ESTIMATION OF NONPARAMETRIC MODELS WITH SIMULTANEITY

Supplementary Material

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Abstract

This supplement presents the statements and proofs of the lemmas that are used in the Appendix of the paper. Lemmas A.1-A.6 are used in the proofs of Theorems 2.4 and 3.2. Lemma B.1 is used in the proof of Theorem 3.1.

Lemmas

Lemma A.1: *Let* $\overline{M}^y$ and $\overline{M}^x$ *be compact and convex sets such that* $\overline{M} = \{y\} \times \overline{M}^t$ *is strictly included in the interior of* $\overline{M}^y \times \overline{M}^x$. *Let* F *denote the set of bounded and continuously differentiable functions* g *on the extension of the support of* (Y,X) *to the whole space and such that the function* $\widetilde{g}$ *defined by* $\widetilde{g}(x) = \int |g(y,x)|\ dy$ *is bounded and continuously differentiable. For all functions* g *in* F, *denote by* $\|g\|$ *the sum of the sup norms of the values and derivatives of* g *and of* $\widetilde{g}$ *on* $\overline{M}^y \times \overline{M}^x$. *For any* $(y,x) \in \overline{M}^y \times \overline{M}^x$, *define the functionals* $\alpha_{y_j}(\cdot)$ *and* $\beta_{x_s}(\cdot)$ *on* F *by* $\alpha_{y_j}(g) = \partial \log g_{Y|X=x}(y)/\partial y_j$ *and* $\beta_{x_s}(g) = \partial \log g_{Y|X=x}(y)/\partial x_s$. *For simplicity we leave the argument* (y,x) *implicit. Assume that the density* $f_{Y,X}$ *belongs to* F *and it is such that for* $\delta > 0$ *and all* $(y,x) \in \overline{M}^y \times \overline{M}^x$, $f_{Y,X}(y,x) > \delta$ *and* $f_X(x) > \delta$. *Then, there exists finite* $a > 0$ *and* $\widetilde{\delta}_0 > 0$ *such that for all* $h \in$F *with* $\|h\| \leq \widetilde{\delta}_0$ and all $(y,x) \in \overline{M}^y \times \overline{M}^x$

$$\alpha_{y_j}(f+h) - \alpha_{y_j}(f) = D\alpha_{y_j}(f;h) + R\alpha_{y_j}(f;h),$$

$$\text{and} \qquad \beta_{x_s}(f+h) - \beta_{x_s}(f) = D\beta_{x_s}(f;h) + R\beta_{x_s}(f;h).$$

where

$$D\alpha_{y_j}(f;h) = \frac{\left[h_{y_j} f - f_{y_j} h\right]}{f^2} \quad ; \quad R\alpha_{y_j}(f;h) = -\frac{h\left[h_{y_j} f - f_{y_j} h\right]}{f^2 (f+h)}$$

$$D\beta_{x_s}(f;h) = \left[\frac{\left[h_{x_s} f - f_{x_s} h\right]}{f^2} - \frac{\left[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}\right]}{\widetilde{f}^2}\right] \quad ;$$

$$R\beta_{x_s}(f;h) = -\left[\frac{h\left[h_{x_s} f - f_{x_s} h\right]}{f^2(f+h)} - \frac{\widetilde{h}\left[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}\right]}{\widetilde{f}^2(\widetilde{f}+\widetilde{h})}\right].$$

Moreover,

$$\left|D\alpha_{y_j}\left(f;h\right)\right| \le a\left\|h\right\|;\ \left|D\beta_{x_s}\left(f;h\right)\right| \le a\left\|h\right\|;\ \left|R\alpha_{y_j}\left(f;h\right)\right| \le a\left\|h\right\|^2;\ \textit{and}\ \left|R\beta_{x_s}\left(f;h\right)\right| \le a\left\|h\right\|^2.$$

Proof: To simplify notation, we will denote $f_{Y,X}(y,x)$ by f, $f_X(x)$ by $\widetilde{f}$, $\partial f_{Y,X}(y,x)/\partial y_j$ by f_{y_j}, $\partial f_{Y,X}(y,x)/\partial x_s$ by f_{x_s}, and $\partial f_X(x)/\partial x_s$ by $\widetilde{f}_{x_s}$, with similar shorthands for functions g and h. By the definitions of $\alpha_{y_j}(g)$ and $\beta_{x_s}(g)$,

$$\alpha_{y_j}(g) = \frac{\frac{\partial g_{Y,X}(y,x)}{\partial y_j}}{g_{Y,X}(y,x)} = \frac{g_{y_j}}{g} \text{ and } \beta_{x_s}(g) = \left(\frac{\frac{\partial g_{Y,X}(y,x)}{\partial x_s}}{g_{Y,X}(y,x)} - \frac{\frac{\partial g_X(x)}{\partial x_s}}{g_X(x)}\right) = \frac{g_{x_s}}{g} - \frac{\widetilde{g}_{x_s}}{\widetilde{g}}.$$

