Econometrica Supplementary Material

SUPPLEMENT TO "GENERALIZED METHOD OF INTEGRATED MOMENTS FOR HIGH-FREQUENCY DATA"

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This supplement appendix contains proofs of results in the main text.

APPENDIX: PROOFS

A.1. *Notations and Preliminary Results*

IN THIS SUBSECTION, we introduce some notations and describe some technical preliminary results which are repeatedly used in the sequel. We denote the conditional expectation operator $\mathbb{E}[\cdot|\mathcal{F}]$ by $\mathbb{E}_{\mathcal{F}}[\cdot]$; similarly, we use $\mathrm{Cov}_{\mathcal{F}}(\cdot,\cdot)$ to denote the $\mathcal{F}$-conditional covariance. For a random variable ξ and a constant $p \geq 1$, we write $\|\xi\|_p = (\mathbb{E}\|\xi\|^p)^{1/p}$ and $\|\xi\|_{\mathcal{F},p} = (\mathbb{E}_{\mathcal{F}}\|\xi\|^p)^{1/p}$. Recall that $N_n \equiv \lfloor T/\Delta_n \rfloor - k_n$. We write $\sum_i$ for $\sum_{i=0}^{N_n}$. We use K to denote a generic finite positive constant that may vary from line to line; we sometimes write K_u to emphasize its dependence on some constant u. We write "w.p.a.1" for "with probability approaching one." As is typical in this type of problems, by a classical localization argument (Jacod and Protter (2012, Section 4.4.1)), we can replace Assumption 1 with the following assumption without loss of generality.

ASSUMPTION A1: *We have Assumption* 1. *The process* (β_t, Z_t) *takes value in some compact set and the process* V_t *takes value in some convex compact set. Moreover, the processes* b_t, $\tilde{b}_t$, *and* $\tilde{\sigma}_t$ *are bounded and, for some* λ*-integrable function* $J : \mathbb{R} \mapsto \mathbb{R}$, *we have* $|\delta(\omega, t, u)|^r \leq J(u)$ *and* $\|\tilde{\delta}(\omega, t, u)\|^2 \leq J(u)$, *for all* $\omega^{(0)} \in \Omega^{(0)}$, $t \geq 0$, *and* $u \in \mathbb{R}$.

We consider a continuous process X'_t given by

$$X'_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sqrt{V_s}\,dW_s.$$

We then set, for each $i = 0, \dots, N_n$,

$$\widehat{V}'_{i\Delta_n} = \frac{1}{k_n\Delta_n}\sum_{j=1}^{k_n}\left(\Delta^n_{i+j}X'\right)^2, \qquad \tilde{v}_{n,i} = \widehat{V}'_{i\Delta_n} - V_{i\Delta_n}. \tag{A.1}$$

Lemma A1, below, collects some known, but nontrivial, estimates from Jacod and Rosenbaum (2013); see (4.8), (4.11), (4.12), Lemma 4.2, and Lemma 4.3 in that paper.

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LEMMA A1: *Suppose that Assumption* A1 *holds for some* $r \in [0, 1)$. *Let* $u \geq 1$ *be a constant and*

$$\eta_{n,i} \equiv \sqrt{\mathbb{E}\Big[\sup_{i\Delta_n \leq u \leq i\Delta_n + k_n\Delta_n} |b_{i\Delta_n+u} - b_{i\Delta_n}|^2 \Big| \mathcal{F}_{i\Delta_n}\Big]}.$$

Then, for some deterministic sequence $a_n \to 0$, *we have*

$$\text{(A.2)} \qquad \begin{cases} \mathbb{E}\big|\widehat{V}_{i\Delta_n} - \widehat{V}'_{i\Delta_n}\big|^u \leq K_u a_n \Delta_n^{(2u-r)\varpi+1-u}, \\ \mathbb{E}|\tilde{v}_{n,i}|^u \leq K_u\big(k_n^{-u/2} + (k_n\Delta_n)^{(u/2)\wedge 1}\big), \\ \big|\mathbb{E}\big[\big(\Delta_{i+1}^n X'\big)^2 - V_{i\Delta_n}\Delta_n | \mathcal{F}_{i\Delta_n}\big]\big| \leq K\Delta_n^{3/2}\big(\Delta_n^{1/2} + \eta_{n,i}\big), \\ \Delta_n \displaystyle\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \eta_{n,i} = o_p(1), \\ \big|\mathbb{E}\big[\tilde{v}_{n,i}^2 - 2k_n^{-1}V_{i\Delta_n}^2 | \mathcal{F}_{i\Delta_n}\big]\big| \leq K\Delta_n^{1/2}\big(k_n^{-1/2} + k_n\Delta_n^{1/2} + \eta_{n,i}\big). \end{cases}$$

Lemma A2 below shows that $\widehat{V}_{i\Delta_n}$ uniformly (with respect to i) approximates the moving average of spot variance given by

$$\text{(A.3)} \qquad \bar{V}_{i\Delta_n} \equiv \frac{1}{k_n\Delta_n} \int_{i\Delta_n}^{i\Delta_n + k_n\Delta_n} V_s\, ds;$$

note that under Assumption A1, the variables $(\bar{V}_{i\Delta_n})_{0 \leq i \leq N_n}$ are uniformly bounded. This lemma extends a result in Li, Todorov, and Tauchen (2014) to the case with overlapping blocks for spot variance estimation.

LEMMA A2: *Suppose Assumptions* A1 *and* 3. *Then* $\sup_{0\leq i \leq N_n} |\widehat{V}_{i\Delta_n} - \bar{V}_{i\Delta_n}| = o_p(1)$. *Consequently, the variables* $(\widehat{V}_{i\Delta_n})_{0\leq i\leq N_n}$ *are uniformly bounded w.p.a.*1.

PROOF: By Itô's formula,

$$\text{(A.4)} \qquad \widehat{V}'_{i\Delta_n} - \bar{V}_{i\Delta_n} = \frac{2}{k_n\Delta_n} \sum_{j=1}^{k_n} \int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n} \big(X'_s - X'_{(i+j-1)\Delta_n}\big)\big(b_s\, ds + \sqrt{V_s}\, dW_s\big).$$

From here, standard estimates for continuous Itô semimartingales yield, for any $p \geq 1$,

$$\text{(A.5)} \qquad \big\|\widehat{V}'_{i\Delta_n} - \bar{V}_{i\Delta_n}\big\|_p \leq K_p k_n^{-1/2}.$$

By using a maximal inequality and picking $p > 2/\varsigma$, we deduce

$$\text{(A.6)} \qquad \Big\|\sup_{0\leq i\leq N_n} \big|\widehat{V}'_{i\Delta_n} - \bar{V}_{i\Delta_n}\big|\Big\|_p \leq K_p \Delta_n^{-1/p} k_n^{-1/2} \to 0.$$

Next, we note that

$$\text{(A.7)}\qquad \mathbb{E}\Big[\sup_{0\le i\le N_n}\big|\widehat{V}_{i\Delta_n}-\widehat{V}'_{i\Delta_n}\big|\Big]\le\frac{1}{k_n\Delta_n}\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\big|\big(\Delta_i^n X\big)^2 1_{\{|\Delta_i^n X|\le u_n\}}-\big(\Delta_i^n X'\big)^2\big|$$
$$=o\big(\Delta_n^{(2-r)\varpi-(1-\varsigma)}\big)\to 0,$$

where the second line is by Lemma 13.2.6 of Jacod and Protter (2012) and the assumption that $\varpi\ge(1-\varsigma)/(2-r)$.

The assertions of the lemma then follow from (A.6) and (A.7). *Q.E.D.*

A.2. *Proof of Theorem 1*

We need two lemmas. Lemma A3 is used to combine stable convergence and convergence in conditional law (see Definition A1 below); it generalizes Proposition 5 of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008) to a functional setting. Lemma A4 generalizes Theorem 3.2 in Jacod and Rosenbaum (2013) in two ways: it concerns a functional setting and it does not require the test function to have polynomial growth in the spot variance. Without further mention, we remark that all variables below take values in Polish spaces.

DEFINITION A1—Convergence in Conditional Law: Let ζ_n be a sequence of random variables defined on the space $(\Omega,\mathcal{F}\otimes\mathcal{G},\mathbb{P})$ and let $\mathbb{L}$ be a transition probability from $(\Omega,\mathcal{F}\otimes\{\emptyset,\Omega^{(1)}\})$ to an extension of $(\Omega,\mathcal{F}\otimes\mathcal{G},\mathbb{P})$. We write $\zeta_n\xrightarrow{\mathcal{L}|\mathcal{F}}\mathbb{L}$ if and only if $\mathbb{E}_{\mathcal{F}}[f(\zeta_n)]\xrightarrow{\mathbb{P}}\int f(z)\mathbb{L}(dz)$ for any bounded continuous function f. If a variable ζ defined on the extension has $\mathcal{F}$-conditional law $\mathbb{L}$, we also write $\zeta_n\xrightarrow{\mathcal{L}|\mathcal{F}}\zeta$.

LEMMA A3: *Let ξ_n and ζ_n be two sequences of random variables defined on $(\Omega,\mathcal{F}\otimes\mathcal{G},\mathbb{P})$ and let ξ and ζ be variables defined on an extension of $(\Omega,\mathcal{F}\otimes\mathcal{G},\mathbb{P})$. Suppose that ξ_n is $\mathcal{F}$-measurable, $\xi_n\xrightarrow{\mathcal{L}\text{-}s}\xi$, and $\zeta_n\xrightarrow{\mathcal{L}|\mathcal{F}}\zeta$. Then $(\xi_n,\zeta_n)\xrightarrow{\mathcal{L}\text{-}s}(\tilde{\xi},\tilde{\zeta})$, where $(\tilde{\xi},\tilde{\zeta})$ is defined on an extension of the space $(\Omega,\mathcal{F}\otimes\mathcal{G},\mathbb{P})$ such that $\tilde{\xi}$ and $\tilde{\zeta}$ are $\mathcal{F}$-conditionally independent and $\tilde{\xi}$ (resp. $\tilde{\zeta}$) has the same $\mathcal{F}$-conditional law as ξ (resp. ζ).*

PROOF: Let f and g be bounded continuous functions and let U be an arbitrary $\mathcal{F}$-measurable real-valued bounded random variable. Denote the $\mathcal{F}$-conditional laws of ξ and ζ by $\mathbb{L}_1$ and $\mathbb{L}_2$, respectively. Since ξ_n is $\mathcal{F}$-measurable,

$$\text{(A.8)}\qquad \mathbb{E}\big[Uf(\xi_n)g(\zeta_n)\big]=\mathbb{E}\big[Uf(\xi_n)\mathbb{E}_{\mathcal{F}}\big[g(\zeta_n)\big]\big].$$

By assumption, $\mathbb{E}_{\mathcal{F}}[g(\zeta_n)] \xrightarrow{\mathbb{P}} \int g(z)\mathbb{L}_2(dz)$. Then, by the bounded convergence theorem,

$$\mathbb{E}\big[Uf(\xi_n)\mathbb{E}_{\mathcal{F}}\big[g(\zeta_n)\big]\big] - \mathbb{E}\bigg[Uf(\xi_n)\int g(z)\mathbb{L}_2(dz)\bigg] \to 0. \tag{A.9}$$

Since $\xi_n \xrightarrow{\mathcal{L}\text{-}s} \xi$,

$$\begin{aligned} &\mathbb{E}\bigg[Uf(\xi_n)\int g(z)\mathbb{L}_2(dz)\bigg] \\ &\quad \to \mathbb{E}\bigg[U\bigg(\int f(x)\mathbb{L}_1(dx)\bigg)\bigg(\int g(z)\mathbb{L}_2(dz)\bigg)\bigg]. \end{aligned} \tag{A.10}$$

Combining (A.8), (A.9), and (A.10), we see

$$\mathbb{E}\big[Uf(\xi_n)g(\zeta_n)\big] \to \mathbb{E}\bigg[U\int\int f(x)g(z)\mathbb{L}_1(dx)\mathbb{L}_2(dz)\bigg]. \tag{A.11}$$

Finally, we realize the product transition probability $\mathbb{L}_1 \otimes \mathbb{L}_2$ as the $\mathcal{F}$-conditional law of $(\tilde{\xi}, \tilde{\zeta})$. The assertion of the lemma readily follows. *Q.E.D.*

LEMMA A4: *Let $f : \mathcal{B} \times \mathcal{Z} \times \mathcal{V} \times \Theta \mapsto \mathbb{R}^d$ be a function in $\mathcal{C}^{2,2,3,1}$. For $\theta, \theta' \in \Theta$, let*

$$\begin{cases} F_n(\theta) \equiv \Delta_n \sum_i \bigg(f(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) \\ \qquad\qquad - \dfrac{1}{k_n}\partial_v^2 f(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta)\widehat{V}_{i\Delta_n}^2 \bigg), \\ F(\theta) \equiv \displaystyle\int_0^T f(\beta_s, Z_s, V_s; \theta)\, ds, \\ S_f(\theta, \theta') \equiv 2\displaystyle\int_0^T \partial_v f(\beta_s, Z_s, V_s; \theta)\partial_v f(\beta_s, Z_s, V_s; \theta')^\intercal V_s^2\, ds. \end{cases}$$

Under Assumptions 1 *and* 3, *we have the following*:

(a) $F_n(\theta) \xrightarrow{\mathbb{P}} F(\theta)$ *uniformly in* $\theta \in \Theta$;

(b) *for each* θ, $\Delta_n^{-1/2}(F_n(\theta) - F(\theta)) \xrightarrow{\mathcal{L}\text{-}s} \mathcal{MN}(0, S_f(\theta, \theta))$;

(c) *if Assumption* 5(ii) *holds in addition, then the sequence* $\Delta_n^{-1/2}(F_n(\cdot) - F(\cdot))$ *of processes converges $\mathcal{F}$-stably in law under the uniform metric to a process $\zeta(\cdot)$ which, conditional on $\mathcal{F}$, is centered Gaussian with covariance function* $S_f(\cdot, \cdot)$.

