Econometrica Supplementary Material

SUPPLEMENT TO "THE ROLE OF AUTOMATIC STABILIZERS IN THE U.S. BUSINESS CYCLE"

(*Econometrica*, Vol. 84, No. 1, January 2016, 141–194)

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Appendices covering: the construction of Table I, the derivation of the model equations, numerical methods, and error analysis.

APPENDIX C: FROM THE NIPA TABLES TO TABLE I

FOR EACH ENTRY IN TABLE I, we construct a sum of one or more entries in the NIPA tables, divide by nominal GDP, and average over 1988 to 2007. Here we describe the components of each entry in Table I.

C.1. *Revenues*

- *Personal income taxes* are the sum of federal and state income taxes (NIPA Table 3.4) plus contributions for government social insurance less contributions to retirement programs (Table 3.6, line 1 minus lines 4, 12, 13, 22, and 29).
- *Corporate income taxes* are from line 5 of Table 3.1.
- *Property taxes* are the sum of business property taxes (Table 3.5) and individual property taxes (Table 3.4).
- *Sales and excise taxes* are state sales taxes (Table 3.5) plus federal excise taxes (Table 3.5).
- *Public deficit* is the residual between the two columns of the table.
- *Customs taxes* are from Table 3.5, line 11.
- *Licenses, fines, fees* are the residual between current tax receipts from Table 3.1 and the other revenue listed in our table.
- *Payroll taxes* are contributions to retirement programs (Table 3.6, lines 4, 12, 13, 22, and 29).

C.2. *Outlays*

- *Unemployment benefits* are from Table 3.12, line 7.
- *Safety net programs* are the sum of the listed sub-components from Table 3.12, where "security income to the disabled" is the sum of lines 23, 29, and 36 and "Others" is the sum of lines 37–39.
- *Government purchases* are current consumption expenditure from Table 3.1.
- *Net interest income* is the difference between interest expense and interest and asset income both from Table 3.1.
- *Health benefits (nonretirement)* are spending on Medicaid (Table 3.12, line 33) multiplied by the share of Medicaid spending that was spent on children,

 DOI: 10.3982/ECTA11574

disabled, and nonelderly adults in 2007 plus other medical care (Table 3.12, line 34).[33]

- *Retirement-related transfers* are the share of Medicaid spent on the elderly plus Social Security, Medicare, pension benefit guarantees, and railroad retirement programs (all from Table 3.12).
- *Other outlays* are the difference between total outlays in Table 3.1 and those listed here.

APPENDIX D: DECISION PROBLEMS AND MODEL EQUATIONS

In this section of the appendix, we derive the optimality conditions that we use to compute the equilibrium of the model.

D.1. *Patient Household's Problem*

The patient household chooses $\{c_t, n_t\}$ to maximize expression (1) subject to equations (2) and (3). Define $\tilde{b}_t = b_t/p_t$ and $\pi_t = p_t/p_{t-1}$ and note that $\hat{p}_t/p_t = 1 + \tau^c$. Then we can rewrite the constraints as

$$(1+\tau^c)c_t + \tilde{b}_{t+1}\pi_{t+1} - \tilde{b}_t = x_t - \bar{\tau}^x(x_t) + T_t^e, \tag{29}$$

$$x_t = I_{t-1}\tilde{b}_t + w_t\bar{s}n_t + d_t. \tag{30}$$

Setting up the Lagrangian, with m_t^1 and m_t^2 as the Lagrange multipliers on constraints (29) and (30), respectively, the optimality conditions are

$$\beta^t c_t^{-1} = m_t^1(1+\tau^c),$$
$$m_t^1\pi_{t+1} = \mathbb{E}_t[m_{t+1}^1 + I_{t+1}m_{t+1}^2],$$
$$m_t^2 = m_t^1(1-\tau^x(x_t)),$$
$$\beta^t\psi_1 n_t^{\psi_2} = m_t^2 w_t\bar{s}.$$

These can be rearranged to give

$$\psi_1 n_t^{\psi_2} = \left(\frac{1}{c_t}\right)\left(\frac{1-\tau^x(x_t)}{1+\tau^c}\right)w_t\bar{s}, \tag{31}$$

$$\frac{1}{c_t} = \beta\mathbb{E}_t\left\{\frac{1+I_t(1-\tau^x(x_{t+1}))}{c_{t+1}\pi_{t+1}}\right\}, \tag{32}$$