Let $\widetilde{\delta}_0 = \min\{\delta/2, 1\}$. Then, for all $h \in \mathtt{F}$ such that $\|h\| \le \widetilde{\delta}_0$, one has that $|h| < \delta/2$ and $\left|\widetilde{h}\right| < \delta/2$. Since $f > \delta$ and $\widetilde{f} > \delta$, it follows that $(f+h) > \delta/2$ and $\left(\widetilde{f}+\widetilde{h}\right) > \delta/2$. By rearranging terms, it follows that

$$\begin{aligned}
&\alpha_{y_j}(f+h) - \alpha_{y_j}(f)\\
&= \left[\frac{f_{y_j}+h_{y_j}}{f+h} - \frac{f_{y_j}}{f}\right] = \left[\frac{\left[h_{y_j}f - f_{y_j}h\right]}{f^2} - \frac{h\left[h_{y_j}f - f_{y_j}h\right]}{f^2(f+h)}\right]\\
&= \frac{\left[h_{y_j}f - f_{y_j}h\right]}{f^2} - \frac{h\left[h_{y_j}f - f_{y_j}h\right]}{f^2(f+h)}\\
&= D\alpha_{y_j}\left(f;h\right) + R\alpha_{y_j}\left(f;h\right) \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\beta_{x_s}(f+h) - \beta_{x_s}(f) &= \left[\frac{f_{x_s}+h_{x_s}}{f+h} - \frac{f_{x_s}}{f}\right] - \left[\frac{\widetilde{f}_{x_s}+\widetilde{h}_{x_s}}{\widetilde{f}+\widetilde{h}} - \frac{\widetilde{f}_{x_s}}{\widetilde{f}}\right]\\
&= \left[\frac{\left[h_{x_s}f - f_{x_s}h\right]}{f^2} - \frac{h\left[h_{x_s}f - f_{x_s}h\right]}{f^2(f+h)}\right] - \left[\frac{\left[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}\right]}{\widetilde{f}^2} - \frac{\widetilde{h}\left[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}\right]}{\widetilde{f}^2(\widetilde{f}+\widetilde{h})}\right]\\
&= \left[\frac{\left[h_{x_s}f - f_{x_s}h\right]}{f^2} - \frac{\left[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}\right]}{\widetilde{f}^2}\right] - \left[\frac{h\left[h_{x_s}f - f_{x_s}h\right]}{f^2(f+h)} - \frac{\widetilde{h}\left[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}\right]}{\widetilde{f}^2(\widetilde{f}+\widetilde{h})}\right]\\
&= D\beta_{x_s}\left(f;h\right) + R\beta_{x_s}\left(f;h\right)
\end{aligned}$$

where the last equalities in each of the two expressions above follow by the definitions of $D\alpha_{y_j}\left(f;h\right)$, $R\alpha_{y_j}\left(f;h\right)$, $D\beta_{x_s}\left(f;h\right)$, and $R\beta_{x_s}\left(f;h\right)$ in the statement of the Lemma.

It remains to show that for some finite $a > 0$, which does not depend on (y,x)

$$\left|D\alpha_{y_j}\left(f;h\right)\right| \le a\left\|h\right\|;\ \left|D\beta_{x_s}\left(f;h\right)\right| \le a\left\|h\right\|;\ \left|R\alpha_{y_j}\left(f;h\right)\right| \le a\left\|h\right\|^2;\ \text{and}\ \left|R\beta_{x_s}\left(f;h\right)\right| \le a\left\|h\right\|^2.$$

Let $a = \max\left\{4\|f\|/\left(\delta^2\right), 4\|f\|/\left(\delta^3\right)\right\}$. By the definition of $\|f\|$ and $\|h\|$, it follows that $|f|, |f_{y_j}|, |f_{x_s}|, |\widetilde{f}|$, and $|\widetilde{f}_{x_s}|$ are not larger than $\|f\|$ and that $|h|, |h_{y_j}|, |h_{x_s}|, |\widetilde{h}|$, and $|\widetilde{h}_{x_s}|$ are not larger than $\|h\|$. Moreover, as was shown above, for all h such that $\|h\| < \delta/2$, $(f+h)$ and $\left(\widetilde{f}+\widetilde{h}\right)$ are larger than $\delta/2$. By the assumption that $f, \widetilde{f} > \delta$, it then follows that

$$\left|D\alpha_{y_j}(f;h)\right| = \left|\frac{\left[h_{y_j} f - f_{y_j} h\right]}{f^2}\right| \leq \frac{2\|f\|}{\delta^2}\|h\| \leq a\|h\|$$