PROOF: *Step 1*. By the standard localization procedure (Jacod and Protter (2012, Section 4.4.1)), we suppose that Assumption A1 holds without loss of generality. In this step, we prove the assertions of the theorem under an additional assumption that $f(\cdot)$ is supported on a compact set. In step 2, we show that this additional assumption can indeed be imposed without loss of generality by a spatial localization argument. For notational simplicity, we set $h(\beta, z, v; \theta) = \partial_v^2 f(\beta, z, v; \theta) v^2$. We also denote $\tilde{Z}_t = (\beta_t, Z_t, V_t)$ and $\alpha_{n,i} = (\Delta_{i+1}^n X')^2 - V_{i\Delta_n}\Delta_n$.

The proof relies on the decomposition

$$\Delta_n^{-1/2}\big(F_n(\theta) - F(\theta)\big) = \sum_{j=1}^{5} R_{j,n}(\theta),$$

where

$$\begin{aligned} R_{1,n}(\theta) \equiv{}& \Delta_n^{1/2} \sum_i \big(f(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) - f\big(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}'_{i\Delta_n}; \theta\big)\big) \\ &- \Delta_n^{1/2} k_n^{-1} \sum_i \big(h(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) - h\big(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}'_{i\Delta_n}; \theta\big)\big), \end{aligned} \tag{A.12}$$

$$\begin{aligned} R_{2,n}(\theta) \equiv{}& \Delta_n^{-1/2} \sum_i \int_{i\Delta_n}^{(i+1)\Delta_n} \big(f(\tilde{Z}_{i\Delta_n}; \theta) - f(\tilde{Z}_s; \theta)\big)\, ds \\ &- \Delta_n^{-1/2} \int_{(N_n+1)\Delta_n}^{T} f(\tilde{Z}_s; \theta)\, ds, \end{aligned} \tag{A.13}$$

$$R_{3,n}(\theta) \equiv \Delta_n^{1/2} \sum_i \partial_v f(\tilde{Z}_{i\Delta_n}; \theta) k_n^{-1} \sum_{u=1}^{k_n} (V_{(i+u-1)\Delta_n} - V_{i\Delta_n}), \tag{A.14}$$

$$\begin{aligned} R_{4,n}(\theta) \equiv{}& \Delta_n^{1/2} \sum_i \big(f(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n} + \tilde{v}_{n,i}; \theta) - f(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n}; \theta) \\ &- \partial_v f(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n}; \theta)\tilde{v}_{n,i} - k_n^{-1} h\big(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}'_{i\Delta_n}; \theta\big)\big), \end{aligned} \tag{A.15}$$

$$R_{5,n}(\theta) \equiv \Delta_n^{-1/2} k_n^{-1} \sum_i \left(\partial_v f(\tilde{Z}_{i\Delta_n}; \theta) \sum_{u=1}^{k_n} \alpha_{n,i+u-1}\right). \tag{A.16}$$

The assertions of the lemma follow from the following claims for $R_{j,n}(\cdot)$, $1 \le j \le 5$:

$$\begin{cases} \sup_{\theta\in\Theta} \big\|R_{j,n}(\theta)\big\| = o_p(1), \quad \text{for } j = 1, 2, \\ \sup_{\theta\in\Theta} \big\|\Delta_n^{1/2} R_{j,n}(\theta)\big\| = o_p(1) \text{ and } R_{j,n}(\theta) = o_p(1), \quad \text{for } j = 3, 4, \\ \sup_{\theta\in\Theta} \big\|R_{j,n}(\theta)\big\| = o_p(1), \quad \text{for } j = 3, 4, \text{ under Assumption 5(ii)}, \\ R_{5,n}(\cdot) \xrightarrow{\mathcal{L}\text{-}s} \zeta(\cdot). \end{cases} \tag{A.17}$$

We now show these claims.

Case $j=1$: Since $f(\cdot)$ is compactly supported, so is $h(\cdot)$. Hence, $\partial_v f(\cdot)$ and $\partial_v h(\cdot)$ are uniformly bounded. By a mean value expansion,

$$\sup_{\theta\in\Theta}\|R_{1,n}(\theta)\| \leq K\Delta_n^{1/2}\sum_i\big|\widehat{V}_{i\Delta_n}-\widehat{V}'_{i\Delta_n}\big|. \tag{A.18}$$

By Lemma A1, the majorant side of (A.18) is $o_p(\Delta_n^{(2-r)\varpi-1/2})$. Note that $\varpi \geq 1/2(2-r)$ under Assumption 3. Hence, (A.18) further implies (A.17) for the case $j=1$.

Case $j=2$: Since $f(\cdot)$ is uniformly bounded, it is easy to see that

$$\sup_{\theta\in\Theta}\left\|\Delta_n^{-1/2}\int_{(N_n+1)\Delta_n}^{T} f(\tilde{Z}_s;\theta)\,ds\right\| \leq Kk_n\Delta_n^{1/2}\to 0. \tag{A.19}$$

Moreover, by a standard estimate (see, e.g., pp. 153–154 in Jacod and Protter (2012)) for the Riemann approximation error of Itô semimartingales, we deduce that for each θ,

$$\Delta_n^{-1/2}\sum_i\int_{i\Delta_n}^{(i+1)\Delta_n}\big(f(\tilde{Z}_{i\Delta_n};\theta)-f(\tilde{Z}_s;\theta)\big)\,ds = o_p(1). \tag{A.20}$$

Next, we verify that the term on the left-hand side of (A.20) is stochastically equicontinuous.

We decompose the left-hand side of (A.20) as $R'_{2,n}(\theta)+R''_{2,n}(\theta)$, where

$$\begin{aligned} R'_{2,n}(\theta) \equiv{}& \Delta_n^{-1/2}\sum_i\int_{i\Delta_n}^{(i+1)\Delta_n}\big(f(\tilde{Z}_{i\Delta_n};\theta)-f(\tilde{Z}_s;\theta)\\ &-\partial_{\tilde{Z}}f(\tilde{Z}_{i\Delta_n};\theta)(\tilde{Z}_{i\Delta_n}-\tilde{Z}_s)\big)\,ds\\ &+\Delta_n^{-1/2}\sum_i\int_{i\Delta_n}^{(i+1)\Delta_n}\partial_{\tilde{Z}}f(\tilde{Z}_{i\Delta_n};\theta)\mathbb{E}[\tilde{Z}_{i\Delta_n}-\tilde{Z}_s|\mathcal{F}_{i\Delta_n}]\,ds, \end{aligned} \tag{A.21}$$

$$\begin{aligned} R''_{2,n}(\theta) \equiv{}& \Delta_n^{-1/2}\sum_i\int_{i\Delta_n}^{(i+1)\Delta_n}\partial_{\tilde{Z}}f(\tilde{Z}_{i\Delta_n};\theta)\\ &\times\big(\tilde{Z}_{i\Delta_n}-\tilde{Z}_s-\mathbb{E}[\tilde{Z}_{i\Delta_n}-\tilde{Z}_s|\mathcal{F}_{i\Delta_n}]\big)\,ds. \end{aligned} \tag{A.22}$$

Note that for $s\in[i\Delta_n,(i+1)\Delta_n]$,

$$\begin{aligned} &\mathbb{E}\Big[\sup_{\theta\in\Theta}\big\|f(\tilde{Z}_{i\Delta_n};\theta)-f(\tilde{Z}_s;\theta)-\partial_{\tilde{Z}}f(\tilde{Z}_{i\Delta_n};\theta)(\tilde{Z}_{i\Delta_n}-\tilde{Z}_s)\big\|\Big]\\ &\quad\leq K\mathbb{E}\|\tilde{Z}_{i\Delta_n}-\tilde{Z}_s\|^2\leq K\Delta_n, \end{aligned} \tag{A.23}$$

and

$$\sup_{\theta\in\Theta}\left\|\partial_{\tilde{Z}} f(\tilde{Z}_{i\Delta_n};\theta)\mathbb{E}[\tilde{Z}_{i\Delta_n}-\tilde{Z}_s|\mathcal{F}_{i\Delta_n}]\right\| \leq K\Delta_n. \tag{A.24}$$

From (A.23) and (A.24), it is easy to see that $\sup_{\theta\in\Theta}\|R'_{2,n}(\theta)\| = o_p(1)$, so $R'_{2,n}(\cdot)$ is stochastically equicontinuous. Next, we observe that $R''_{2,n}(\theta)$ is a sum of martingale differences. By the Burkhölder–Davis–Gundy inequality and the boundedness of the partial derivatives of $f(\cdot)$, we derive, for any $p\geq 1$,

$$\begin{aligned}
&\left\|R''_{2,n}(\theta)-R''_{2,n}(\theta')\right\|_p^p \\
&\quad\leq K_p\Delta_n^{-p/2}\mathbb{E}\left\|\sum_i\left(\int_{i\Delta_n}^{(i+1)\Delta_n}\left(\partial_{\tilde{Z}} f(\tilde{Z}_{i\Delta_n};\theta)-\partial_{\tilde{Z}} f\left(\tilde{Z}_{i\Delta_n};\theta'\right)\right)\right.\right. \\
&\qquad\left.\left.\times\left(\tilde{Z}_{i\Delta_n}-\tilde{Z}_s-\mathbb{E}[\tilde{Z}_{i\Delta_n}-\tilde{Z}_s|\mathcal{F}_{i\Delta_n}]\right)ds\right)^2\right\|^{p/2} \\
&\quad\leq K_p\left\|\theta-\theta'\right\|^p.
\end{aligned}\tag{A.25}$$

In particular, by taking $p>\dim(\theta)$, we deduce that $R''_{2,n}(\cdot)$ is stochastically equicontinuous (see, e.g., Theorem 2.2.4 in van der Vaart and Wellner (1996)). The stochastic equicontinuity of the term in (A.20) then readily follows. Hence, (A.20) holds uniformly in $\theta\in\Theta$. In view of (A.19), we deduce (A.17) for the case $j=2$.

Case $j=3$: We set $\zeta_{3,n,i}\equiv k_n^{-1}\sum_{u=1}^{k_n}(V_{(i+u-1)\Delta_n}-V_{i\Delta_n})$, $\zeta'_{3,n,i}\equiv\mathbb{E}[\zeta_{3,n,i}|\mathcal{F}_{i\Delta_n}]$, and $\zeta''_{3,n,i}\equiv\zeta_{3,n,i}-\zeta'_{3,n,i}$. We then decompose $R_{3,n}(\theta)=R'_{3,n}(\theta)+R''_{3,n}(\theta)$, where

$$\begin{aligned}
R'_{3,n}(\theta)&\equiv\Delta_n^{1/2}\sum_i\partial_v f(\tilde{Z}_{i\Delta_n};\theta)\zeta'_{3,n,i},\\
R''_{3,n}(\theta)&\equiv\Delta_n^{1/2}\sum_i\partial_v f(\tilde{Z}_{i\Delta_n};\theta)\zeta''_{3,n,i}.
\end{aligned}$$

Under Assumption A1, it is easy to see $\mathbb{E}\|\zeta'_{3,n,i}\|\leq Kk_n\Delta_n$. Since $\partial_v f(\cdot)$ is uniformly bounded, we further have $\mathbb{E}[\sup_{\theta\in\Theta}\|R'_{3,n}(\theta)\|]\leq Kk_n\Delta_n^{1/2}\to 0$. Hence,

$$\sup_{\theta\in\Theta}\left\|R'_{3,n}(\theta)\right\|=o_p(1). \tag{A.26}$$

For $R''_{3,n}(\theta)$, we decompose it as

$$\begin{aligned}
&R''_{3,n}(\theta)=\sum_{j=0}^{k_n-1}R''_{3,j,n}(\theta),\\
&R''_{3,j,n}(\theta)\equiv\Delta_n^{1/2}\sum_{\substack{u\geq 0\\ 0\leq j+uk_n\leq N_n}}\partial_v f(\tilde{Z}_{(j+uk_n)\Delta_n};\theta)\zeta''_{3,n,j+uk_n}.
\end{aligned}\tag{A.27}$$

Since $\mathbb{E}[\zeta''_{3,n,i}|\mathcal{F}_{i\Delta_n}] = 0$ and $\zeta''_{3,n,i}$ is $\mathcal{F}_{(i+k_n-1)\Delta_n}$ measurable, each $R''_{3,j,n}(\theta)$ is a sum of martingale differences. Moreover, by a standard estimate for Itô semimartingales, we have, for any $p \geq 2$ and all $u = 1, \ldots, k_n$,

$$\|V_{(i+u-1)\Delta_n} - V_{i\Delta_n}\|_p \leq K_p \bar{a}_{n,p}, \quad \text{where} \tag{A.28}$$
$$\bar{a}_{n,p} \equiv \begin{cases} (k_n\Delta_n)^{1/p} & \text{in general,} \\ (k_n\Delta_n)^{1/2} & \text{when } V_t \text{ is continuous.} \end{cases}$$

From (A.28), we use the triangle inequality to deduce

$$\left\|\zeta''_{3,n,i}\right\|_p \leq K_p \|\zeta_{3,n,i}\|_p \leq K_p \bar{a}_{n,p}. \tag{A.29}$$