[33]See Table 2 in the 2008 actuarial report of the Centers for Medicare and Medicaid Services (https://www.cms.gov/ActuarialStudies/downloads/MedicaidReport2008.pdf).

which are the patient household's labor-supply and Euler conditions. Finally, notice that the patient household's stochastic discount factor is

$$\lambda_{t,s} = \frac{m^2_{t+s}}{m^2_t} = \frac{\beta^s c^{-1}_{t+s}\left(1-\tau^x(x_{t+s})\right)}{c^{-1}_t\left(1-\tau^x(x_t)\right)}. \tag{33}$$

D.2. *Impatient Households' Problem*

The idiosyncratic state of a household is its real bond holdings $\tilde{b}$, its employment status e, and its skill level s. Let $\mathcal{S}$ be the collection of aggregate state variables. Then the problem of a household with real assets $\tilde{b}$ and labor market states e and s can be written as

$$V(\tilde{b}, e, s, \mathcal{S}) = \max_{c,n}\left\{\log(c) - \psi_1 \frac{n^{1+\psi_2}}{1+\psi_2} + \hat{\beta}\mathbb{E}V\left(\tilde{b}', e', s', \mathcal{S}'\right)\right\}$$

subject to

$$\begin{aligned}
&\left(1+\tau^c\right)c + \tilde{b}'\pi' - \tilde{b} = x - \bar{\tau}^x(x) + T^s(j),\\
&x = I(\mathcal{S}_{-1})\tilde{b} + s(j)w(\mathcal{S})n + T^u(j),\\
&n = 0 \quad \text{if} \quad e \neq 2,
\end{aligned}$$

where $I(\mathcal{S}_{-1})$ refers to the interest rate determined in the previous period. Here the expectation operator is over aggregate and idiosyncratic shocks. From this problem, one can derive a Euler equation and a labor supply condition that are analogous to those for the patient household's problem. One difference, however, is that, in these analogous expressions, the expectation operator reflects an expectation over idiosyncratic uncertainty as well as over aggregate uncertainty.

D.3. *Intermediate-Goods Firm*

A firm that sets its price at date t chooses $p^*_t, \{y_s(j), k_s(j), l_s(j)\}^\infty_{s=t}$ to solve

$$\begin{aligned}
&\max \mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s}(1-\theta)^{s-t}\left\{\left(1-\tau^k\right)\right.\\
&\quad \times \left.\left[\frac{p^*_t}{p_s} y_s(j) - w_s l_s(j) - (\upsilon r_s + \delta)k_s(j) - \xi\right] - (1-\upsilon) r_s k_s(j)\right\},
\end{aligned}$$

subject to

$$\begin{aligned}
&y_s(j) = \left(\frac{p^*_t}{p_s}\right)^{\mu/(1-\mu)} y_s,\\
&y_s(j) = a_s k_s(j)^\alpha l(j)^{1-\alpha},
\end{aligned}$$

where the first constraint is the demand for the firm's good and the second its production function. By defining $\hat{r}_t \equiv (1-\upsilon\tau^k)/(1-\tau^k)r_t$, we can rewrite the objective function as if all capital costs were deductible, but the cost of capital were higher ($\hat{r}_t > r_t$ if $\upsilon < 1$). Dropping the constant $1-\tau^K$ and substituting in the demand curve gives the modified problem:

$$\max_{p_t^*,\{k_s(j),l_s(j)\}_{s=t}^{\infty}} E_t \sum_{s=t}^{\infty} \left[\left(\frac{p_t^*}{p_s}\right)^{1/(1-\mu)} y_s - w_s l_s(j) - (\hat{r}_s+\delta)k_s(j) - \xi \right] \times \lambda_{t,s}(1-\theta)^{s-t}$$

subject to

$$\left(\frac{p_t^*}{p_s}\right)^{\mu/(1-\mu)} y_s = a_s k_s(j)^{\alpha} l(j)^{1-\alpha}.$$

The first-order conditions with respect to $k_s(j)$ and $l_s(j)$ are

$$(\hat{r}_s+\delta) = M_s \alpha a_s k_s(j)^{\alpha-1} l_s(j)^{1-\alpha}, \tag{34}$$

$$w_s = M_s(1-\alpha)a_s k_s(j)^{\alpha} l_s(j)^{-\alpha}, \tag{35}$$

where M_s is the Lagrange multiplier on the production function constraint at date s, which is real marginal cost at date s.