$$\left|R\alpha_{y_j}(f;h)\right| = \left|\frac{h\left[h_{y_j} f - f_{y_j} h\right]}{f^2(f+h)}\right| \leq \frac{2\ \|f\|}{\delta^3}\|h\|^2 \leq a\|h\|^2$$

$$\left|D\beta_{x_s}(f;h)\right| = \left|\frac{\left[h_{x_s} f - f_{x_s} h\right]}{f^2} - \frac{\left[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}\right]}{\widetilde{f}^2}\right| \leq \left(\frac{4\|f\|}{\delta^2}\right)\|h\| \leq a\|h\|, \text{ and}$$

$$\left|R\beta_{x_s}(f;h)\right| = \left|\frac{h\left[h_{x_s} f - f_{x_s} h\right]}{f^2(f+h)} - \frac{\widetilde{h}\left[\widetilde{h}_{x_s}\widetilde{f} - \widetilde{f}_{x_s}\widetilde{h}\right]}{\widetilde{f}^2(\widetilde{f}+\widetilde{h})}\right| \leq \frac{4\ \|f\|}{\delta^3}\|h\|^2 \leq a\|h\|^2.$$

This completes the proof of the Lemma.

Lemma A.2: *Let $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\|\cdot\|$, $\alpha_{y_j}(\cdot)$ and $\beta_{x_s}(\cdot)$ be as in the statement of Lemma A.1. Assume that the density $f_{Y,X}$ belongs to $\mathtt{F}$ and it is such that for $\delta > 0$ and all $(y,x) \in \overline{M}^y \times \overline{M}^x$, $f_{Y,X}(y,x) > \delta$ and $f_X(x) > \delta$. Let μ denote a bounded, nonnegative, continuously differentiable function, with values equal to zero when any coordinate of $x \in \overline{M}^t$ is on the boundary of $\overline{M}^t$, positive values when x is in the interior of $\overline{M}^t$, and such that $\int_{\overline{M}} \mu(y,x)\, dx = 1$. For simplicity we will leave the argument (y,x) implicit. Define the functional Φ_{y_j,x_s} on F by*

$$\Phi_{y_j,x_s}(g) = \int_{\overline{M}} \alpha_{y_j}(g)\,\beta_{x_s}(g)\,\mu(y,x)\,dx - \left(\int_{\overline{M}} \alpha_{y_j}(g)\,\mu(y,x)\,dx\right)\left(\int_{\overline{M}} \beta_{x_s}(g)\,\mu(y,x)\,dx\right).$$

Then, there exist finite $b_1 > 0$ and $\widetilde{\delta}_1 > 0$ such that for all $h \in \mathtt{F}$ with $\|h\| \leq \widetilde{\delta}_1$

$$\Phi_{y_j,x_s}(f+h) - \Phi_{y_j,x_s}(f) = D\Phi_{y_j,x_s}(f;h) + R\Phi_{y_j,x_s}(f;h)$$

where, employing the notation of Lemma A.1 for the definitions of $D\alpha_{y_j}(f;h)$, $R\alpha_{y_j}(f;h)$, $D\beta_{x_s}(f;h)$, and $R\beta_{x_s}(f;h)$,

$$D\Phi_{y_j,x_s}(f;h) = \int_{\overline{M}} D\alpha_{y_j}(f;h)\left(\beta_{x_s}(f) - \int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\ \ dx$$

$$+ \int_{\overline{M}} D\beta_{x_s}(f;h)\ \left(\alpha_{y_j}(f) - \int_{\overline{M}}\alpha_{y_j}(f)\mu\,dx\right)\ \mu\ \ dx, \text{ and}$$

$$R\Phi_{y_j,x_s}(f;h) = \int_{\overline{M}} R\alpha_{y_j}(f;h)\left(\beta_{x_s}(f) - \int_{\overline{M}}\beta_{x_s}(f)\mu\,dx\right)\mu\ \ dx$$

$$+\int_{\overline{M}} R\beta_{x_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\ dx\right)\ \mu\ \ dx$$

$$+\int_{\overline{M}}\left(D\alpha_{y_j}\left(f;h\right)+R\alpha_{y_j}\left(f;h\right)\right)\left(D\beta_{x_s}\left(f;h\right)+R\beta_{x_s}\left(f;h\right)\right)\mu\ \ dx$$

$$-\left(\int_{\overline{M}}\left(D\alpha_{y_j}\left(f;h\right)+R\alpha_{y_j}\left(f;h\right)\right)\mu\ \ dx\right)\left(\int_{\overline{M}}\left(D\beta_{x_s}\left(f;h\right)+R\beta_{x_s}\left(f;h\right)\right)\mu\ \ dx\right).$$