Then, by using the martingale structure of $R''_{3,j,n}(\theta)$, we derive $\|R''_{3,j,n}(\theta)\|_2 \leq K\Delta_n^{1/2}$. From here, we see that

$$\left\|R''_{3,n}(\theta)\right\|_2 \leq K k_n \Delta_n^{1/2} \to 0, \tag{A.30}$$

which further implies that, for each θ,

$$R''_{3,n}(\theta) = o_p(1). \tag{A.31}$$

Given (A.26) and (A.31), to show (A.17) for the case $j = 3$, it remains to verify that $\Delta_n^{1/2} R''_{3,n}(\cdot)$ is stochastically equicontinuous and that, under Assumption 5(ii), so is $R''_{3,n}(\cdot)$. Observe that for any $p \geq 2$ and $\theta, \theta' \in \Theta$,

$$\begin{aligned} &\left\|R''_{3,n}(\theta) - R''_{3,n}(\theta')\right\|_p \\ &\quad \leq \sum_{j=0}^{k_n-1} \left\|R''_{3,j,n}(\theta) - R''_{3,j,n}(\theta')\right\|_p \\ &\quad \leq K_p \Delta_n^{1/2} \sum_{j=0}^{k_n-1} \left\{\mathbb{E}\left[(k_n\Delta_n)^{-(p/2-1)} \sum_{\substack{u \geq 0 \\ 0 \leq j+uk_n \leq N_n}} \left\|\theta - \theta'\right\|^p \left\|\zeta''_{3,n,j+uk_n}\right\|^p\right]\right\}^{1/p} \\ &\quad \leq K_p k_n^{1/2} \bar{a}_{n,p} \left\|\theta - \theta'\right\|, \end{aligned} \tag{A.32}$$

where the first inequality is by the triangle inequality; the second inequality is by the Burkhölder–Davis–Gundy inequality, Hölder's inequality, and the fact that the map $\theta \mapsto \partial_v f(\cdot; \theta)$ is Lipschitz; the third inequality is due to (A.29). From here, it is easy to see that for any $p > \dim(\theta)$, $\Delta_n^{1/2}\|R''_{3,n}(\theta) - R''_{3,n}(\theta')\|_p \leq K_p\|\theta - \theta'\|$, which further implies the stochastic equicontinuity of $\Delta_n^{1/2} R''_{3,n}(\cdot)$. Next, under Assumption 5(ii), we can take p such that

$$\begin{cases} \dim(\theta) < p \leq \dfrac{2(1-\varsigma)}{\varsigma} & \text{in general,} \\ \dim(\theta) < p & \text{when } V_t \text{ is continuous.} \end{cases} \tag{A.33}$$

Combining (A.32) and (A.33), we deduce $\|R''_{3,n}(\theta) - R''_{3,n}(\theta')\|_p \leq K_p\|\theta - \theta'\|$ and, hence, $R''_{3,n}(\cdot)$ is stochastically equicontinuous under Assumption 5(ii). The proof of (A.17) for the case $j = 3$ is now complete.

Case $j = 4$: We set

$$
\begin{aligned}
\zeta'_{4,n,i}(\theta) &= \frac{1}{2}\partial_v^2 f(\tilde{Z}_{i\Delta_n};\theta)\big(\tilde{v}_{n,i}^2 - 2k_n^{-1}V_{i\Delta_n}^2\big),\\
\zeta''_{4,n,i}(\theta) &= f(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n} + \tilde{v}_{n,i};\theta) - f(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n};\theta)\\
&\quad - \partial_v f(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n};\theta)\tilde{v}_{n,i} - \frac{1}{2}\partial_v^2 f(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n};\theta)\tilde{v}_{n,i}^2\\
&\quad + k_n^{-1}h(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n};\theta) - k_n^{-1}h\big(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}'_{i\Delta_n};\theta\big).
\end{aligned}
$$

We can then decompose $R_{4,n}(\theta) = R'_{4,n}(\theta) + R''_{4,n}(\theta)$, where

$$
\begin{cases}
R'_{4,n}(\theta) \equiv \Delta_n^{1/2}\sum_i \big(\mathbb{E}\big[\zeta'_{4,n,i}(\theta)|\mathcal{F}_{i\Delta_n}\big] + \zeta''_{4,n,i}(\theta)\big),\\
R''_{4,n}(\theta) \equiv \Delta_n^{1/2}\sum_i \big(\zeta'_{4,n,i}(\theta) - \mathbb{E}\big[\zeta'_{4,n,i}(\theta)|\mathcal{F}_{i\Delta_n}\big]\big).
\end{cases}
$$

By the mean value theorem, we deduce

$$
\text{(A.34)}\qquad \sup_{\theta\in\Theta}\big\|\zeta''_{4,n,i}(\theta)\big\| \leq K|\tilde{v}_{n,i}|^3 + Kk_n^{-1}|\tilde{v}_{n,i}|.
$$

By (A.34) and Lemma A1, we further deduce that

$$
\text{(A.35)}\qquad \sup_{\theta\in\Theta}\left\|\Delta_n^{1/2}\sum_i \zeta''_{4,n,i}(\theta)\right\| = O_p\big(\Delta_n^{-1/2}\big(k_n^{-3/2} + k_n\Delta_n\big)\big) = o_p(1).
$$

In addition, by the boundedness of $\partial_v^2 f(\cdot)$ and Lemma A1, we derive

$$
\begin{aligned}
\text{(A.36)}\qquad \sup_{\theta\in\Theta}\left\|\Delta_n^{1/2}\sum_i \mathbb{E}\big[\zeta'_{4,n,i}(\theta)|\mathcal{F}_{i\Delta_n}\big]\right\| &\leq K\big(k_n^{-1/2} + k_n\Delta_n^{1/2}\big) + K\Delta_n\sum_i \eta_{n,i}\\
&= o_p(1).
\end{aligned}
$$

From (A.35) and (A.36), we derive

$$
\text{(A.37)}\qquad \sup_{\theta\in\Theta}\big\|R'_{4,n}(\theta)\big\| = o_p(1).
$$

Turning to $R''_{4,n}(\theta)$, we first note that for any $p \geq 2$ (recall $\bar{a}_{n,p}$ from (A.28)),

$$
\begin{aligned}
\|\tilde{v}_{n,i}\|_p &\leq K_p\big\|\widehat{V}'_{i\Delta_n} - \bar{V}_{i\Delta_n}\big\|_p + K_p\|\bar{V}_{i\Delta_n} - V_{i\Delta_n}\|_p\\
&\leq K_p\big(k_n^{-1/2} + \bar{a}_{n,p}\big).
\end{aligned}
$$

Hence, $\|\tilde{v}_{n,i}^2\|_p = \|\tilde{v}_{n,i}\|_{2p}^2 \leq K_p(k_n^{-1} + \bar{a}_{n,2p}^2)$. It is then easy to see

$$\begin{aligned}\text{(A.38)}\qquad &\big\|\zeta'_{4,n,i}(\theta) - \mathbb{E}\big[\zeta'_{4,n,i}(\theta)|\mathcal{F}_{i\Delta_n}\big]\big\|_p \\ &\leq K_p\big\|\zeta'_{4,n,i}(\theta)\big\|_p \\ &\leq \begin{cases} K_p\big(k_n^{-1} + (k_n\Delta_n)^{1/p}\big) & \text{in general,} \\ K_p\big(k_n^{-1} + k_n\Delta_n\big) & \text{when } V_t \text{ is continuous.}\end{cases}\end{aligned}$$

Now, we can follow the same steps that are used for analyzing $R''_{3,n}(\theta)$ in the analysis of $R''_{4,n}(\theta)$, while using (A.38) in place of (A.29); we can show that $R''_{4,n}(\theta) = o_p(1)$ for each θ, $\Delta_n^{1/2}R_{4,n}(\theta) = o_p(1)$ uniformly and, under Assumption 5(ii), $R''_{4,n}(\theta) = o_p(1)$ uniformly. Combining these results with (A.37), we finish the proof for (A.17) with $j = 4$.

Case $j = 5$: We now show $R_{5,n}(\cdot) \xrightarrow{\mathcal{L}\text{-}s} \zeta(\cdot)$. We first note that the finite-dimensional convergence follows essentially the same proof as that of Lemma 4.5 in Jacod and Rosenbaum (2013). (To be precise, the only modification needed is to replace the weight $\partial_{lm}g(c_i^n)$ in their definition of $V_t^{n,5}$ by $\partial_v f(\tilde{Z}_{i\Delta_n};\theta)$.) The key here to show that $R_{5,n}(\cdot)$ is stochastically equicontinuous. Let $\alpha'_{n,i} \equiv \mathbb{E}[\alpha_{n,i}|\mathcal{F}_{i\Delta_n}]$ and $\alpha''_{n,i} \equiv \alpha_{n,i} - \alpha'_{n,i}$. We can then decompose

$$\text{(A.39)}\qquad R_{5,n}(\theta) = R'_{5,n}(\theta) + R''_{5,n}(\theta),$$

where

$$\text{(A.40)}\qquad \begin{cases} R'_{5,n}(\theta) \equiv \Delta_n^{-1/2}k_n^{-1}\displaystyle\sum_i\left(\partial_v f(\tilde{Z}_{i\Delta_n};\theta)\sum_{u=1}^{k_n}\alpha'_{n,i+u-1}\right), \\ R''_{5,n}(\theta) \equiv \Delta_n^{-1/2}k_n^{-1}\displaystyle\sum_i\left(\partial_v f(\tilde{Z}_{i\Delta_n};\theta)\sum_{u=1}^{k_n}\alpha''_{n,i+u-1}\right).\end{cases}$$

Note that

$$\begin{aligned}\text{(A.41)}\qquad \sup_{\theta\in\Theta}\big\|R'_{5,n}(\theta)\big\| &\leq K\Delta_n^{-1/2}k_n^{-1}\sum_i\sum_{u=1}^{k_n}\big|\alpha'_{n,i+u-1}\big| \\ &\leq K\Delta_n^{1/2} + K\Delta_n\sum_i \eta_{n,i} \xrightarrow{\mathbb{P}} 0,\end{aligned}$$

where the second inequality and the convergence follow from Lemma A1.

Moreover, by rearranging the summands in (A.40), we can rewrite $R''_{5,n}(\theta)$ as a sum of martingale differences:

$$R''_{5,n}(\theta) = \Delta_n^{-1/2}\sum_{i=0}^{N_n+k_n-1}\left(\frac{1}{k_n}\sum_{j=0\vee(i-N_n)}^{(k_n-1)\wedge i}\partial_v f(\tilde{Z}_{(i-j)\Delta_n};\theta)\right)\alpha''_{n,i}.$$

Note that for any $p \geq 2$, $\|\alpha''_{n,i}\|_p \leq K_p \Delta_n$. By using the Lipschitz continuity of the map $\theta \mapsto \partial_v f(\cdot; \theta)$ and the Burkhölder–Davis–Gundy inequality, we deduce, for any $p > \dim(\theta)$,

$$\left\| R''_{5,n}(\theta) - R''_{5,n}(\theta') \right\|_p \leq K_p \left\| \theta - \theta' \right\|.$$

Hence, $R''_{5,n}(\cdot)$ is stochastically equicontinuous. In view of (A.41), $R_{5,n}(\cdot)$ is also stochastically equicontinuous. The proof of (A.17) for case $j = 5$ is now complete.

Step 2. We now prove the assertions of the lemma with the compact support condition on f relaxed. By Assumption A1, the variables $\{(\beta_{i\Delta_n}, Z_{i\Delta_n}, \bar{V}_{i\Delta_n}) : 0 \leq i \leq N_n\}$ take value in a compact subset $\tilde{\mathcal{K}} \subseteq \mathcal{B} \times \mathcal{Z} \times \mathcal{V}$. Fix some $\varepsilon > 0$ arbitrarily small and denote the ε-enlargement of $\tilde{\mathcal{K}}$ by $\tilde{\mathcal{K}}^\varepsilon$, that is,

$$\tilde{\mathcal{K}}^\varepsilon \equiv \left\{ (\beta, z, v) \in \mathcal{B} \times \mathcal{Z} \times \mathcal{V} : \sup_{(\beta', z', v') \in \tilde{\mathcal{K}}} \left\| (\beta, z, v) - \left(\beta', z', v'\right) \right\| < \varepsilon \right\}.$$

By Lemma A2, we see that $\{(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}) : 0 \leq i \leq N_n\} \subseteq \tilde{\mathcal{K}}^\varepsilon$ w.p.a.1. By the $\mathcal{C}^\infty$ Urysohn's lemma, we can find a $\mathcal{C}^\infty$ function ϕ which takes value 1 on the closure of $\tilde{\mathcal{K}}^\varepsilon$ and 0 on the complement of $\tilde{\mathcal{K}}^{2\varepsilon}$. Let $f^*(\beta, z, v; \theta) = \phi(\beta, z, v) f(\beta, z, v; \theta)$. We observe that f^* is compactly supported. Hence, the assertions of the lemma hold for f^* as shown in step 1.