We can derive several useful features of the solution from these two optimality conditions. First, taking their ratio:

$$\frac{w_s}{\hat{r}_s+\delta} = \frac{1-\alpha}{\alpha}\frac{k_s(j)}{l_s(j)},$$

so that all firms have the same capital–labor ratio and, by market clearing, $k_s(j)/l_s(j) = k_s/l_s$ for all firms.

Second, these optimality conditions allow us already to derive the expression for dividends as a function of factor prices. Total factor payments are

$$(\hat{r}_s+\delta)k_s = M_s \alpha a_s k_s^{\alpha} l_s^{1-\alpha}, \tag{36}$$

$$w_s l_s = M_s(1-\alpha) a_s k_s^{\alpha} l_s^{1-\alpha}. \tag{37}$$

The aggregate after-tax dividend of the intermediate goods firms is then

$$\int_0^1 d_t^i(j)\,dj = \left(1-\tau^k\right)\int_0^1 \left[\frac{p_t(j)}{p_t} y_t(j) - w_t(j)l_t(j) - (\hat{r}_t+\delta)k_t(j) - \xi\right] dj,$$

and by market clearing, this becomes

$$\int_0^1 d_t^i(j)\,dj = \left(1-\tau^k\right)\left[y_t - M_t a_t k_t^{\alpha} l_t^{1-\alpha} - \xi\right]. \tag{38}$$

Similarly, total profits are

$$\begin{aligned}
&\int_0^1 \left[\frac{p_t(j)}{p_t} y_t(j) - w_t l_t(j) - (r_t+\delta)k_t(j) - \xi\right] dj \\
&\quad = \int_0^1 \left[\frac{p_t(j)}{p_t} y_t(j) - w_t l_t(j) - (\hat{r}_t+\delta+r_t-\hat{r}_t)k_t(j) - \xi\right] dj \\
&\quad = y_t - M_t a_t k_t^{\alpha} l_t^{1-\alpha} - \xi + \tau^k\left(\frac{1-\upsilon}{1-\upsilon\tau^k}\right)\hat{r}_t k_t.
\end{aligned} \tag{39}$$

And revenue from the corporate income tax is the difference between (39) and (38).

Finally, we turn to the optimality condition with respect to p_t^*:

$$\begin{aligned}
&\mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s}(1-\theta)^{s-t} \\
&\quad \times \left[\frac{1}{1-\mu}\left(\frac{p_t^*}{p_s}\right)^{1/(1-\mu)-1} \frac{y_s}{p_s} - M_s \frac{\mu}{1-\mu}\left(\frac{p_t^*}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s}\right] = 0,
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
&\mathbb{E}_t \sum_{s=t}^{\infty} \frac{1}{1-\mu}\left(\frac{p_t^*}{p_s}\right)^{1/(1-\mu)-1} \frac{y_s}{p_s}\lambda_{t,s}(1-\theta)^{s-t} \\
&\quad = \mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s}(1-\theta)^{s-t} M_s \frac{\mu}{1-\mu}\left(\frac{p_t^*}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s}, \\
&\frac{p_t^*}{p_t} = \frac{p_t \mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s}(1-\theta)^{s-t} M_s \mu_t \left(\frac{p_t}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s}}{p_t \mathbb{E}_t \sum_{s=t}^{\infty} \left(\frac{p_t}{p_s}\right)^{\mu/(1-\mu)} \frac{y_s}{p_s}\lambda_{t,s}(1-\theta)^{s-t}} \\
&\quad \equiv \frac{\bar{p}_t^A}{\bar{p}_t^B}.
\end{aligned} \tag{40}$$