Moreover,

$$\left|D\Phi_{y_j,x_s}\left(f;h\right)\right|\leq b_1\left\|h\right\|\ \ and\ \ \left|R\Phi_{y_j,x_s}\left(f;h\right)\right|\leq b_1\left\|h\right\|^2.$$

Proof: Let $\widetilde{\delta}_1=\min\{\widetilde{\delta}_0/2,1\}$. To show that for all h such that $\|h\|\leq\widetilde{\delta}_1$

$$\Phi_{y_j,x_s}\left(f+h\right)-\Phi_{y_j,x_s}\left(f\right)=D\Phi_{y_j,x_s}\left(f;h\right)+R\Phi_{y_j,x_s}\left(f;h\right)$$

we note that by the definitions of Φ_{y_j,x_s}, α_{y_j}, and β_{x_s}, it can be shown by adding and subtracting terms that

$$\Phi_{y_j,x_s}\left(f+h\right)-\Phi_{y_j,x_s}\left(f\right)$$

$$\begin{aligned}
=\ & \int_{\overline{M}}\alpha_{y_j}(f+h)\ \beta_{x_s}(f+h)\ \mu\ dx\ -\left(\int_{\overline{M}}\ \alpha_{y_j}(f+h)\ \mu\ dx\right)\left(\int_{\overline{M}}\ \beta_{x_s}(f+h)\ \mu\ dx\right)\\
& -\int_{\overline{M}}\ \alpha_{y_j}(f)\ \beta_{x_s}(f)\ \mu\ dx\ +\left(\int_{\overline{M}}\ \alpha_{y_j}(f)\ \mu\ dx\right)\left(\int_{\overline{M}}\ \beta_{x_s}(f)\ \mu\ dx\right)\\
&=\int_{\overline{M}}\ \left(\alpha_{y_j}(f+h)-\alpha_{y_j}(f)\right)\left(\beta_{x_s}(f)-\int_{\overline{M}}\ \beta_{x_s}(f)\mu dx\right)\ \mu\ dx\\
&+\int_{\overline{M}}\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\ \alpha_{y_j}(f)\mu dx\right)\left(\beta_{x_s}(f+h)-\beta_{x_s}(f)\right)\mu\ dx\\
&+\int_{\overline{M}}\ \left(\alpha_{y_j}(f+h)-\alpha_{y_j}(f)\right)\left(\beta_{x_s}(f+h)-\beta_{x_s}(f)\right)\mu\ dx\\
&-\left(\int_{\overline{M}}\ \left(\alpha_{y_j}(f+h)-\alpha_{y_j}(f)\right)\mu\ dx\right)\left(\int_{\overline{M}}\ \left(\beta_{x_s}(f+h)-\beta_{x_s}(f)\right)\mu\ dx\right).
\end{aligned}$$

Employing the expressions of $\left(\alpha_{y_j}(f+h)-\alpha_{y_j}(f)\right)$ and of $\left(\beta_{x_s}(f+h)-\beta_{x_s}(f)\right)$ in terms of $D\alpha_{y_j}(f;h)$, $R\alpha_{y_j}(f;h)$, $D\beta_{x_s}(f;h)$, and $R\beta_{x_s}(f;h)$, guaranteed by Lemma A.1, it follows from the last equality that

$$\Phi_{y_j,x_s}\left(f+h\right)-\Phi_{y_j,x_s}\left(f\right)$$

$$=\int_{\overline{M}}\ D\alpha_{y_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}}\ \beta_{x_s}(f)\ \mu dx\right)\ \ \mu\ dx$$

$$+\int_{\overline{M}}\ D\beta_{x_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\ \alpha_{y_j}(f)\ \mu dx\right)\ \ \mu\ dx$$

$$+\int_{\overline{M}} R\alpha_{y_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\ \mu dx\right)\ \mu\ dx$$
$$+\int_{\overline{M}} R\beta_{x_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\ \mu\ dx$$
$$+\int_{\overline{M}}\left(D\alpha_{y_j}(f;h)+R\alpha_{y_j}(f;h)\right)\left(D\beta_{x_s}(f;h)+R\beta_{x_s}(f;h)\right)\ \mu\ dx$$
$$-\left(\int_{\overline{M}}\left(D\alpha_{y_j}(f;h)+R\alpha_{y_j}(f;h)\right)\ \mu\ dx\right)\left(\int_{\overline{M}}\left(D\beta_{x_s}(f;h)+R\beta_{x_s}(f;h)\right)\ \mu\ dx\right)$$

Note that the sum of the first two terms is $D\Phi_{y_j,x_s}(f;h)$ and the sum of the last four terms is $R\Phi_{y_j,x_s}(f;h)$. Hence,

$$\Phi_{y_j,x_s}(f+h)-\Phi_{y_j,x_s}(f)=D\Phi_{y_j,x_s}(f;h)+R\Phi_{y_j,x_s}(f;h)$$