Finally, we observe that for $j = 0, 1, 2$, we have (i) $\partial_v^j f^*(\beta_t, Z_t, V_t; \cdot) = \partial_v^j f(\beta_t, Z_t, V_t; \cdot)$ for all $t \in [0, T]$; (ii) $\partial_v^j f^*(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \cdot) = \partial_v^j f(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \cdot)$ for all $0 \leq i \leq N_n$ w.p.a.1. Therefore, $F_n(\cdot)$, $F(\cdot)$, and $S_f(\cdot, \cdot)$ coincide w.p.a.1 with those defined for f^*. The assertions of the lemma then readily follow. *Q.E.D.*

PROOF OF THEOREM 1: *Step 1*. We outline the proof in this step. We set

$$h(y, z, v; \theta) = \partial_v^2 g(y, z, v; \theta) v^2, \quad \bar{h}(\beta, z, v; \theta) = \partial_v^2 \bar{g}(\beta, z, v; \theta) v^2, \tag{A.42}$$

where we note that the definition of $\bar{h}(\cdot)$ in (A.42) is consistent with (3.5) because of Assumption 2(iii). The proof relies on the decomposition

$$\Delta_n^{-1/2}\left(G_n(\theta) - G(\theta)\right) = R_{1,n}(\theta) + R_{2,n}(\theta) + R_{3,n}(\theta), \tag{A.43}$$

where

$$R_{1,n}(\theta) \equiv \Delta_n^{-1/2} \Bigg[\Delta_n \sum_i \big(\bar{g}(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) - k_n^{-1} \bar{h}(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) \big) - G(\theta) \Bigg],$$

$$R_{2,n}(\theta) \equiv \Delta_n^{1/2} \sum_i \bigl(g(Y_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) - \bar{g}(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta)\bigr),$$

$$R_{3,n}(\theta) \equiv \Delta_n^{1/2} k_n^{-1} \sum_i \bigl(\bar{h}(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) - h(Y_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta)\bigr).$$

By applying Lemma A4 with $f(\cdot) = \bar{g}(\cdot)$, we have

$$\text{(A.44)} \qquad \begin{cases} \sup_{\theta\in\Theta} \bigl\| \Delta_n^{1/2} R_{1,n}(\theta) \bigr\| = o_p(1), & \\ R_{1,n}(\theta) \xrightarrow{\mathcal{L}\text{-}s} \mathcal{MN}\bigl(0, \bar{S}_g(\theta, \theta)\bigr), & \text{for each } \theta \in \Theta, \\ R_{1,n}(\cdot) \xrightarrow{\mathcal{L}\text{-}s} \zeta_1(\cdot) & \text{under Assumption 5,} \end{cases}$$

where $\zeta_1(\cdot)$ is a centered $\mathcal{F}$-conditional Gaussian process with $\mathcal{F}$-conditional covariance function $\bar{S}_g(\cdot, \cdot)$.

In step 2, we show

$$\text{(A.45)} \qquad \begin{cases} \sup_{\theta\in\Theta} \bigl\| \Delta_n^{1/2} R_{2,n}(\theta) \bigr\| = o_p(1), & \\ R_{2,n}(\theta) \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{MN}\bigl(0, \bar{\Gamma}_g(\theta, \theta)\bigr), & \text{for each } \theta \in \Theta, \\ R_{2,n}(\cdot) \xrightarrow{\mathcal{L}|\mathcal{F}} \zeta_2(\cdot) & \text{under Assumption 5,} \end{cases}$$

where $\zeta_2(\cdot)$ is an $\mathcal{F}$-conditionally centered Gaussian process with conditional covariance function $\bar{\Gamma}_g(\cdot, \cdot)$.

In step 3, we show

$$\text{(A.46)} \qquad \begin{cases} \sup_{\theta\in\Theta} \bigl\| \Delta_n^{1/2} R_{3,n}(\theta) \bigr\| = o_p(1), & \\ R_{3,n}(\theta) = o_p(1), & \text{for each } \theta \in \Theta, \\ \sup_{\theta\in\Theta} \bigl\| R_{3,n}(\theta) \bigr\| = o_p(1) & \text{under Assumption 5.} \end{cases}$$

With an appeal to Lemma A3, the assertion of the theorem then follows from (A.44), (A.45), and (A.46).

Step 2. In this step, we show (A.45). By localization, we can suppose Assumption A1 without loss of generality. Furthermore, in view of Lemma A2, we can restrict attention to the w.p.a.1 event on which $\widehat{V}_{i\Delta_n}$ is uniformly bounded.

By Assumption 2(vi), it is easy to see that $\Delta_n^{1/2} R_{2,n}(\cdot)$ is stochastically equicontinuous. We further show that $R_{2,n}(\cdot)$ is stochastically equicontinuous under the $\mathcal{F}$-conditional probability under Assumption 5(i). This is done by verifying the conditions of Theorem 3 in Hansen (1996) as follows. First observe that, by Assumption 2(iv), for each θ,

$$\limsup_{n\to\infty} \biggl(\Delta_n \sum_i \bigl\| g(Y_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) \bigr\|_{\mathcal{F},k}^2 \biggr)^{1/2} \le K.$$

Second, by Assumption 2(vi),

$$\begin{cases} \left\| g(y,z,v;\theta) - g\big(y,z,v;\theta'\big) \right\| \le B(y,z,v;\theta) \left\| \theta - \theta' \right\|, \\ \limsup_{n\to\infty} \left(\Delta_n \sum_i \left\| B(Y_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}) \right\|_{\mathcal{F},k}^2 \right)^{1/2} \le K, \\ \limsup_{n\to\infty} \left(\Delta_n \sum_i \left\| g(Y_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n};\theta) - g\big(Y_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n};\theta'\big) \right\|_{\mathcal{F},k}^2 \right)^{1/2} \\ \quad \le K \left\| \theta - \theta' \right\|. \end{cases}$$

By Assumption 5(i), $\sum_{j\ge 1} \alpha_{\mathrm{mix}}(j)^{1/k'-1/k} < \infty$, $2 \le k' < k$ and $k' > \dim(\theta)$. Now, we can apply Theorem 3 of Hansen (1996) to deduce the stochastic equicontinuity of $R_{2,n}(\cdot)$ with respect to the Euclidean metric under the $\mathcal{F}$-conditional probability.

We now note that, to show (A.45), it remains to show the finite-dimensional convergence for $R_{2,n}(\cdot) \xrightarrow{\mathcal{L}|\mathcal{F}} \zeta_2(\cdot)$. By the Cramer–Wold device, it suffices to establish the following: for each θ,

$$R_{2,n}(\theta) \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{MN}\big(0, \bar{\Gamma}_g(\theta,\theta)\big), \tag{A.47}$$

for scalar-valued $g(\cdot)$. Since θ is fixed, we suppress it in our notations for simplicity in the remaining part of this step.

The key step is to show that $\widehat{\Gamma}_{g,n}^* \equiv \mathbb{E}_{\mathcal{F}}[R_{2,n}^2] \xrightarrow{\mathbb{P}} \bar{\Gamma}_g$. For notational simplicity, we set

$$\begin{aligned} \hat{z}_{n,i} &= (\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}), \\ \bar{z}_{n,i} &= (\beta_{i\Delta_n}, Z_{i\Delta_n}, \bar{V}_{i\Delta_n}), \\ \tilde{z}_{n,i} &= (\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n}), \\ \xi_i(\beta,z,v) &= g\big(\mathscr{Y}(\beta,\chi_i), z, v\big) - \bar{g}(\beta,z,v). \end{aligned}$$

We can then rewrite

$$R_{2,n} = \Delta_n^{1/2} \sum_i \xi_i(\hat{z}_{n,i}). \tag{A.48}$$

Note that

$$\widehat{\Gamma}_{g,n}^* = \Delta_n \sum_i \mathbb{E}_{\mathcal{F}}\big[\xi_i(\hat{z}_{n,i})^2\big] + 2\Delta_n \sum_{j=1}^{N_n} \sum_{i=j}^{N_n} \mathbb{E}_{\mathcal{F}}\big[\xi_i(\hat{z}_{n,i})\xi_{i-j}(\hat{z}_{n,i-j})\big]. \tag{A.49}$$

We consider an approximation between $\widehat{\Gamma}^*_{g,n}$ and $\bar{\Gamma}_g$ given by

$$\Gamma_{g,n} \equiv \Delta_n \sum_i \mathbb{E}_{\mathcal{F}}\big[\xi_i(\tilde{z}_{n,i})^2\big] + 2\Delta_n \sum_{j=1}^{N_n} \sum_{i=j}^{N_n} \mathbb{E}_{\mathcal{F}}\big[\xi_i(\tilde{z}_{n,i})\xi_{i-j}(\tilde{z}_{n,i})\big].$$

Below, we show that $\widehat{\Gamma}^*_{g,n} - \Gamma_{g,n}$ and $\Gamma_{g,n} - \bar{\Gamma}_g$ are both $o_p(1)$ terms.

We start with $\widehat{\Gamma}^*_{g,n} - \Gamma_{g,n}$. Observe that, for $i, j \geq 0$,

$$\begin{aligned} &\big|\mathbb{E}_{\mathcal{F}}\big[\xi_i(\hat{z}_{n,i})\xi_{i-j}(\hat{z}_{n,i-j}) - \xi_i(\tilde{z}_{n,i})\xi_{i-j}(\tilde{z}_{n,i})\big]\big| \\ &\quad \leq K\alpha_{\mathrm{mix}}(j)^{1-2/k}\big\|\xi_i(\hat{z}_{n,i})\big\|_{\mathcal{F},k}\big\|\xi_{i-j}(\hat{z}_{n,i-j}) - \xi_{i-j}(\tilde{z}_{n,i})\big\|_{\mathcal{F},k} \\ &\qquad + K\alpha_{\mathrm{mix}}(j)^{1-2/k}\big\|\xi_{i-j}(\tilde{z}_{n,i})\big\|_{\mathcal{F},k}\big\|\xi_i(\hat{z}_{n,i}) - \xi_i(\tilde{z}_{n,i})\big\|_{\mathcal{F},k} \\ &\quad \leq K\alpha_{\mathrm{mix}}(j)^{1-2/k}\big(\bar{g}_k(\hat{z}_{n,i})\rho_k(\hat{z}_{n,i-j}, \tilde{z}_{n,i}) + \bar{g}_k(\tilde{z}_{n,i})\rho_k(\hat{z}_{n,i}, \tilde{z}_{n,i})\big), \end{aligned} \tag{A.50}$$

where the first inequality is obtained by using the triangle inequality and then the mixing inequality; the second inequality follows from $\|\xi_i(\cdot)\|_{\mathcal{F},k} \leq K\bar{g}_k(\cdot)$ and (3.6). Note that $\bar{g}_k(\hat{z}_{n,i})$ and $\bar{g}_k(\tilde{z}_{n,i})$ are uniformly (w.r.t. i) bounded and $\rho_k(\hat{z}_{n,i-j}, \tilde{z}_{n,i}) \leq K\|\hat{z}_{n,i-j} - \tilde{z}_{n,i}\|^\kappa$. Therefore, (A.50) further implies

$$\begin{aligned} &\big|\mathbb{E}_{\mathcal{F}}\big[\xi_i(\hat{z}_{n,i})\xi_{i-j}(\hat{z}_{n,i-j}) - \xi_i(\tilde{z}_{n,i})\xi_{i-j}(\tilde{z}_{n,i})\big]\big| \\ &\quad \leq K\alpha_{\mathrm{mix}}(j)^{1-2/k}\big(\|\hat{z}_{n,i-j} - \tilde{z}_{n,i}\|^\kappa + \|\hat{z}_{n,i} - \tilde{z}_{n,i}\|^\kappa\big) \\ &\quad \leq K\alpha_{\mathrm{mix}}(j)^{1-2/k}\Big(\sup_{0\leq i\leq N_n}|\widehat{V}_{i\Delta_n} - \bar{V}_{i\Delta_n}|^\kappa + \bar{A}^\kappa_{n,i,j}\Big), \end{aligned} \tag{A.51}$$

where $\bar{A}_{n,i,j} \equiv \|\bar{z}_{n,i-j} - \tilde{z}_{n,i}\| + \|\bar{z}_{n,i} - \tilde{z}_{n,i}\|$. Note that Assumption 4 implies that $\sum_{j\geq 1}\alpha_{\mathrm{mix}}(j)^{1-2/k} < \infty$. By (A.51) and the triangle inequality, we then derive

$$\begin{aligned} \big|\widehat{\Gamma}^*_{g,n} - \Gamma_{g,n}\big| \leq{}& K\sup_{0\leq i\leq N_n}|\widehat{V}_{i\Delta_n} - \bar{V}_{i\Delta_n}|^\kappa + K\Delta_n\sum_i \bar{A}^\kappa_{n,i,0} \\ &+ K\Delta_n\sum_{j=1}^{N_n}\sum_{i=j}^{N_n}\alpha_{\mathrm{mix}}(j)^{1-2/k}\bar{A}^\kappa_{n,i,j}. \end{aligned} \tag{A.52}$$

It is easy to see that $\mathbb{E}|\bar{A}_{n,i,j}|^2 \leq K(k_n\Delta_n + 1 \wedge j\Delta_n)$, which implies

$$\Delta_n \sum_i \bar{A}^\kappa_{n,i,0} = o_p(1), \tag{A.53}$$

and

$$\mathbb{E}\left|\Delta_n\sum_{j=1}^{N_n}\sum_{i=j}^{N_n}\alpha_{\mathrm{mix}}(j)^{1-2/k}\bar{A}^\kappa_{n,i,j}\right| \leq K\sum_{j=1}^{N_n}\alpha_{\mathrm{mix}}(j)^{1-2/k}(k_n\Delta_n + 1\wedge j\Delta_n)^{\kappa/2}. \tag{A.54}$$

By Kronecker's lemma, we see $\Delta_n^{\kappa/2}\sum_{j=1}^{N_n} j^{\kappa/2}\alpha_{\text{mix}}(j)^{1-2/k} \to 0$. From here, it follows that the right-hand side of (A.54) is $o(1)$ and, hence,

$$\Delta_n \sum_{j=1}^{N_n}\sum_{i=j}^{N_n} \alpha_{\text{mix}}(j)^{1-2/k}\bar{A}^{\kappa}_{n,i,j} = o_p(1). \tag{A.55}$$

By Lemma A2, (A.53), and (A.55), we see that the terms on the majorant side of (A.52) are $o_p(1)$. Hence,

$$\widehat{\Gamma}^*_{g,n} - \Gamma_{g,n} \xrightarrow{\mathbb{P}} 0. \tag{A.56}$$

