theta}} \\
&= -2\pi T^{-1/2}\sum_{j=1}^{T-1}W(\omega_j)\frac{\partial}{\partial\bar{\theta}}\operatorname{tr}\big\{f_{\bar{\theta}_0^m}^{-1}(\omega_j)(I_T(\omega_j)-f_0(\omega))\big\} \\
&\quad + 2T^{-1/2}\sum_{t=1}^{T}\frac{\partial\mu(\bar{\theta}_0^m)'}{\partial\bar{\theta}}f_{\bar{\theta}_0^m}^{-1}(0)(Y_t-\mu_0) + o_p(1) \\
&= (\mathrm{M1}) + (\mathrm{M2}) + o_p(1).
\end{aligned}$$

Terms (M1) and (M2) satisfy a central limit theorem and can be analyzed in the same way as terms (I) and (II) in (A.8). The limiting covariance matrix can be verified accordingly. The detail is omitted. □

Proof of Theorem 4. It suffices to verify that Assumptions 1–4 in Chernozhukov and Hong (2003) hold under our set of conditions. Relabel these assumptions as CH1–CH4. CH1 and CH2 are trivial. CH3 is implied by Lemma 1(i), (ii), and (iv). To verify CH4, applying a second order Taylor expansion of $L_T(\theta)$ around θ_0 (cf. CH4(i)):

$$L_T(\theta) - L_T(\theta_0) = (\theta-\theta_0)'\frac{\partial L_T(\theta_0)}{\partial\theta} + \frac{1}{2}(\theta-\theta_0)'\frac{\partial^2 L_T(\theta_0)}{\partial\theta\,\partial\theta'}(\theta-\theta_0) + R_T(\theta)$$

with

$$R_T(\theta) = (\theta-\theta_0)'\left\{\frac{\partial^2 L_T(\tilde{\theta}_T)}{\partial\theta\,\partial\theta'} - \frac{\partial^2 L_T(\theta_0)}{\partial\theta\,\partial\theta'}\right\}(\theta-\theta_0),$$

where $\tilde{\theta}_T$ lies between θ and θ_0. Now

$$T^{-1/2}\frac{\partial L_T(\theta_0)}{\partial\theta} \to^d N(0,V).$$

Therefore, CH4(ii) is satisfied (V corresponds to Ω_n in CH4). For CH4(iii), note that V is nonrandom and positive definite, and that

$$\begin{aligned}
&-T^{-1}\frac{\partial^2 L_T(\theta_0)}{\partial\theta\,\partial\theta'} \\
&= T^{-1}\sum_{j=1}^{T-1}W(\omega_j)\left(\frac{\partial\operatorname{vec}(f_{\theta_0}(\omega_j)')}{\partial\theta'}\right)'\big\{f_{\theta_0}^{-1}(\omega_j)'\otimes f_{\theta_0}^{-1}(\omega_j)\big\} \\
&\quad\times\left(\frac{\partial\operatorname{vec}(f_{\theta_0}(\omega_j))}{\partial\theta'}\right)
\end{aligned} \tag{A.11}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega_j)\left(\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega)')}{\partial\theta'}\right)'\{f_{\theta_0}^{-1}(\omega)'\otimes f_{\theta_0}^{-1}(\omega)\}$$
$$\times\left(\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega))}{\partial\theta'}\right)d\omega + o(1),$$

where the leading term on the right hand side is nonrandom and positive definite because $f_{\theta_0}^{-1}(\omega)$, and

$$\int_{-\pi}^{\pi} W(\omega_j)\left(\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega)')}{\partial\theta'}\right)'\left(\frac{\partial\,\mathrm{vec}(f_{\theta_0}(\omega))}{\partial\theta'}\right)d\omega$$

are positive definite by Assumption 5 and local identification. It is $O(1)$ because the integrand is bounded; see Assumption 5. Therefore, CH4(iii) is satisfied. CH4(iv.a) holds because

$$|R_T(\theta)| \le \|T^{1/2}(\theta-\theta_0)\|^2\left\|T^{-1}\frac{\partial^2 L_T(\tilde{\theta}_T)}{\partial\theta\,\partial\theta'} - T^{-1}\frac{\partial^2 L_T(\theta_0)}{\partial\theta\,\partial\theta'}\right\|,$$

where the second term can be made arbitrarily small by choosing $\|\theta-\theta_0\|$ small because of (A.11) and the boundedness and continuity of $\partial\,\mathrm{vec}(f_\theta(\omega))/\partial\theta'$ and $f_\theta^{-1}(\omega)$ in θ (Assumptions 3 and 5(ii)). CH4(iv.b) holds because of the preceding argument and the fact that $\|T^{1/2}(\theta-\theta_0)\|^2 = O(1)$.

The proof for $\widehat{\widehat{\theta}}_T$ involves the same argument and is therefore omitted. □

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