This equation gives the solution for p_t^*. It is useful to write $\bar{p}_t^A$ and $\bar{p}_t^B$ recursively. To that end,

$$\begin{aligned}(41)\qquad \bar{p}_t^A &= p_t \mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s}(1-\theta)^{s-t} M_s \mu_t \left(\frac{p_t}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s} \\ &= M_t \mu_t y_t + \mathbb{E}_t p_{t+1} \pi_{t+1}^{-1} \mathbb{E}_{t+1} \lambda_{t,t+1}(1-\theta)\left(\frac{p_t}{p_{t+1}}\right)^{\mu/(1-\mu)-1} \\ &\quad \times \sum_{s=t+1}^{\infty} \lambda_{t+1,s}(1-\theta)^{s-t-1} M_s \mu \left(\frac{p_{t+1}}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s} \\ &= M_t \mu_t y_t + \mathbb{E}_t\left[\lambda_{t,t+1}(1-\theta)\pi_{t+1}^{-\mu/(1-\mu)} \bar{p}_{t+1}^A\right],\end{aligned}$$

where $\pi_{t+1} \equiv p_{t+1}/p_t$. Similar logic for $\bar{p}_t^B$ yields

$$(42)\qquad \bar{p}_t^B = y_t + \mathbb{E}_t\left[\lambda_{t,t+1}(1-\theta)\pi_{t+1}^{-\mu/(1-\mu)-1} \bar{p}_{t+1}^B\right].$$

Next comes the relationship between p_t^* and inflation. The price index is

$$p_t = \left(\int_0^1 p_t(j)^{1/(1-\mu)}\, dj\right)^{1-\mu},$$

and with Calvo pricing, we have

$$\begin{aligned}p_t &= \left((1-\theta)\int_0^1 \big(p_{t-1}(j)\big)^{1/(1-\mu)}\, dj + \theta\big(p_t^*\big)^{1/(1-\mu)}\right)^{1-\mu} \\ &= \left((1-\theta) p_{t-1}^{1/(1-\mu)} + \theta\big(p_t^*\big)^{1/(1-\mu)}\right)^{1-\mu}.\end{aligned}$$

Therefore

$$(43)\qquad \pi_t = \left(\frac{1-\theta}{1-\theta\left(\frac{p_t^*}{p_t}\right)^{1/(1-\mu)}}\right)^{1-\mu}.$$

Finally, note that because the capital–labor ratio is constant across firms, the production of variety j follows:

$$y_t(j) = a_t\left(\frac{k_t}{l_t}\right)^{\alpha} l_t(j).$$

The demand for variety j can be written in terms of the relative price to arrive at

$$\left(\frac{p_t(j)}{p_t}\right)^{\mu/(1-\mu)} y_t = a_t\left(\frac{k_t}{\ell_t}\right)^{\alpha} \ell_t(j).$$

Integrating both sides yields

$$\int_0^1 \left(\frac{p_t(j)}{p_t}\right)^{\mu/(1-\mu)} dj y_t = a_t\left(\frac{k_t}{\ell_t}\right)^{\alpha} \int_0^1 \ell_t(j)\, dj.$$

By market clearing, we have then that

$$S_t y_t = a_t k_t^{\alpha} \ell_t^{1-\alpha}, \tag{44}$$

where

$$S_t = \int_0^1 \left(\frac{p_t(j)}{p_t}\right)^{\mu/(1-\mu)} dj.$$

S_t reflects the efficiency loss due to price dispersion and it evolves according to

$$S_t = (1-\theta)S_{t-1}\pi_t^{-\mu/(1-\mu)} + \theta\left(\frac{p_t^*}{p_t}\right)^{\mu/(1-\mu)}. \tag{45}$$

Throughout this subsection, we have dropped most of the t subscripts on μ_t. When the equations in this subsection are linearized around the zero-inflation steady state, the markup shock only enters equation (41).

D.4. *Capital Goods Firm*

The capital goods firm chooses a sequence $\{k_{t+1}, k_{t+2}, \dots\}$ to maximize

$$\mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s}\left(1+\tau^p\right)^{-(s-t+1)}\left[r_s k_s - k_{s+1} + k_s - \frac{\zeta}{2}\left(\frac{k_{s+1}-k_s}{k_s}\right)^2 k_s\right].$$

The discounting by $1/(1+\tau^P)$ comes from the property tax since

$$v_t = \frac{1}{1+\tau^P} d_t^k + \frac{1}{1+\tau^P}\mathbb{E}_t[\lambda_{t,t+1}v_{t+1}]. \tag{46}$$