It remains to show that for some finite $b_1>0$,

$$\left|D\Phi_{y_j,x_s}(f;h)\right|\le b_1\|h\|\ \text{ and }\ \left|R\Phi_{y_j,x_s}(f;h)\right|\le b_1\|h\|^2.$$

Let $a=\max\left\{4\|f\|/\left(\delta^2\right),4\|f\|/\left(\delta^3\right)\right\}$, as defined in the proof of Lemma A.1, and let $b_1=\max\left\{a\frac{16\ \|f\|}{\delta},16\ a^2\right\}$. Since by assumption $f>\delta$ and $\widetilde{f}>\delta$, it follows by the definitions of $\alpha_{y_j}(f)$ and $\beta_{x_s}(f)$, the assumption that $\int_{\overline{M}}\mu(y,x)dx=1$, and the conclusion of Lemma A.1 that

$$\begin{aligned}
&\left|\int_{\overline{M}} D\alpha_{y_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\ \mu dx\right)\ \mu\ dx\right|\\
\le\ &\int_{\overline{M}}\left|D\alpha_{y_j}(f;h)\right|\left|\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\ \mu dx\right)\right|\ \mu\ dx\\
\le\ &\int_{\overline{M}} a\|h\|\left|\left(\left|\frac{4\|f\|}{\delta}\right|\right)\right|\ \mu\ dx\le a\|h\|\frac{4\|f\|}{\delta}\le\frac{b_1}{2}\|h\|
\end{aligned}$$

and that

$$\begin{aligned}
&\left|\int_{\overline{M}} D\beta_{x_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\ \mu\ dx\right|\\
\le\ &\int_{\overline{M}}\left|D\beta_{x_s}(f;h)\right|\ \left|\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\right|\ \mu\ dx\\
\le\ &\int_{\overline{M}} a\|h\|\left|\left(\left|\frac{2\|f\|}{\delta}\right|\right)\right|\ \mu\ dx\le a\|h\|\frac{2\|f\|}{\delta}\le\frac{b_1}{2}\|h\|
\end{aligned}$$

Hence, $\left|D\Phi_{y_j,x_s}(f;h)\right|\le b_1\|h\|$. Analogously, for the four terms of $R\Phi_{y_j,x_s}(f;h)$,

$$\left|\int_{\overline{M}} R\alpha_{y_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}} \beta_{x_s}(f)\ \mu dx\right)\ \ \mu\ dx\right|$$
$$\leq \int_{\overline{M}} a\left\|h\right\|^2\left|\left(\left|\frac{4\left\|f\right\|}{\delta}\right|\right)\right|\ \ \mu\ dx\leq\frac{b_1}{4}\left\|h\right\|^2$$

$$\left|\int_{\overline{M}} R\beta_{x_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}} \alpha_{y_j}(f)\ \mu dx\right)\ \ \mu\ dx\right|$$
$$\leq \int_{\overline{M}} a\left\|h\right\|^2\left|\left(\left|\frac{2\left\|f\right\|}{\delta}\right|\right)\right|\ \ \mu\ dx\leq\frac{b_1}{4}\left\|h\right\|^2$$

$$\left|\int_{\overline{M}} \left(D\alpha_{y_j}\left(f;h\right)+R\alpha_{y_j}\left(f;h\right)\right)\left(D\beta_{x_s}\left(f;h\right)+R\beta_{x_s}\left(f;h\right)\right)\ \mu\ dx\right|$$
$$\leq \int_{\overline{M}} a^2\left(\left\|h\right\|+\left\|h\right\|^2\right)^2\ \mu\ dx\leq a^2\left(\left\|h\right\|+\left\|h\right\|^2\right)^2\leq 4\ a^2\left\|h\right\|^2\leq\frac{b_1}{4}\left\|h\right\|^2$$

$$\left|\left(\int_{\overline{M}} \left(D\alpha_{y_j}\left(f;h\right)+R\alpha_{y_j}\left(f;h\right)\right)\ \mu\ dx\right)\left(\int_{\overline{M}} \left(D\beta_{x_s}\left(f;h\right)+R\beta_{x_s}\left(f;h\right)\right)\ \mu\ dx\right)\right|$$
$$\leq\ a^2\left(\left\|h\right\|+\left\|h\right\|^2\right)^2\leq\frac{b_1}{4}\left\|h\right\|^2$$

Hence, $\left|R\Phi_{y_j,x_s}\left(f;h\right)\right|\leq b_1\left\|h\right\|^2$. This completes the proof.