Next, we show

$$\Gamma_{g,n} \xrightarrow{\mathbb{P}} \bar{\Gamma}_g. \tag{A.57}$$

To simplify notation, we denote $\gamma_{g,j,s} \equiv \gamma_{g,j}(\beta_s, Z_s, V_s)$ and $\bar{\gamma}_{g,s} \equiv \bar{\gamma}_g(\beta_s, Z_s, V_s)$ for $j \geq 0$ and $s \geq 0$. Hence, we can rewrite $\Gamma_{g,n}$ as

$$\Gamma_{g,n} = \Delta_n \sum_i \gamma_{g,0,i\Delta_n} + 2\Delta_n \sum_{j=1}^{\infty}\sum_{i=j}^{N_n} \gamma_{g,j,i\Delta_n},$$

where empty sums are set to zero by convention. Therefore,

$$\begin{aligned}\Gamma_{g,n} - \bar{\Gamma}_g = {}& \left(\Delta_n \sum_{i=0}^{N_n} \gamma_{g,0,i\Delta_n} - \int_0^T \gamma_{g,0,s}\,ds\right) \\ & + 2\sum_{j=1}^{\infty}\left(\Delta_n \sum_{i=j}^{N_n} \gamma_{g,j,i\Delta_n} - \int_0^T \gamma_{g,j,s}\,ds\right).\end{aligned} \tag{A.58}$$

It is easy to see that $\gamma_{g,j}(\beta, z, v)$ is continuous in (β, z, v), so the process $(\gamma_{g,j,t})_{t\geq 0}$ is càdlàg. Hence, by invoking the Riemann approximation, we deduce that, for each $j \geq 0$, $\Delta_n \sum_{i=j}^{N_n} \gamma_{g,j,i\Delta_n} - \int_0^T \gamma_{g,j,s}\,ds \to 0$. Moreover, observe that

$$\sum_{j=1}^{\infty}\left|\Delta_n \sum_{i=j}^{N_n} \gamma_{g,j,i\Delta_n} - \int_0^T \gamma_{g,j,s}\,ds\right| \leq K \sup_{t\in[0,T]} \bar{g}_k(\beta_t, Z_t, V_t)^2 \leq K, \tag{A.59}$$

where the first inequality is derived by using the mixing inequality and $\sum_{j\geq 1}\alpha_{\text{mix}}(j)^{1-2/k} < \infty$, and the second inequality holds because (β_t, Z_t, V_t) is bounded under Assumption A1 and $\bar{g}_k(\cdot)$ is bounded on bounded sets. This dominance condition (i.e., (A.59)) allows us to use the dominated convergence theorem to obtain the limit of the right-hand side of (A.58). From here, (A.57) readily follows. Combining (A.56) and (A.57), we derive

$$\text{(A.60)} \quad \mathbb{E}_{\mathcal{F}}\big[R_{2,n}^2\big] = \widehat{\Gamma}_{g,n}^{*} \xrightarrow{\mathbb{P}} \bar{\Gamma}_g.$$

We now show that $R_{2,n} \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{MN}(0, \bar{\Gamma}_g)$. We need to adapt Theorem 5.20 of White (2001) so as to accommodate the fact that the convergence of the conditional variance in (A.60) is only in probability. Consider a subset $\bar{\Omega}$ of Ω given by $\bar{\Omega} \equiv \{\bar{\Gamma}_g > 0\}$ and let $\bar{\Omega}^c$ be the complement of $\bar{\Omega}$. Clearly, $\bar{\Omega}$ is $\mathcal{F}$-measurable. In restriction to $\bar{\Omega}^c$, $\mathbb{E}_{\mathcal{F}}[R_{2,n}^2] = o_p(1)$ and, thus, the $\mathcal{F}$-conditional law of $R_{n,2}$ converges to the degenerate distribution at zero. Below, we restrict attention on the event $\bar{\Omega}$, so we can assume $\bar{\Gamma}_g > 0$.

We consider an arbitrary subsequence $\mathbb{N}_1 \subseteq \mathbb{N}$. By the subsequence characterization of convergence in probability, it is enough to show that there exists a further subsequence $\mathbb{N}_2 \subseteq \mathbb{N}_1$ such that, as $n \to \infty$ along $\mathbb{N}_2$, the $\mathcal{F}$-conditional distribution function of $R_{2,n}$ converges uniformly to the $\mathcal{F}$-conditional distribution function of $\mathcal{MN}(0, \bar{\Gamma}_g)$ on $\mathbb{P}$-almost every path in $\bar{\Omega}$. By (A.60), we can extract a subsequence $\mathbb{N}_2 \subseteq \mathbb{N}_1$ such that, along $\mathbb{N}_2$, $\mathbb{E}_{\mathcal{F}}[R_{2,n}^2] \to \bar{\Gamma} > 0$ for almost every path in $\bar{\Omega}$. Recall from (A.48) that $R_{2,n} = \Delta_n^{1/2} \sum_i \xi_i(\hat{z}_{n,i})$. Under Assumption 4, $\xi_i(\hat{z}_{n,i})$ forms a sequence with zero mean and α-mixing coefficients bounded by $\alpha_{\mathrm{mix}}(\cdot)$ under the transition probability $\mathbb{P}^{(1)}$. Moreover, $\|\xi_i(\hat{z}_{n,i})\|_{\mathcal{F},k} \leq K\bar{g}_k(\hat{z}_{n,i}) \leq K$. We are now ready to apply Theorem 5.20 in White (2001) and Pólya's theorem under the transition probability $\mathbb{P}^{(1)}$ and deduce that, along $\mathbb{N}_2$, the $\mathcal{F}$-conditional distribution function of $R_{2,n}$ converges uniformly to the $\mathcal{F}$-conditional distribution function of $\mathcal{MN}(0, \bar{\Gamma}_g)$ for almost every path in $\bar{\Omega}$. We can then use a subsequence argument to further deduce that $R_{2,n} \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{MN}(0, \bar{\Gamma}_g)$. The proof of (A.45) is now complete.

Step 3. It remains to show (A.46). By a componentwise argument, we can consider $R_{3,n}(\theta)$ to be scalar-valued without loss of generality. We denote

$$\tilde{h}_{n,i}(\theta) \equiv \bar{h}(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) - h(Y_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta),$$

so that $R_{3,n}(\theta) = \Delta_n^{1/2} k_n^{-1} \sum_i \tilde{h}_{n,i}(\theta)$. We first show that $R_{3,n}(\theta) = o_p(1)$ for fixed θ under the $\mathcal{F}$-conditional probability. By Assumption 2(iii) and the $\mathcal{F}$-measurability of $(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n})$, we have $\mathbb{E}_{\mathcal{F}}[\tilde{h}_{n,i}(\theta)] = 0$. Furthermore, since $\bar{g}_k(\cdot)$ is bounded on bounded sets,

$$\text{(A.61)} \quad \big\|\tilde{h}_{n,i}(\theta)\big\|_{\mathcal{F},k} \leq K\bar{g}_k(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n})\widehat{V}_{i\Delta_n}^2 \leq K.$$

By Assumption 4, conditional on $\mathcal{F}$, the α-mixing coefficient of the sequence $(\tilde{h}_{n,i}(\theta))_{i\geq 0}$ is bounded by $\alpha_{\text{mix}}(\cdot)$. Observe that, w.p.a.1,

$$\begin{aligned}\mathbb{E}_{\mathcal{F}}\big[R_{3,n}^2(\theta)\big] &\leq \Delta_n k_n^{-2}\sum_{i,j=0}^{N_n}\big|\mathbb{E}_{\mathcal{F}}\big[\tilde{h}_{n,i}(\theta)\tilde{h}_{n,j}(\theta)\big]\big| \\ &\leq K\Delta_n k_n^{-2}\sum_{i,j=0}^{N_n}\alpha_{\text{mix}}\big(|i-j|\big)^{1-2/k}\big\|\tilde{h}_{n,i}(\theta)\big\|_{\mathcal{F},k}\big\|\tilde{h}_{n,j}(\theta)\big\|_{\mathcal{F},k} \\ &\leq Kk_n^{-2},\end{aligned}$$

where the first inequality is by the triangle inequality; the second inequality follows from the mixing inequality; the third inequality follows from (A.61) and Assumption 4. From here, we deduce that $R_{3,n}(\theta) = o_p(1)$ under the $\mathcal{F}$-conditional probability. Next, by using an argument similar to step 2, we can show that $\Delta_n^{1/2}R_{3,n}(\cdot)$ is stochastically equicontinuous under the $\mathcal{F}$-conditional probability and that, under Assumption 5(i), so is $R_{3,n}(\cdot)$. From here, (A.46) readily follows. This finishes the proof. *Q.E.D.*

A.3. *Proof of Theorem 2*

We first prove two lemmas. Lemma A5 is a general uniform law of large numbers for integrated volatility functionals. This lemma is then used to prove Lemma A6, which establishes a uniform consistency result for the estimation of asymptotic covariance functions. Lemma A6 is also used in the proof of Theorem 3.

LEMMA A5: *Suppose* (i) *Assumptions* A1 *and* 3; (ii) *the function* $(\beta, z, v, \theta) \mapsto f(\beta, z, v; \theta)$ *is continuous and continuously differentiable in* θ. *Then*

$$\Delta_n \sum_i f(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) \xrightarrow{\mathbb{P}} \int_0^T f(\beta_s, Z_s, V_s; \theta)\,ds$$

uniformly in θ *on the compact set* Θ.

PROOF: By using a spatial localization procedure as in step 2 of the proof of Lemma A4, we can assume that f is compactly supported without loss of generality. Denote $\tilde{\beta}_t \equiv (\beta_t, Z_t)$ for notational simplicity. Construct two processes, $\tilde{\beta}_t^+$ and $\widehat{V}_t^+$, as follows: for each $i \geq 1$ and $t \in [(i-1)\Delta_n, i\Delta_n)$, set $\tilde{\beta}_t^+ \equiv \tilde{\beta}_{i\Delta_n}$ and $\widehat{V}_t^+ \equiv \widehat{V}_{i\Delta_n}$. Observe that, since f is bounded,

$$\begin{aligned}&\mathbb{E}\left|\Delta_n\sum_i f(\tilde{\beta}_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) - \int_0^T f(\tilde{\beta}_s, V_s; \theta)\,ds\right| \\ &\quad\leq Kk_n\Delta_n + \int_0^{N_n\Delta_n}\mathbb{E}\big|f\big(\tilde{\beta}_s^+, \widehat{V}_s^+; \theta\big) - f(\tilde{\beta}_s, V_s; \theta)\big|\,ds.\end{aligned}$$

By Theorem 9.3.2 in Jacod and Protter (2012), we have $\widehat{V}_s^+ \xrightarrow{\mathbb{P}} V_s$ for each $s \geq 0$. By the right continuity of the process $\tilde{\beta}$, we have $\tilde{\beta}_s^+ \to \tilde{\beta}_s$ for each $s \geq 0$, which further implies $(\tilde{\beta}_s^+, \widehat{V}_s^+) \xrightarrow{\mathbb{P}} (\tilde{\beta}_s, V_s)$. By the continuity of $f(\cdot)$, $f(\tilde{\beta}_s^+, \widehat{V}_s^+; \theta) \xrightarrow{\mathbb{P}} f(\tilde{\beta}_s, V_s; \theta)$. By the bounded convergence theorem, we deduce $\int_0^{N_n \Delta_n} \mathbb{E}|f(\tilde{\beta}_s^+, \widehat{V}_s^+; \theta) - f(\tilde{\beta}_s, V_s; \theta)|\, ds \to 0$, which further yields

$$\Delta_n \sum_i f(\tilde{\beta}_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta) \xrightarrow{\mathbb{P}} \int_0^T f(\tilde{\beta}_s, V_s; \theta)\, ds. \tag{A.62}$$

By condition (ii), the mapping $\theta \mapsto f(\cdot; \theta)$ is Lipschitz continuous. From here, it is easy to see that $\Delta_n \sum_i f(\tilde{\beta}_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \cdot)$ is stochastically equicontinuous. The pointwise convergence (A.62) then implies the asserted uniform convergence. *Q.E.D.*

For Lemma A6 below, we need some additional notations. We consider two $\mathbb{R}$-valued functions ϕ_1 and ϕ_2 on $\mathbb{R}$ and set, for $l \geq 0$,

$$\begin{cases} \widehat{\Sigma}_n(\theta, \tau, \tau') \equiv \widehat{S}_n(\theta, \tau, \tau') + \widehat{\Gamma}_n(\theta, \tau, \tau'), \\ \widehat{S}_n(\theta, \tau, \tau') \equiv 2\Delta_n \displaystyle\sum_{i=0}^{N_n} \hat{m}'_{n,i}(g, \theta) \hat{m}'_{n,i}(g, \theta)^\intercal \widehat{V}_{i\Delta_n}^2 \phi_1(i\Delta_n \tau) \phi_2(i\Delta_n \tau'), \\ \widehat{\Gamma}_n(\theta, \tau, \tau') \equiv \widehat{\Gamma}_{0,n}(\theta, \tau, \tau') \\ \qquad\qquad + \displaystyle\sum_{l=1}^{B_n} w(l, B_n)\bigl(\widehat{\Gamma}_{l,n}(\theta, \tau, \tau') + \widehat{\Gamma}_{l,n}(\theta, \tau', \tau)^\intercal\bigr), \\ \widehat{\Gamma}_{l,n}(\theta, \tau, \tau') \equiv \Delta_n \displaystyle\sum_{i=l}^{N_n} \hat{\delta}_{n,i}(g, \theta) \hat{\delta}_{n,i-l}(g, \theta)^\intercal \phi_1(i\Delta_n \tau) \phi_2\bigl((i-l)\Delta_n \tau'\bigr). \end{cases}$$