This problem leads to the first-order condition

$$1+\zeta\left(\frac{k_{t+1}-k_t}{k_t}\right) = \mathbb{E}_t\left\{\frac{\lambda_{t,t+1}}{1+\tau^p}\left[r_{t+1} + 1 - \frac{\zeta}{2}\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right)^2 + \zeta\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}}\right)\frac{k_{t+2}}{k_{t+1}}\right]\right\}. \tag{47}$$

This expression can be transformed into one that only includes variables dated t and $t+1$ by writing it in terms of $\hat{k}_t = k_{t+1}$ and introducing $\hat{k}_t^{\text{lag}} = \hat{k}_{t-1}$. Dividends paid by the capital goods firm are the term in brackets in the objective function less τ^p times the value of the firm, which follows equation (46).

APPENDIX E: NUMERICAL SOLUTION ALGORITHM

As the main text described, the key steps involved in solving the model are: (i) to discretize the cross-sectional distributions and decision rules, (ii) to solve for the stationary equilibrium, (iii) to collect all of the many equations defining an approximate equilibrium and linearizing them, and (iv) to solve the system with a linear rational expectations solver. We elaborate on each of these steps next.

E.1. *Discretizing the Model*

For each discrete type of impatient household characterized by a skill level and an employment status, we approximate the distribution of wealth by a histogram with 250 bins. We approximate the policy rules for savings and labor supply by two piece-wise linear splines with 100 knot points each. We deal with the borrowing constraint in the approximation of the policy functions by, following Reiter (2010), parameterizing the point at which the borrowing constraint is just binding, and then constructing a grid for higher levels of assets. As a result of these approximations, there are now 450 variables for employed workers, and 350 variables for nonemployed workers (who do not choose labor supply).

E.2. *Solving for the Stationary Equilibrium*

Solving for the steady state of the model requires two steps: first, solving for the impatient household policy rules and distribution of wealth; and second, solving for the aggregate variables including the assets and consumption of the representative patient household. These two steps are interrelated, as the equilibrium interest rate depends on the patient household's marginal tax rate, which depends on the patient household's income and therefore wealth, which in turn depends on the level of wealth held by impatient households.

We use an iterative procedure to find the equilibrium income of the patient households. Given a guess of the patient household's income and therefore marginal tax rate, we find the equilibrium interest rate from the patient household's Euler equation and then the solution of the intermediate goods firm's problem to find the equilibrium wage. With these objects, we solve the impatient households' problem to find their consumption and asset positions. With these in hand, we use standard techniques from the analysis of representative-agent models to find the rest of the aggregate variables. Finally, we check our guess of the patient household's income and iterate from here.

E.3. *System of Equations*

Keeping Track of the Wealth Distribution

We track real assets at the beginning of the period using Reiter's (2010) procedure to allocate impatient households to the discrete grid in a way that preserves total assets. As we have nominal bonds in the model, we account for the effect of inflation in the evolution of the household's asset position. For each discrete type of household, this provides 250 equations.

Solving for Household Decision Rules

We use the impatient households' Euler equations and labor supply conditions to solve for their decision rules by imposing that these equations hold with equality at the spline knot points. This provides 100 equations for nonemployed households and 200 for employed households.

Aggregate Equations

In addition to those equations that relate to the solution of the impatient household's problem and the distribution of wealth across households, we have equations that correspond to the patient household's savings and labor supply decisions, as well as those that correspond to the firms' problems. These equations are discussed in more detail in Appendix B. We use equations (29), (31), (32), (40), (41), (42), (43), (36), (37), (44), (45), (46), (47). We introduce an auxiliary variable that carries an extra lag of capital, $k_t^{\text{lag}} = k_{t-1}$. In addition, from the main text we have equations (22), (23), (24), (26), (25), (27), and exogenous AR(1) processes for ε_t, a_t, and μ_t. We use these equations to solve for c_t, n_t, b_t M_t, p_t^*/p_t, $\bar{p}_t^A$, $\bar{p}_t^B$, S_t, π_t, y_t, w_t, r_t, v_t, k_t, k_t^{lag}, d_t, B_t, T_t^e, g_t, and I_t.