Lemma A.3: *Let $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\left\|\cdot\right\|$, $\alpha_{y_j}(\cdot)$, $\beta_{x_s}(\cdot)$, and μ be as in the statement of Lemma A.3. Assume that the density $f_{Y,X}$ belongs to $\mathtt{F}$ and it is such that for $\delta>0$ and all $(y,x)\in\overline{M}^y\times\overline{M}^x$, $f_{Y,X}(y,x)>\delta$ and $f_X(x)>\delta$. For simplicity we will leave the argument* (y,x) implicit. *Define the functional Φ_{y_j,y_s} and Φ_{x_j,x_s}* on *F by*

$$\Phi_{y_j,y_s}\left(g\right)=\int_{\overline{M}} \alpha_{y_j}(g)\,\alpha_{y_s}(g)\ \mu(y,x)\,dx-\left(\int_{\overline{M}} \alpha_{y_j}(g)\ \mu(y,x)\ dx\right)\left(\int_{\overline{M}} \alpha_{y_s}(g)\ \mu(y,x)\ dx\right).$$

$$\Phi_{x_j,x_s}\left(g\right)=\int_{\overline{M}} \beta_{x_j}(g)\,\beta_{x_s}(g)\ \mu(y,x)\,dx-\left(\int_{\overline{M}} \beta_{x_j}(g)\ \mu(y,x)\ dx\right)\left(\int_{\overline{M}} \beta_{x_s}(g)\ \mu(y,x)\ dx\right).$$

Then, there exist finite $b_2,b_3>0$ and $\widetilde{\delta}_2,\widetilde{\delta}_3>0$ such that for all $h\in\mathtt{F}$ with $\left\|h\right\|\leq\min\left\{\widetilde{\delta}_2,\widetilde{\delta}_3\right\}$

$$\Phi_{y_j,y_s}\left(f+h\right)-\Phi_{y_j,y_s}\left(f\right)=D\Phi_{y_j,y_s}\left(f;h\right)+R\Phi_{y_j,y_s}\left(f;h\right)$$
$$\Phi_{x_j,x_s}\left(f+h\right)-\Phi_{x_j,x_s}\left(f\right)=D\Phi_{x_j,x_s}\left(f;h\right)+R\Phi_{x_j,x_s}\left(f;h\right)$$

where, employing the notation of Lemma A.1 for the definitions of $D\alpha_{y_j}(f;h)$, $R\alpha_{y_j}(f;h)$, $D\alpha_{y_s}(f;h)$, *and* $R\alpha_{y_s}(f;h)$,

$$D\Phi_{y_j,y_s}(f;h)=\int_{\overline{M}}D\alpha_{y_j}(f;h)\left(\alpha_{y_s}(f)-\int_{\overline{M}}\alpha_{y_s}(f)\mu\ dx\right)\ \mu\ \ dx$$

$$+\int_{\overline{M}}D\alpha_{y_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\ dx\right)\ \mu\ \ dx,\ \textit{and}$$

$$R\Phi_{y_j,y_s}(f;h)=\int_{\overline{M}}R\alpha_{y_j}(f;h)\left(\alpha_{y_s}(f)-\int_{\overline{M}}\alpha_{y_s}(f)\mu\ dx\right)\ \mu\ \ dx$$

$$+\int_{\overline{M}}R\alpha_{y_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\mu\ dx\right)\ \mu\ \ dx$$

$$+\int_{\overline{M}}\left(D\alpha_{y_j}(f;h)+R\alpha_{y_j}(f;h)\right)\left(D\alpha_{y_s}(f;h)+R\alpha_{y_s}(f;h)\right)\mu\ \ dx$$

$$-\left(\int_{\overline{M}}\left(D\alpha_{y_j}(f;h)+R\alpha_{y_j}(f;h)\right)\mu\ \ dx\right)\left(\int_{\overline{M}}\left(D\alpha_{y_s}(f;h)+R\alpha_{y_s}(f;h)\right)\mu\ \ dx\right).$$

$$D\Phi_{x_j,x_s}(f;h)=\int_{\overline{M}}D\beta_{x_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\ dx\right)\ \mu\ \ dx$$

$$+\int_{\overline{M}}D\beta_{x_s}(f;h)\ \left(\beta_{x_j}(f)-\int_{\overline{M}}\beta_{x_j}(f)\mu\ dx\right)\ \mu\ \ dx,\ \textit{and}$$

$$R\Phi_{x_j,x_s}(f;h)=\int_{\overline{M}}R\beta_{x_j}(f;h)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\mu\ dx\right)\ \mu\ \ dx$$

$$+\int_{\overline{M}}R\beta_{x_s}(f;h)\ \left(\beta_{x_j}(f)-\int_{\overline{M}}\beta_{x_j}(f)\mu\ dx\right)\ \mu\ \ dx$$

$$+\int_{\overline{M}}\left(D\beta_{x_j}(f;h)+R\beta_{x_j}(f;h)\right)\left(D\beta_{x_s}(f;h)+R\beta_{x_s}(f;h)\right)\mu\ \ dx$$

$$-\left(\int_{\overline{M}}\left(D\beta_{x_j}(f;h)+R\beta_{x_j}(f;h)\right)\mu\ \ dx\right)\left(\int_{\overline{M}}\left(D\beta_{x_s}(f;h)+R\beta_{x_s}(f;h)\right)\mu\ \ dx\right).$$