We also set, for $l \geq 0$,

$$\begin{cases} \Sigma(\theta, \tau, \tau') \equiv S(\theta, \tau, \tau') + \Gamma(\theta, \tau, \tau'), \\ S(\theta, \tau, \tau') \equiv 2 \displaystyle\int_0^T \partial_v \bar{g}(\beta_s, Z_s, V_s; \theta) \partial_v \bar{g}(\beta_s, Z_s, V_s; \theta')^\intercal \\ \qquad\qquad \times V_s^2 \phi_1(s\tau) \phi_2(s\tau')\, ds, \\ \Gamma(\theta, \tau, \tau') \equiv \Gamma_0(\theta, \tau, \tau') + \displaystyle\sum_{l \geq 1} \bigl(\Gamma_l(\theta, \tau, \tau') + \Gamma_l(\theta, \tau', \tau)^\intercal\bigr), \\ \Gamma_l(\theta, \tau, \tau') \equiv \displaystyle\int_0^T \mathrm{Cov}_{\mathcal{F}}\bigl(g(\mathscr{Y}(\beta_s, \chi_i), Z_s, V_s; \theta), \\ \qquad\qquad g(\mathscr{Y}(\beta_s, \chi_{i-l}), Z_s, V_s; \theta)\bigr) \phi_1(s\tau) \phi_2(s\tau')\, ds. \end{cases}$$

LEMMA A6: *Let $\mathcal{T}$ be a compact subset of $\mathbb{R}$, $\tilde{\theta}$ be a Θ-valued $\mathcal{F}$-measurable random variable, and $\tilde{\theta}_n$ be a sequence of Θ-valued estimators. Suppose* (i) *As-*

sumptions 1–4 *and* 7; (ii) *the functions* ϕ_1 *and* ϕ_2 *are Lipschitz continuous*; (iii) $\tilde{\theta}_n - \tilde{\theta} = o_p(B_n^{-1})$. *Then* $\widehat{\Sigma}_n(\tilde{\theta}_n, \tau, \tau') \xrightarrow{\mathbb{P}} \Sigma(\tilde{\theta}, \tau, \tau')$ *uniformly in* $\tau, \tau' \in \mathcal{T}$.

PROOF: *Step 1*. As is typical in this type of problem, by a polarization argument, we can consider a one-dimensional setting without loss of generality. We henceforth suppose that $g(\cdot)$ is scalar-valued. By localization, we also suppose that Assumption A1 holds. In view of Lemma A2, we can restrict attention to the w.p.a.1 event on which the variables $(\widehat{V}_{i\Delta_n})_{1\leq i\leq N_n}$ are uniformly bounded. Since g is fixed, we write $\hat{m}_{n,i}(\theta)$, $\hat{m}'_{n,i}(\theta)$, and $\hat{\delta}_{n,i}(\theta)$ in place of $\hat{m}_{n,i}(g, \theta)$, $\hat{m}'_{n,i}(g, \theta)$, and $\hat{\delta}_{n,i}(g, \theta)$. Below, we also denote

$$\phi_{n,i,j}(\tau, \tau') \equiv \phi_1(i\Delta_n\tau)\phi_2\big((i-j)\Delta_n\tau'\big).$$

We complete the proof by showing

$$\text{(A.63)} \qquad \sup_{\tau,\tau'\in\mathcal{T}} \big|\widehat{S}_n(\tilde{\theta}_n, \tau, \tau') - S(\tilde{\theta}, \tau, \tau')\big| = o_p(1),$$

$$\text{(A.64)} \qquad \sup_{\tau,\tau'\in\mathcal{T}} \big|\widehat{\Gamma}_n(\tilde{\theta}_n, \tau, \tau') - \Gamma(\tilde{\theta}, \tau, \tau')\big| = o_p(1).$$

In this step, we show (A.63). We set

$$\begin{cases} S_n(\theta, \tau, \tau') \equiv 2\Delta_n \displaystyle\sum_{i=0}^{N_n} \hat{\mu}'_{n,i}(\theta)^2 \widehat{V}_{i\Delta_n}^2 \phi_{n,i,0}(\tau, \tau'), \\ \hat{\mu}'_{n,i}(\theta) \equiv \partial_v \bar{g}(\beta_{i\Delta_n}, Z_{i\Delta_n}, \widehat{V}_{i\Delta_n}; \theta), \quad i \geq 0. \end{cases}$$

By applying Lemma A5 to the function $f(\beta, z, v; \theta, \tau, \tau') \equiv 2\partial_v \bar{g}(\beta, z, v; \theta)^2 \times v^2\phi_1(t\tau)\phi_2(t\tau')$, we derive

$$\text{(A.65)} \qquad \sup_{\tau,\tau'\in\mathcal{T}} \big|S_n(\tilde{\theta}, \tau, \tau') - S(\tilde{\theta}, \tau, \tau')\big| = o_p(1).$$

We now show the following claim:

$$\text{(A.66)} \qquad \Delta_n \sum_i \big|\hat{m}'_{n,i}(\tilde{\theta}_n) - \hat{\mu}'_{n,i}(\tilde{\theta})\big|^2 \leq O_p(1)\|\tilde{\theta}_n - \tilde{\theta}\|^2 + O_p\big(k_n^{-1} + k_n\Delta_n\big) = o_p(1).$$

We set, for each $i \geq 0$,

$$\bar{\mu}'_{n,i}(\theta) \equiv \frac{1}{k_n} \sum_{j=0}^{k_n-1} \partial_v \bar{g}(\beta_{(i+j)\Delta_n}, Z_{(i+j)\Delta_n}, \widehat{V}_{i\Delta_n}; \theta).$$

First, by the mean value theorem and a standard estimate for Itô semimartingales, we deduce

$$\begin{aligned}
\text{(A.67)} \quad & \big\|\bar{\mu}'_{n,i}(\tilde{\theta}) - \hat{\mu}'_{n,i}(\tilde{\theta})\big\|_{\mathcal{F},2} \\
& \leq K k_n^{-1} \sum_{j=0}^{k_n-1} \big[\|\beta_{(i+j)\Delta_n} - \beta_{i\Delta_n}\|_{\mathcal{F},2} + \|Z_{(i+j)\Delta_n} - Z_{i\Delta_n}\|_{\mathcal{F},2}\big] \\
& \leq K (k_n \Delta_n)^{1/2}.
\end{aligned}$$

Next, note that

$$\begin{aligned}
& \hat{m}'_{n,i}(\tilde{\theta}) - \bar{\mu}'_{n,i}(\tilde{\theta}) \\
& = \frac{1}{k_n} \sum_{j=0}^{k_n-1} \big(\partial_v g(Y_{(i+j)\Delta_n}, Z_{(i+j)\Delta_n}, \widehat{V}_{i\Delta_n}; \tilde{\theta}) \\
& \quad - \partial_v \bar{g}(\beta_{(i+j)\Delta_n}, Z_{(i+j)\Delta_n}, \widehat{V}_{i\Delta_n}; \tilde{\theta})\big),
\end{aligned}$$

where each term in the sum has zero $\mathcal{F}$-conditional mean because of Assumption 2(iii). By a use of the mixing inequality and Assumption 2(iv), we derive

$$\text{(A.68)} \quad \big\|\hat{m}'_{n,i}(\tilde{\theta}) - \bar{\mu}'_{n,i}(\tilde{\theta})\big\|_{\mathcal{F},2} \leq K k_n^{-1/2}.$$

Finally, by Assumption 2(vi),

$$\begin{aligned}
\text{(A.69)} \quad & \Delta_n \sum_i \big|\hat{m}'_{n,i}(\tilde{\theta}_n) - \hat{m}'_{n,i}(\tilde{\theta})\big|^2 \\
& \leq K \Delta_n \sum_i \left(\frac{1}{k_n} \sum_{j=0}^{k_n-1} B(Y_{(i+j)\Delta_n}, Z_{(i+j)\Delta_n}, \widehat{V}_{i\Delta_n})\right)^2 \|\tilde{\theta}_n - \tilde{\theta}\|^2 \\
& \leq O_p(1) \|\tilde{\theta}_n - \tilde{\theta}\|^2.
\end{aligned}$$

Using (A.67), (A.68), and (A.69), we readily deduce (A.66).

Now, we note that

$$\begin{aligned}
\text{(A.70)} \quad & \sup_{\tau,\tau' \in \mathcal{T}} \big|\widehat{S}_n(\tilde{\theta}_n, \tau, \tau') - S_n(\tilde{\theta}, \tau, \tau')\big| \\
& = \sup_{\tau,\tau' \in \mathcal{T}} \left|\Delta_n \sum_i \big(\hat{m}'_{n,i}(\tilde{\theta}_n)^2 - \hat{\mu}'_{n,i}(\tilde{\theta})^2\big) \widehat{V}^2_{i\Delta_n} \phi_{n,i,0}(\tau, \tau')\right| \\
& \leq K \Delta_n \sum_i \big|\hat{m}'_{n,i}(\tilde{\theta}_n)^2 - \hat{\mu}'_{n,i}(\tilde{\theta})^2\big|.
\end{aligned}$$

From (A.66), it is easy to see that the majorant side of (A.70) is $o_p(1)$ by using the Cauchy–Schwarz inequality. Combining this with (A.65), we have (A.63).

Step 2. In this step, we show (A.64). We first complement the notations $\hat{g}_{n,i}(\theta)$, $\hat{m}_{n,i}(\theta)$, and $\hat{\delta}_{n,i}(\theta)$ with the following:

$$\text{(A.71)}\quad \begin{cases} g_{n,i}(\theta) \equiv g(Y_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n}; \theta), \\ m_{n,i}(\theta) \equiv k_n^{-1} \sum_{j=0}^{k_n-1} g(Y_{(i+j)\Delta_n}, Z_{(i+j)\Delta_n}, V_{i\Delta_n}; \theta), \\ \mu_{n,i}(\theta) \equiv \mathbb{E}_{\mathcal{F}}\left[g(Y_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n}; \theta)\right] = \bar{g}(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n}; \theta), \\ \delta_{n,i}(\theta) \equiv g_{n,i}(\theta) - m_{n,i}(\theta), \quad \bar{\delta}_{n,i}(\theta) \equiv g_{n,i}(\theta) - \mu_{n,i}(\theta). \end{cases}$$

We note that $\widehat{\Gamma}_n(\tilde{\theta}_n, \tau, \tau')$ can be written as

$$\text{(A.72)}\quad \begin{aligned} \widehat{\Gamma}_n(\tilde{\theta}_n, \tau, \tau') &= \Delta_n \sum_{i=0}^{N_n} \hat{\delta}_{n,i}(\tilde{\theta}_n)^2 \phi_{n,i,0}(\tau, \tau') \\ &\quad + 2 \sum_{j=1}^{B_n} w(j, B_n) \Delta_n \sum_{i=j}^{N_n} \hat{\delta}_{n,i}(\tilde{\theta}_n) \hat{\delta}_{n,i-j}(\tilde{\theta}_n) \phi_{n,i,j}(\tau, \tau'). \end{aligned}$$

We consider a progressive list of approximations between $\widehat{\Gamma}_n(\tilde{\theta}_n, \tau, \tau')$ and $\Gamma(\tilde{\theta}, \tau, \tau')$ as follows:

$$\text{(A.73)}\quad \begin{cases} \widehat{\Gamma}_n^{(1)}(\tilde{\theta}, \tau, \tau') \equiv \Delta_n \sum_{i=0}^{N_n} \bar{\delta}_{n,i}(\tilde{\theta})^2 \phi_{n,i,0}(\tau, \tau') \\ \qquad + 2 \sum_{j=1}^{B_n} w(j, B_n) \Delta_n \sum_{i=j}^{N_n} \bar{\delta}_{n,i}(\tilde{\theta}) \bar{\delta}_{n,i-j}(\tilde{\theta}) \phi_{n,i,j}(\tau, \tau'), \\ \widehat{\Gamma}_n^{(2)}(\tilde{\theta}, \tau, \tau') \equiv \Delta_n \sum_{i=0}^{N_n} \mathbb{E}_{\mathcal{F}}\left[\bar{\delta}_{n,i}(\tilde{\theta})^2\right] \phi_{n,i,0}(\tau, \tau') \\ \qquad + 2 \sum_{j=1}^{B_n} w(j, B_n) \Delta_n \\ \qquad \times \sum_{i=j}^{N_n} \mathbb{E}_{\mathcal{F}}\left[\bar{\delta}_{n,i}(\tilde{\theta}) \bar{\delta}_{n,i-j}(\tilde{\theta})\right] \phi_{n,i,j}(\tau, \tau'), \\ \widehat{\Gamma}_n^{(3)}(\tilde{\theta}, \tau, \tau') \equiv \Delta_n \sum_{i=0}^{N_n} \mathbb{E}_{\mathcal{F}}\left[\bar{\delta}_{n,i}(\tilde{\theta})^2\right] \phi_{n,i,0}(\tau, \tau') \\ \qquad + 2\Delta_n \sum_{j=1}^{N_n} \sum_{i=j}^{N_n} \mathbb{E}_{\mathcal{F}}\left[\bar{\delta}_{n,i}(\tilde{\theta}) \bar{\delta}_{n,i-j}(\tilde{\theta})\right] \phi_{n,i,j}(\tau, \tau'). \end{cases}$$

First consider $\widehat{\Gamma}_n(\tilde{\theta}_n, \tau, \tau') - \widehat{\Gamma}_n^{(1)}(\tilde{\theta}, \tau, \tau')$. Observe that

$$
\begin{aligned}
\text{(A.74)} \quad & \sup_{\tau,\tau' \in \mathcal{T}} \big| \widehat{\Gamma}_n\big(\tilde{\theta}_n, \tau, \tau'\big) - \widehat{\Gamma}_n^{(1)}\big(\tilde{\theta}, \tau, \tau'\big) \big| \\
& \le K \sum_{j=0}^{B_n} \Delta_n \sum_{i=j}^{N_n} \big| \hat{\delta}_{n,i}(\tilde{\theta}_n) \hat{\delta}_{n,i-j}(\tilde{\theta}_n) - \bar{\delta}_{n,i}(\tilde{\theta}) \bar{\delta}_{n,i-j}(\tilde{\theta}) \big| \\
& \le K \sum_{j=0}^{B_n} \Delta_n \sum_{i=j}^{N_n} \big(\big| \hat{\delta}_{n,i}(\tilde{\theta}_n) \big| \big| \hat{\delta}_{n,i-j}(\tilde{\theta}_n) - \bar{\delta}_{n,i-j}(\tilde{\theta}) \big| \\
& \quad + \big| \hat{\delta}_{n,i}(\tilde{\theta}_n) - \bar{\delta}_{n,i}(\tilde{\theta}) \big| \big| \bar{\delta}_{n,i-j}(\tilde{\theta}) \big| \big) \\
& \le K B_n \bigg(\Delta_n \sum_i \big(\hat{\delta}_{n,i}(\tilde{\theta}_n)^2 + \bar{\delta}_{n,i}(\tilde{\theta})^2 \big) \bigg)^{1/2} \\
& \quad \times \bigg(\Delta_n \sum_i \big| \hat{\delta}_{n,i}(\tilde{\theta}_n) - \bar{\delta}_{n,i}(\tilde{\theta}) \big|^2 \bigg)^{1/2},
\end{aligned}
$$

where the first inequality follows from the triangle inequality and the uniform boundedness of $w(j, B_n)$ and $\phi_{n,i,j}(\tau, \tau')$; the second inequality is by the triangle inequality; the third inequality follows from the Cauchy–Schwarz inequality.