E.4. *Linearization and Solution*

At this stage, we have a large system of nonlinear equations that the discretized model must satisfy. We follow Reiter (2009, 2010) in linearizing this system around the stationary equilibrium using automatic differentiation and then solving the linearized system as a linear rational expectations model using the algorithm from Sims (2002).

APPENDIX F: NUMERICAL ERROR ANALYSIS

Here we discuss the accuracy of our numerical calculations for the main results, in Section 4. There are two sources of errors, both of which commonly arise in related algorithms. First, there are errors in the decision rules of the impatient households between the points at which the household optimality conditions are imposed. These errors are present even in the stationary equilibrium. Away from the stationary equilibrium (the point around which we linearize), there are errors due to nonlinear responses to aggregate states, as is the case with other applications of perturbation methods.

To assess the accuracy of our solution, we calculate unit-free Euler equation errors.[34] We calculate the Euler equation errors for the patient household as well as for impatient households. For impatient households, we use a test grid over asset holdings that is finer than the grid on which we solve for household decision rules.[35] For a given aggregate state of the economy, $\mathcal{S}_t$, the distribution of bond holdings, the capital stock, and exogenous variables are predetermined:

Predetermined and exogenous:

$k_t, B_t, b_t, a_t, \varepsilon_t, \mu_t$, distribution of households.

We then use the computed solutions to determine

Approx. solutions:

$M_t, c_t, v_t, \bar{p}_t^A, \bar{p}_t^B$, impatient hhld. savings and labor supply rules.

We then use the nonlinear, static relationships and market clearing conditions to determine the remaining variables. Table S.I lists the equations we impose

TABLE S.I

EQUATIONS THAT HOLD EXACTLY IN ERROR ANALYSIS[a]

Description	Number	Variable(s) Determined
Price-setting equations	(40), (43), (45)	$p_t^*/p_t, \pi_t, S_t$
Production function	(14)	y_t
Firm F.O.C.s and definition of $\hat{r}$	(34), (35)	$\hat{r}_t, r_t, w_t$
Government policy rules	(25), (26), (27)	I_t, T_t^e, g_t
Impatient budget constraints	(6)	$c_t(i)$
Aggregate resource constraint	(48)	k_{t+1}
Accounting definitions	(3), (7), (16), (19), (22)	$x_t, x_t(i), d_t(j), d_t^k, d_t$
Progressive tax rule	(4)	Tax rev. and marginal rates
Government budget constraint	(23)	B_{t+1}
Bond market clearing	(24)	b_{t+1}

[a]Due to the capital adjustment cost, there are two values of k_{t+1} that solve the aggregate resource constraint, the relevant solution is the larger of the two.

[34]See Judd (1992) for an explanation of this accuracy check and the interpretation of the errors in terms of bounded rationality.

[35]Specifically, we use the same 250 point grid for $b(i)$ as we use to approximate the distribution of wealth.

and the variables that we solve for. In addition to those equations listed, the patient household budget constraint, equation (2), holds by Walras's Law. The aggregate resource constraint is

$$k_{t+1} + c_t + \int_0^{\nu} c_t(i)\,di + g_t = y_t + (1-\delta)k_t - \xi - \frac{\zeta}{2}\left(\frac{k_{t+1}-k_t}{k_t}\right)^2 k_t. \tag{48}$$

Notice that all budget constraints and market clearing conditions are forced to hold. From these calculations and a given set of aggregate shocks, we can compute the next state of the economy, $\mathcal{S}_{t+1}$, and repeat these steps to find c_{t+1}, and so on. To compute expectations, we use Gaussian quadrature over the three aggregate shocks using a grid that has 11 nodes in each dimension. For a given household (i.e., a patient household or an impatient household with particular idiosyncratic states), we can compute the level of consumption implied by the right-hand side of the Euler equation as

$$\hat{c}_t \equiv \left[\beta \mathbb{E}_t \left\{ \frac{1 + I_t\big(1 - \tau^x(x_{t+1})\big)}{c_{t+1}\pi_{t+1}} \right\}\right]^{-1}, \tag{49}$$

where the expectation is over aggregate and idiosyncratic shocks in the case of impatient households. The unit-free Euler equation error for a given type of household is then $\hat{c}_t/c_t - 1$, where c_t is the level of consumption implied by the approximated decision rules.[36] Here we have used the Euler equation for bond holdings, which is the relevant Euler equation for impatient households. For patient households, we could alternatively use equation (47) to construct $\hat{c}$. We will refer to these two versions as the "bond" error and the "investment" error.