Moreover,

$$\left|D\Phi_{y_j,y_s}(f;h)\right|\leq b_2\left\|h\right\|\ \textit{and}\ \left|R\Phi_{y_j,y_s}(f;h)\right|\leq b_2\left\|h\right\|^2.$$

$$\left|D\Phi_{x_j,x_s}(f;h)\right|\leq b_3\left\|h\right\|\ \textit{and}\ \left|R\Phi_{x_j,x_s}(f;h)\right|\leq b_3\left\|h\right\|^2.$$

Proof: The proof follows similarly to the proof of Lemma A.2, after replacing β_{x_s} with α_{x_s} in the results related to $\Phi_{y_j,y_s}(g)$ and replacing α_{x_j} with β_{x_j} in the results related to $\Phi_{x_j,x_s}(g)$, with the obvious modification of the upperbounds.

Lemma A.4: *Let* $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\|\cdot\|$, $\alpha_{y_j}(\cdot)$, $\beta_{x_s}(\cdot)$, μ, $\Phi_{y_j,x_s}(\cdot)$, $\Phi_{y_j,y_s}(\cdot)$ *and* $\Phi_{x_j,x_s}(\cdot)$ *be as in the statements of Lemmas A.2 and A.3. Assume that the density* $f_{Y,X}$

belongs to F *and it is such that for* $\delta > 0$ *and all* $(y,x) \in \overline{M}^y \times \overline{M}^x$, $f_{Y,X}(y,x) > \delta$ *and* $f_X(x) > \delta$. *For simplicity we will leave the argument* (y,x) implicit. *Define the functional* Ξ *on* F *by*

$$\Xi(g) = \frac{\Phi_{y_1,y_2}(g)\ \Phi_{x,x}(g) - \Phi_{y_1,x}(g)\ \Phi_{y_2,x}(g)}{\Phi_{y_1,y_1}(g)\ \Phi_{x,x}(g) - \Phi^2_{y_1,x}(g)}$$

and assume that for some $\delta^* > 0$, $\left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right] > \delta^*$. *Then, there exist finite* $c > 0$ *and* $\widetilde{\delta} > 0$ *such that for all* $h \in$F *with* $\|h\| \leq \widetilde{\delta}$

$$\Xi(f+h) - \Xi(f) = D\Xi(f;h) + R\Xi(f;h)$$

where

$$\begin{aligned}
& D\Xi(f;h) \\
= \ & \frac{\left[D\Phi_{y_1,y_2}(f;h)\ \Phi_{x,x}(f) - D\Phi_{y_1,x}(f;h)\ \Phi_{y_2,x}(f)\right]}{\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)} \\
& + \frac{\left[\Phi_{y_1,y_2}(f)\ D\Phi_{x,x}(f;h) - \Phi_{y_1,x}(f)\ D\Phi_{y_2,x}(f;h)\right]}{\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)} \\
& - \frac{\left[\Phi_{y_1,y_2}(f)\ \Phi_{x,x}(f) - \Phi_{y_1,x}(f)\ \Phi_{y_2,x}(f)\right]\left[D\Phi_{y_1,y_1}(f;h)\ \Phi_{x,x} + \Phi_{y_1,y_1}(f)\ D\Phi_{x,x}(f;h)\right]}{\left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right]^2} \\
& + \frac{\left[2\ D\Phi_{y_1,x}(f;h)\ \Phi_{y_1,x}(f)\right]\left[\Phi_{y_1,y_2}(f)\ \Phi_{x,x}(f) - \Phi_{y_1,x}(f)\ \Phi_{y_2,x}(f)\right]}{\left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right]^2},
\end{aligned}$$

$$\left|D\Xi_{y_j,y_s}(f;h)\right| \leq c\,\|h\|\,,\ \textit{and}\ \left|R\Xi_{y_j,y_s}(f;h)\right| \leq c\,\|h\|^2\,.$$