We observe that

$$
\begin{aligned}
\text{(A.75)} \quad \Delta_n \sum_i \big| \hat{\delta}_{n,i}(\tilde{\theta}_n) - \bar{\delta}_{n,i}(\tilde{\theta}) \big|^2 & \le K \Delta_n \sum_i \big| \hat{\delta}_{n,i}(\tilde{\theta}_n) - \hat{\delta}_{n,i}(\tilde{\theta}) \big|^2 \\
& \quad + K \Delta_n \sum_i \big| \hat{\delta}_{n,i}(\tilde{\theta}) - \delta_{n,i}(\tilde{\theta}) \big|^2 \\
& \quad + K \Delta_n \sum_i \big| \delta_{n,i}(\tilde{\theta}) - \bar{\delta}_{n,i}(\tilde{\theta}) \big|^2.
\end{aligned}
$$

From Assumption 2(vi), it is easy to see

$$
\text{(A.76)} \quad \Delta_n \sum_i \big| \hat{\delta}_{n,i}(\tilde{\theta}_n) - \hat{\delta}_{n,i}(\tilde{\theta}) \big|^2 \le O_p(1) \| \tilde{\theta}_n - \tilde{\theta} \|^2.
$$

By Assumption 2(v), we have, for $0 \le i \le N_n$,

$$
\begin{aligned}
\text{(A.77)} \quad & \mathbb{E}_{\mathcal{F}} \big| \hat{\delta}_{n,i}(\tilde{\theta}) - \delta_{n,i}(\tilde{\theta}) \big|^2 \\
& \le K |\widehat{V}_{i\Delta_n} - V_{i\Delta_n}|^{2\kappa} + K k_n^{-1} \sum_{j=0}^{k_n - 1} |\widehat{V}_{(i+j)\Delta_n} - V_{(i+j)\Delta_n}|^{2\kappa}.
\end{aligned}
$$

We note that $|\widehat{V}_{i\Delta_n} - V_{i\Delta_n}|^{2\kappa} \leq K|\widehat{V}_{i\Delta_n} - \widehat{V}'_{i\Delta_n}|^{2\kappa\wedge 1} + K|\widehat{V}'_{i\Delta_n} - V_{i\Delta_n}|^{2\kappa}$ for all i w.p.a.1 and, by (A.2) and Assumption 3,

$$\text{(A.78)}\quad \mathbb{E}\left[\left|\widehat{V}_{i\Delta_n} - \widehat{V}'_{i\Delta_n}\right|^{2\kappa\wedge 1} + \left|\widehat{V}'_{i\Delta_n} - V_{i\Delta_n}\right|^{2\kappa}\right] \leq K\left(\Delta_n^{(2-r)\varpi} + k_n^{-\kappa} + (k_n\Delta_n)^{\kappa}\right) \leq Kk_n^{-\kappa}.$$

Combining (A.77) and (A.78), we deduce

$$\text{(A.79)}\quad \Delta_n \sum_i \left|\hat{\delta}_{n,i}(\tilde{\theta}) - \delta_{n,i}(\tilde{\theta})\right|^2 = O_p\left(k_n^{-\kappa}\right).$$

Third, following the same argument for (A.66), we deduce

$$\text{(A.80)}\quad \Delta_n \sum_i \left|\delta_{n,i}(\tilde{\theta}) - \bar{\delta}_{n,i}(\tilde{\theta})\right|^2 = O_p\left(k_n^{-1} + k_n\Delta_n\right).$$

Plugging (A.76), (A.79), and (A.80) into (A.75), we deduce

$$\text{(A.81)}\quad \Delta_n \sum_i \left|\hat{\delta}_{n,i}(\tilde{\theta}_n) - \bar{\delta}_{n,i}(\tilde{\theta})\right|^2 \leq O_p(1)\|\tilde{\theta}_n - \tilde{\theta}\|^2 + O_p\left(k_n^{-\kappa}\right).$$

By Assumption 2(iv), it is easy to see that $\mathbb{E}|\bar{\delta}_{n,i}(\tilde{\theta})|^2 \leq K\mathbb{E}|g_{n,i}(\tilde{\theta})|^2 \leq K$. From this estimate and (A.81), we further deduce that

$$\text{(A.82)}\quad \Delta_n \sum_i \left(\left|\hat{\delta}_{n,i}(\tilde{\theta}_n)\right|^2 + \left|\bar{\delta}_{n,i}(\tilde{\theta})\right|^2\right) = O_p(1).$$

Plugging (A.81) and (A.82) into (A.74), we deduce

$$\text{(A.83)}\quad \sup_{\tau,\tau'\in\mathcal{T}} \left|\widehat{\Gamma}_n\left(\tilde{\theta}_n, \tau, \tau'\right) - \widehat{\Gamma}_n^{(1)}\left(\tilde{\theta}, \tau, \tau'\right)\right| \leq O_p(B_n)\|\tilde{\theta}_n - \tilde{\theta}\| + O_p\left(B_n k_n^{-\kappa/2}\right) = o_p(1),$$

where the equality follows from our assumptions that $\tilde{\theta}_n - \tilde{\theta} = o_p(B_n^{-1})$ and $B_n k_n^{-\kappa/2} = o(1)$.

Next, we consider $\widehat{\Gamma}_n^{(1)}(\tilde{\theta}, \tau, \tau') - \widehat{\Gamma}_n^{(2)}(\tilde{\theta}, \tau, \tau')$. We denote

$$\text{(A.84)}\quad \begin{cases} \zeta_{n,i,j} \equiv \bar{\delta}_{n,i}(\tilde{\theta})\bar{\delta}_{n,i-j}(\tilde{\theta}) - \mathbb{E}_{\mathcal{F}}\left[\bar{\delta}_{n,i}(\tilde{\theta})\bar{\delta}_{n,i-j}(\tilde{\theta})\right], \\ \bar{\zeta}_{n,j}\left(\tau, \tau'\right) \equiv \Delta_n \sum_{i=j}^{N_n} \zeta_{n,i,j}\phi_{n,i,j}\left(\tau, \tau'\right). \end{cases}$$

We can then rewrite

$$\text{(A.85)}\quad \widehat{\Gamma}_n^{(1)}\left(\tilde{\theta}, \tau, \tau'\right) - \widehat{\Gamma}_n^{(2)}\left(\tilde{\theta}, \tau, \tau'\right) = \bar{\zeta}_{n,0}\left(\tau, \tau'\right) + 2\sum_{j=1}^{B_n} w(j, B_n)\bar{\zeta}_{n,j}\left(\tau, \tau'\right).$$

By the mixing inequality, for $i, j \geq 0$ and $l \geq i$,

$$\left|\mathbb{E}_{\mathcal{F}}[\zeta_{n,i,j}\zeta_{n,l,j}]\right| \leq K\alpha_{\text{mix}}\big((l-i-j)^+\big)^{1-2/k}\|\zeta_{n,i,j}\|_{\mathcal{F},k}\|\zeta_{n,l,j}\|_{\mathcal{F},k}, \tag{A.86}$$

where $(\cdot)^+$ denotes the positive part. By the Cauchy–Schwarz inequality and Assumption 2(iv),

$$\begin{aligned}\|\zeta_{n,i,j}\|_{\mathcal{F},k} &\leq K\bar{g}_{2k}(\beta_{i\Delta_n}, Z_{i\Delta_n}, V_{i\Delta_n}; \tilde{\theta})\bar{g}_{2k}(\beta_{(i-j)\Delta_n}, Z_{(i-j)\Delta_n}, V_{(i-j)\Delta_n}; \tilde{\theta}) \\ &\leq K.\end{aligned} \tag{A.87}$$

Therefore, $|\mathbb{E}_{\mathcal{F}}[\zeta_{n,i,j}\zeta_{n,l,j}]| \leq K\alpha_{\text{mix}}((l-i-j)^+)^{1-2/k}$. From here, it follows that, for all $j \geq 0$ and $\tau, \tau' \in \mathcal{T}$,

$$\begin{aligned}\big\|\bar{\zeta}_{n,j}(\tau,\tau')\big\|_2 &= \left\|\Delta_n \sum_{i=j}^{N_n} \zeta_{n,i,j}\phi_{n,i,j}(\tau,\tau')\right\|_2 \\ &\leq K\Delta_n\left(\sum_{i=j}^{N_n}\sum_{l=i}^{N_n}\mathbb{E}\left|\mathbb{E}_{\mathcal{F}}[\zeta_{n,i,j}\zeta_{n,l,j}]\right|\right)^{1/2} \\ &\leq K\Delta_n\left(\sum_{i=j}^{N_n}\sum_{l=i}^{N_n}\alpha_{\text{mix}}\big((l-i-j)^+\big)^{1-2/k}\right)^{1/2} \\ &\leq K\Delta_n^{1/2}(j+1)^{1/2}.\end{aligned} \tag{A.88}$$

Then, by the triangle inequality and the boundedness of the kernel function $w(\cdot,\cdot)$, we derive from (A.85) that

$$\mathbb{E}\big|\widehat{\Gamma}_n^{(1)}(\tilde{\theta},\tau,\tau') - \widehat{\Gamma}_n^{(2)}(\tilde{\theta},\tau,\tau')\big| \leq K\Delta_n^{1/2}\sum_{j=0}^{B_n}(j+1)^{1/2} = O\big(\Delta_n^{1/2}B_n^{3/2}\big).$$

Since $B_n = o(k_n^{\kappa/2})$ by Assumption 7 and $k_n \leq K\Delta_n^{-1/2}$ by Assumption 3, we have $B_n = o(\Delta_n^{-1/4})$. Hence, for any $\tau, \tau' \in \mathcal{T}$,

$$\widehat{\Gamma}_n^{(1)}(\tilde{\theta},\tau,\tau') - \widehat{\Gamma}_n^{(2)}(\tilde{\theta},\tau,\tau') = o_p(1). \tag{A.89}$$

We further show that $\widehat{\Gamma}_n^{(1)}(\tilde{\theta},\cdot,\cdot) - \widehat{\Gamma}_n^{(2)}(\tilde{\theta},\cdot,\cdot)$ is stochastically equicontinuous. Let τ_1, τ_1', τ_2, and τ_2' be generic elements in $\mathcal{T}$. We observe

$$\begin{aligned}&\big\|\bar{\zeta}_{n,j}(\tau_1,\tau_1') - \bar{\zeta}_{n,j}(\tau_2,\tau_2')\big\|_4 \\ &\quad= \left\|\Delta_n\sum_{i=j}^{N_n}\zeta_{n,i,j}\big(\phi_{n,i,j}(\tau_1,\tau_1') - \phi_{n,i,j}(\tau_2,\tau_2')\big)\right\|_4 \\ &\quad\leq K\Delta_n^{1/2}(j+1)^{1/2}\big\|(\tau_1,\tau_1') - (\tau_2,\tau_2')\big\|,\end{aligned}$$

where the inequality is derived by using an argument similar to (A.88), but generalized to the case with L_4-norm by using Theorem 3 of Yoshihara (1978); the condition $\sum_{j\geq 1} j\alpha_{\text{mix}}(j)^{(k-2)/k} < \infty$ is used here. It then follows that

$$\begin{aligned}
&\big\|\widehat{\Gamma}_n^{(1)}\big(\tilde{\theta},\tau_1,\tau_1'\big)-\widehat{\Gamma}_n^{(2)}\big(\tilde{\theta},\tau_1,\tau_1'\big)-\big(\widehat{\Gamma}_n^{(1)}\big(\tilde{\theta},\tau_2,\tau_2'\big)-\widehat{\Gamma}_n^{(2)}\big(\tilde{\theta},\tau_2,\tau_2'\big)\big)\big\|_4 \\
&\quad\leq K\big\|\big(\tau_1,\tau_1'\big)-\big(\tau_2,\tau_2'\big)\big\|.
\end{aligned}$$