Using the steps above, we can compute the Euler equation error for each type of household. As a summary statistic, we integrate $\hat{c}$ across households using the distribution of wealth at the given state of the economy to compute aggregate consumption implied by the right-hand side of the Euler equation. We similarly can integrate the consumptions implied by the approximate policy rules to find aggregate consumption as implied by the left-hand side of household Euler equations. We can then express an aggregate Euler equation error for all impatient households as $\int \hat{c}(i)\,di / \int c(i)\,di - 1$ and an aggregate Euler equation error for all households as $[\int \hat{c}(i)\,di + \hat{c}]/[\int c(i)\,di + c] - 1$, where c is the consumption of the patient households and $\hat{c}$ can be calculated from either (47) or (49). We choose to focus on these aggregate Euler equation errors as opposed to the disaggregated errors for each type of household because this is what is relevant to our results on aggregate dynamics. Nonetheless, the

[36]For impatient households, we approximate their policy rules for savings as opposed to consumption, so c is computed from their budget constraint and depends on the approximate policy rule for labor supply and the market clearing prices.

TABLE S.II

LARGEST AND MEAN ABSOLUTE ERRORS ACROSS 50 RANDOMLY DRAWN POINTS IN THE STATE SPACE

	Euler Equation Errors (Log Base 10)				
	Patient			Aggregate	
	Investment	Bond	Impatient	Investment	Bond
Largest	−2.15	−2.09	−2.05	−2.50	−2.08
Mean	−3.10	−2.78	−2.78	−3.08	−2.80

	Labor Supply Errors (Log Base 10)		
	Patient	Impatient	Aggregate
Largest	−3.19	−3.14	−3.15
Mean	−3.88	−3.82	−3.86

disaggregated Euler errors do not show large differences in magnitude across households.

We can also assess the errors in household labor supply decision rules. In the course of the steps listed above, we have solved for everything on the right-hand side of equation (31) and the analogous equations for impatient households. Specifically, we use the approximate solutions to find c and the value of x that follows from plugging the approximate policy rule for n into equation (3). We can then solve for the implied value of n, call it $\hat{n}$, from the right-hand side of equation (31) and express the error in this equation as $\hat{n}/n - 1$. Again, we summarize these errors by integrating n and $\hat{n}$ over the distribution of households.

The Euler equation and labor supply errors vary over the state space. We randomly draw points in the state space by simulating the model for 50,000 periods, and we compute the errors every 1000 simulated periods. We describe the distribution of errors across the 50 resulting points by reporting the largest absolute error and the mean absolute error in Table S.II.

APPENDIX G: METHODS FOR TRANSITION DYNAMICS

In Sections 5 and 6, we discuss perfect foresight transition experiments. Unlike above, we do not linearize the model equations, but instead compute the transition using the fully nonlinear model equations.

Initial Guess

We assume that the economy has returned to steady state after $T = 250$ periods and look for equilibrium values for endogenous variables between dates

$t = 0$ and T. In this explanation of our methods, we use variables without subscripts to represent sequences from 0 to T. Let X denote a path for all endogenous aggregate variables from date 0 to date T. These variables include aggregate quantities and prices. Specifically, in Appendix E.3 under the heading "Aggregate Equations," we list 20 endogenous aggregate variables and three exogenous variables. In addition to those, we also include the exogenous path for the preference shock for the zero-lower-bound experiment. In addition, let $c_t^I \equiv \int_0^\nu c_t(i)\,di$ be the aggregate consumption of impatient households. Define b_t^I similarly as the aggregate bond holding of the impatient households and $n_t^I \equiv \int_0^\nu s_t(i) n_t(i)\,di$ as the aggregate effective labor supply. Finally, define $T_t^I \equiv \int \bar{\tau}^x(x_t(i))\,di$ as the aggregate income tax payment of the impatient households. X contains the sequences $\{c_t^I, n_t^I, b_t^I, T_t^I\}_{t=0}^T$. Importantly, X does not include the distribution of wealth or the household decision rules. So in total, X represents time paths for 28 variables, four of which are exogenous.