Hence, $\widehat{\Gamma}_n^{(1)}(\tilde{\theta},\cdot,\cdot)-\widehat{\Gamma}_n^{(2)}(\tilde{\theta},\cdot,\cdot)$ is stochastically equicontinuous. In view of (A.89), we deduce

$$\sup_{\tau,\tau'\in\mathcal{T}}\big|\widehat{\Gamma}_n^{(1)}\big(\tilde{\theta},\tau,\tau'\big)-\widehat{\Gamma}_n^{(2)}\big(\tilde{\theta},\tau,\tau'\big)\big|=o_p(1). \tag{A.90}$$

Turning to $\widehat{\Gamma}_n^{(2)}(\tilde{\theta},\tau,\tau')-\widehat{\Gamma}_n^{(3)}(\tilde{\theta},\tau,\tau')$, we first note that

$$\begin{aligned}
&\sup_{\tau,\tau'\in\mathcal{T}}\big|\widehat{\Gamma}_n^{(3)}\big(\tilde{\theta},\tau,\tau'\big)-\widehat{\Gamma}_n^{(2)}\big(\tilde{\theta},\tau,\tau'\big)\big| \\
&\quad\leq K\sum_{j=B_n+1}^{N_n}\Delta_n\sum_{i=j}^{N_n}\big|\mathbb{E}_{\mathcal{F}}\big[\bar{\delta}_{n,i}(\tilde{\theta})\bar{\delta}_{n,i-j}(\tilde{\theta})\big]\big| \\
&\qquad+K\sum_{j=1}^{B_n}\big|1-w(j,B_n)\big|\Delta_n\sum_{i=j}^{N_n}\big|\mathbb{E}_{\mathcal{F}}\big[\bar{\delta}_{n,i}(\tilde{\theta})\bar{\delta}_{n,i-j}(\tilde{\theta})\big]\big|.
\end{aligned}$$

Observe

$$\mathbb{E}\left[\sum_{j=B_n+1}^{N_n}\Delta_n\sum_{i=j}^{N_n}\big|\mathbb{E}_{\mathcal{F}}\big[\bar{\delta}_{n,i}(\tilde{\theta})\bar{\delta}_{n,i-j}(\tilde{\theta})\big]\big|\right]\leq K\sum_{j=B_n+1}^{N_n}\alpha_{\text{mix}}(j)^{1-2/k}\to 0,$$

where the inequality is by the triangle inequality and the mixing inequality and the convergence is due to $\sum_{j\geq 1}\alpha_{\text{mix}}(j)^{1-2/k}<\infty$ and $B_n\to\infty$. Similarly,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=1}^{B_n}\big|1-w(j,B_n)\big|\Delta_n\sum_{i=j}^{N_n}\big|\mathbb{E}_{\mathcal{F}}\big[\bar{\delta}_{n,i}(\tilde{\theta})\bar{\delta}_{n,i-j}(\tilde{\theta})\big]\big|\right] \\
&\quad\leq K\sum_{j=1}^{B_n}\big|1-w(j,B_n)\big|\alpha_{\text{mix}}(j)^{1-2/k}.
\end{aligned}$$

Note that for each j, $1-w(j,B_n)\to 0$ as $n\to\infty$. Since $\sum_{j\geq 1}|1-w(j,B_n)|\times\alpha_{\text{mix}}(j)^{1-2/k}\leq K\sum_{j\geq 1}\alpha_{\text{mix}}(j)^{1-2/k}<\infty$, the majorant side of the above inequality converges to zero as $n \to \infty$ by the dominated convergence theorem. By these convergence results, we deduce

$$\sup_{\tau,\tau' \in \mathcal{T}} \left| \widehat{\Gamma}_n^{(2)}(\tilde{\theta}, \tau, \tau') - \widehat{\Gamma}_n^{(3)}(\tilde{\theta}, \tau, \tau') \right| = o_p(1). \tag{A.91}$$

We now show that

$$\widehat{\Gamma}_n^{(3)}(\tilde{\theta}, \tau, \tau') \xrightarrow{\mathbb{P}} \Gamma(\tilde{\theta}, \tau, \tau'). \tag{A.92}$$

By essentially the same argument as step 2 of the proof of Theorem 1, we can show (A.92) for fixed $\tau, \tau' \in \mathcal{T}$. By using the mixing inequality and the Lipschitz continuity of $\phi_1(\cdot)$ and $\phi_2(\cdot)$, it is easy to see that $\widehat{\Gamma}_n^{(3)}(\tilde{\theta}, \cdot, \cdot)$ is stochastically equicontinuous. Therefore, (A.92) holds uniformly in $\tau, \tau' \in \mathcal{T}$. Combining this with (A.83), (A.90), and (A.91), we deduce (A.64). The proof of the lemma is now complete. *Q.E.D.*

PROOF OF THEOREM 2: (a) From the uniform consistency of $G_n(\cdot)$ shown in Theorem 1(a), we deduce $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta^*$ under Assumption 6 by a standard argument for extreme estimation (see, e.g., Newey and McFadden (1994)). From the proof of Theorem 1, we note that it only requires the function $\bar{g}(\cdot)$ to be in $\mathcal{C}^{2,2,3,1}$. Therefore, we can also apply Theorem 1(a) with $g(\cdot)$ replaced by $\partial_\theta g(\cdot)$ and deduce that, uniformly in $\theta \in \Theta$,

$$\partial_\theta G_n(\theta) \xrightarrow{\mathbb{P}} \int_0^T \overline{\partial_\theta g}(\beta_s, Z_s, V_s; \theta)\, ds = H(\theta), \tag{A.93}$$

where the equality follows from Assumption 2(iii). From (A.93), after some routine manipulation, it is easy to see that the estimator $\hat{\theta}_n$ has the asymptotically linear representation

$$\Delta_n^{-1/2}(\hat{\theta}_n - \theta^*) = -(H^\intercal \Xi H)^{-1} H^\intercal \Xi \Delta_n^{-1/2} G_n(\theta^*) + o_p(1). \tag{A.94}$$

The assertion in part (a) then follows from (A.94) and Theorem 1(b).

(b) By (A.93) and $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta^*$, we see $H_n \xrightarrow{\mathbb{P}} H$. By Lemma A6 (with $\phi_1(\cdot)$ and $\phi_2(\cdot)$ being identically 1), we see $\widehat{\Sigma}_{g,n}(\hat{\theta}_n) \xrightarrow{\mathbb{P}} \Sigma_g(\theta^*, \theta^*)$. The assertion of part (b) then readily follows from (3.14) and (3.16).

(c) Observe that, with $A \equiv (\mathbf{I}_q - H(H^\intercal \Xi H)^{-1} H^\intercal \Xi)\Sigma_g(\theta^*, \theta^*)^{1/2}$, we have

$$\Delta_n^{-1/2} G_n(\hat{\theta}_n) = A \Sigma_g(\theta^*, \theta^*)^{-1/2} \Delta_n^{-1/2} G_n(\theta^*) + o_p(1).$$

The assertion of part (c) then follows from the fact that $\Sigma_g(\theta^*, \theta^*)^{-1/2} \Delta_n^{-1/2} \times G_n(\theta^*) \xrightarrow{\mathcal{L}\text{-}s} \mathcal{N}(0, \mathbf{I}_q)$ and $A^\intercal \Xi A$ is idempotent with rank q-$\dim(\theta)$. *Q.E.D.*

A.4. *Proof of Theorem 3*

PROOF OF THEOREM 3: *Step 1*. In this step, we consider the asymptotic property of the test in restriction to Ω_{H_0}. By a mean value expansion, we have

$$M_n(\hat{\theta}_n, \tau) = M_n(\theta^*, \tau) + \partial_\theta M_n(\bar{\theta}_{n,\tau}, \tau)(\hat{\theta}_n - \theta^*),$$

where $\bar{\theta}_{n,\tau}$ is some mean value between $\hat{\theta}_n$ and θ^*. Since $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta^*$, $\sup_{\tau \in \mathcal{T}} \|\bar{\theta}_{n,\tau} - \theta^*\|$ converges to zero in (outer) probability. By Theorem 1(a), we have uniformly in θ and τ,

$$\partial_\theta M_n(\theta, \tau) \xrightarrow{\mathbb{P}} \int_0^T \partial_\theta \bar{\psi}(\beta_s, Z_s, V_s; \theta)\phi(\tau s)\, ds.$$

Here, we have used the fact that because $\psi(\cdot)$ satisfies Assumption 2(iii), $\overline{\partial_\theta \psi}(\cdot) = \partial_\theta \bar{\psi}(\cdot)$. Therefore, uniformly in $\tau \in \mathcal{T}$,

$$\partial_\theta M_n(\bar{\theta}_{n,\tau}, \tau) \xrightarrow{\mathbb{P}} \int_0^T \partial_\theta \bar{\psi}(\beta_s, Z_s, V_s; \theta^*)\phi(\tau s)\, ds. \tag{A.95}$$

From (A.94), we derive the following representation for $M_n(\hat{\theta}_n, \tau)$: uniformly in τ,

$$\Delta_n^{-1/2} M_n(\hat{\theta}_n, \tau) = [\mathbf{I}_{q_1} \vdots -D(\tau)] \Delta_n^{-1/2} \begin{pmatrix} M_n(\theta^*, \tau) \\ G_n(\theta^*) \end{pmatrix} + o_p(1).$$

Recall that

$$\tilde{g}(y, z, v; \tau) \equiv \big(\psi(y, z, v; \theta^*)\phi(t\tau), g(y, z, v; \theta^*)\big).$$

Note that the τ-indexed process $(M_n(\theta^*, \tau), G_n(\theta^*))_{\tau \in \mathcal{T}}$ is associated with $\tilde{g}(\cdot; \tau)$ as in (3.3). Since the index τ is a scalar, we verify Assumption 5 (with θ there replaced by τ), so we can use Theorem 1(c) to show that the sequence $\Delta_n^{-1/2}(M_n(\theta^*, \tau), G_n(\theta^*))$ of τ-indexed processes converges stably in law to a process which, conditional on $\mathcal{F}$, is centered Gaussian with covariance function $\Sigma_{\tilde{g}}(\tau, \tau')$. From here, it follows that

$$\Delta_n^{-1/2} M_n(\hat{\theta}_n, \cdot) \xrightarrow{\mathcal{L}\text{-}s} \tilde{\zeta}(\cdot),$$

where the process $\tilde{\zeta}(\cdot)$ is, conditional on $\mathcal{F}$, centered Gaussian with covariance function

$$C(\tau, \tau') \equiv [\mathbf{I}_{q_1} \vdots -D(\tau)] \Sigma_{\tilde{g}}(\tau, \tau') [\mathbf{I}_{q_1} \vdots -D(\tau')]^\intercal.$$

By Lemma A6, we see that $\widehat{\Sigma}_{\tilde{g},n}(\hat{\theta}_n, \tau, \tau') \xrightarrow{\mathbb{P}} \Sigma_{\tilde{g}}(\tau, \tau')$ uniformly in $\tau, \tau' \in \mathcal{T}$. Similarly to (A.95), we also have $\widehat{D}_n(\tau) \xrightarrow{\mathbb{P}} D(\tau)$ uniformly in $\tau \in \mathcal{T}$. Hence,

$$\widehat{C}_n(\tau, \tau') \xrightarrow{\mathbb{P}} C(\tau, \tau') \quad \text{uniformly.} \tag{A.96}$$

By the continuous mapping theorem, we have, in restriction to Ω_{H_0},

$$\widehat{K}_n \xrightarrow{\mathcal{L}\text{-}s} \sup_{\tau\in\mathcal{T}} \max_{1\leq j\leq q_1} \frac{|\tilde{\zeta}_j(\tau)|}{\sqrt{C_{jj}(\tau, \tau)}}. \tag{A.97}$$

Moreover, the $\mathcal{F}$-conditional law of the simulated Gaussian process $\tilde{\zeta}_n^{\mathrm{Sim}}(\cdot)$ converges (under any metric for the weak convergence of probability measures) in probability to the $\mathcal{F}$-conditional law of $\tilde{\zeta}(\cdot)$. Therefore, $cv_n^\alpha \xrightarrow{\mathbb{P}} cv^\alpha$, where cv^α denotes the $\mathcal{F}$-conditional $(1-\alpha)$-quantile of the limiting variable in (A.97). From here, it follows that $\mathbb{P}(\widehat{K}_n > cv_n^\alpha | \Omega_{H_0}) \to \alpha$.

Step 2. We now consider the case with misspecification. By condition (iv), $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta^\dagger$ and, hence, by Theorem 1(a),

$$M_n(\hat{\theta}_n, \tau) \xrightarrow{\mathbb{P}} M(\theta^\dagger, \tau) \quad \text{uniformly.}$$

In restriction to Ω_{H_a}, under Assumption 8, there exists some $\tau \in \mathcal{T}$ such that $M(\theta^\dagger, \tau) \neq 0$ by Proposition 4 in Li, Todorov, and Tauchen (2016). Hence, $\sup_{\tau\in\mathcal{T}} |M_n(\hat{\theta}_n, \tau)| \xrightarrow{\mathbb{P}} \sup_{\tau\in\mathcal{T}} |M(\theta^\dagger, \tau)| > 0$. Moreover, we note that (A.96) is valid in restriction to Ω_{H_a} as well. Hence, the sequence $cv_n^\alpha = O_p(1)$. From here, we see that $\widehat{K}_n$ diverges to ∞ in probability and $\mathbb{P}(\widehat{K}_n > cv_n^\alpha | \Omega_{H_a}) \to 1$. *Q.E.D.*

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