We require an initial guess X^0. We start with a scaled down version of the exogenous variables so that they differ from their steady-state values by a small amount. For this starting point, the steady-state values for quantities and prices are good initial guesses. After computing an equilibrium for this scaled down problem, we then gradually scale up the exogenous variables to the full version of the transition experiment.

Solving the Household's Problem

The impatient household's decision problem depends on X through the prices. For a given X^i, we solve the household's problem using the endogenous gridpoint method (Carroll (2006)).

Simulating the Population of Households

We simulate the population of households in order to compute aggregate consumption and aggregate labor supply. We use a nonstochastic simulation method. We approximate the distribution of wealth with a histogram with 250 unequally spaced wealth levels for each value of (e, s), placing more bins at low asset levels. We then update the distribution of wealth according to the household savings policies and the exogenous transitions across skill and employment states. When households choose levels of savings between the center of two bins, we allocate these households to the adjacent bins in a way that preserves total savings. See Young (2010) for a description of nonstochastic simulation in this manner.

Checking the Equilibrium Conditions

In Appendix E.3 under the heading "Aggregate Equations," we list 20 equations.[37] These equations depend on the distribution of wealth and the impatient household decision rules only through c^I, n^I, b^I, and T_t^I. So we can directly check whether these 20 equations hold at X. In addition, we need to verify that c^I, n^I, b^I, T^I are part of an equilibrium. This requires solving for the household decision rules and simulating the population of impatient households. We do that using the methods described in the previous two paragraphs and check whether the aggregate behavior of impatient households that is implied matches the values listed in X.

Updating X^i

The difficult part of the solution method arises when our guess X^i is not an equilibrium. In this case, we need to find a new guess X^{i+1} that moves us towards an equilibrium. To do this, we construct an auxiliary model by replacing the computational equilibrium conditions with additional analytical equilibrium conditions that approximate the behavior of the population of impatient households but are easier to analyze. Specifically, we use the equations

$$(c_t^I)^{-\gamma} = \eta_t^1 \beta (1 + r_t)(c_{t+1}^I)^{-\gamma}, \tag{50}$$

$$(c_t^I)^{-\gamma} w_t = \eta_t^2 (n_t^I)^{\psi}, \tag{51}$$

$$T_t^I = \eta_t^3 (w_t n_t^I + I_{t-1} b_t^I / \pi_t + T^{u,I} U_t), \tag{52}$$

$$(1 + \tau^c) c_t^I + b_{t+1}^I = (1 + I_{t-1}) b_t / \pi_t + w_t n_t^I + T^{u,I} U_t + T^{n,I} N_t - \tau_t^I, \tag{53}$$

where η^1, η^2, and η^3 are treated as parameters of the auxiliary model, and U_t and N_t are masses of unemployed and needy households. $T^{u,I}$ and $T^{n,I}$ are parameters that determine the aggregate transfer payments as a function of the number of unemployed and needy. These are constant parameters, as the distribution of skills is stationary. U_t and N_t evolve exogenously in line with the Markov chain transition matrix Π_t.

For a given X^i, we have computed c^I, n^I, b^I, and T^I from the computational equilibrium conditions. We then calibrate η^1, η^2, η^3 from the above equations. We then solve for a new value of X from the 20 analytical equilibrium conditions and this system of equations. This is a problem of solving for 26 unknowns at each date from 26 nonlinear equations at each date.[38] We solve this system using the method described by Juillard (1996) for computing perfect foresight

[37]For the zero-lower-bound experiment, we modify equation (27) such that the nominal interest rate is the maximum of the value implied by the Taylor rule and zero.

[38]There are 26 equations and endogenous variables as opposed to 24 because we include N and U and the associated equations.

transition paths for nonlinear models. This method is a variant of Newton's method that exploits the sparsity of the Jacobian matrix. Call this solution $X^{i'}$. We then form X^{i+1} by updating partially from X^i towards $X^{i'}$.

In essence, we are computing an equilibrium as if there were a representative impatient household whose behavior were described by a standard Euler equation with the wedge η^1, a standard labor supply equation with the wedge η^2, and so on. We use this equilibrium under a representative impatient household to construct our next guess X^{i+1}. We iterate on these steps until the values η^1, η^2, and η^3 converge.

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Manuscript received April, 2013; final revision received July, 2015.