Proof: Let $b = \max\left\{b_1, b_2, b_3, \left(8\,\|f\|^2/\delta^2\right)\right\}$ and $\widetilde{\delta} = \min\left\{\widetilde{\delta}_1, \widetilde{\delta}_2, \widetilde{\delta}_3, \delta^*/\left(32\ b^2\right)\right\}$, where $\widetilde{\delta}_1$ and b_1 are defined in Lemma A.2 and $\widetilde{\delta}_2, \widetilde{\delta}_3, b_2, b_3$ are defined in Lemma A.3. Since the assumptions of Lemmas A.2 and A.3 are satisfied, it follows from those lemmas that for all h in F with $\|h\| \leq \widetilde{\delta}$, and for $j,s = 1,2$

$$\Phi_{y_j,y_s}(f+h) - \Phi_{y_j,y_s}(f) = D\Phi_{y_j,y_s}(f;h) + R\Phi_{y_j,y_s}(f;h)$$

$$\Phi_{y_j,x}(f+h) - \Phi_{y_j,x}(f) = D\Phi_{y_j,x}(f;h) + R\Phi_{y_j,x}(f;h)\ \text{ and}$$

$$\Phi_{x,x}(f+h) - \Phi_{x,x}(f) = D\Phi_{x,x}(f;h) + R\Phi_{x,x}(f;h)$$

where $D\Phi_{y_j,x}(f;h)\,, R\Phi_{y_j,x}(f;h)\,, D\Phi_{y_j,y_s}(f;h)\,, R\Phi_{y_j,y_s}(f;h)\,, D\Phi_{x,x}(f;h)\,,$ and $R\Phi_{x,x}(f;h)$ are as defined in Lemmas A.2 and A.3. By those Lemmas, these terms satisfy

$$\left|D\Phi_{y_j,x_s}(f;h)\right| \leq b\,\|h\|\ \textit{ and }\ \left|R\Phi_{y_j,x_s}(f;h)\right| \leq b\,\|h\|^2\,,$$

$$\left|D\Phi_{y_j,y_s}\left(f;h\right)\right| \leq b\left\|h\right\| \;\; and \;\; \left|R\Phi_{y_j,y_s}\left(f;h\right)\right| \leq b\left\|h\right\|^2, \text{ and}$$

$$\left|D\Phi_{x,x}\left(f;h\right)\right| \leq b\left\|h\right\| \;\; and \;\; \left|R\Phi_{x,x}\left(f;h\right)\right| \leq b\left\|h\right\|^2.$$

We seek a first order expansion

$$\Xi\left(f+h\right)-\Xi\left(f\right)=D\Xi\left(f;h\right)+R\Xi\left(f;h\right)$$

with

$$\left|D\Xi\left(f;h\right)\right| \leq c\left\|h\right\| \text{ and } \left|R\Xi\left(f;h\right)\right| \leq c\left\|h\right\|^2$$

Denote D', N', D, and N by

$$N'=\Phi_{y_1,y_2}\left(f+h\right)\;\Phi_{x,x}\left(f+h\right)-\Phi_{y_1,x}\left(f+h\right)\;\Phi_{y_2,x}\left(f+h\right)$$
$$N=\Phi_{y_1,y_2}\left(f\right)\;\Phi_{x,x}\left(f\right)-\Phi_{y_1,x}\left(f\right)\;\Phi_{y_2,x}\left(f\right)$$
$$D'=\Phi_{y_1,y_1}\left(f+h\right)\;\Phi_{x,x}\left(f+h\right)-\left[\Phi_{y_1,x}\left(f+h\right)\right]^2$$
$$D=\Phi_{y_1,y_1}\left(f\right)\;\Phi_{x,x}\left(f\right)-\left[\Phi_{y_1,x}\left(f\right)\right]^2$$

Then,

$$\begin{aligned}\Xi\left(f+h\right)-\Xi\left(f\right) &= \frac{\Phi_{y_1,y_2}\left(f+h\right)\;\Phi_{x,x}\left(f+h\right)-\Phi_{y_1,x}\left(f+h\right)\;\Phi_{y_2,x}\left(f+h\right)}{\Phi_{y_1,y_1}\left(f+h\right)\;\Phi_{x,x}\left(f+h\right)-\left[\Phi_{y_1,x}\left(f+h\right)\right]^2} \\ &\quad -\frac{\Phi_{y_1,y_2}\left(f\right)\;\Phi_{x,x}\left(f\right)-\Phi_{y_1,x}\left(f\right)\;\Phi_{y_2,x}\left(f\right)}{\Phi_{y_1,y_1}\left(f\right)\;\Phi_{x,x}\left(f\right)-\left[\Phi_{y_1,x}\left(f\right)\right]^2} \\ &= \frac{N'}{D'}-\frac{N}{D}\end{aligned}$$

We will make use of the equality

$$\frac{N'}{D'}-\frac{N}{D}=\frac{N'D-N\;D'}{D^2}-\frac{\left(D'-D\right)\;\left(N'\;D-N\;D'\right)}{D'\;D^2}$$

Lemmas A.2 and A.3 and the definitions of N', D', N, and D imply that

$$\begin{aligned}N' &= \Phi_{y_1,y_2}\left(f+h\right)\;\Phi_{x,x}\left(f+h\right)-\Phi_{y_1,x}\left(f+h\right)\;\Phi_{y_2,x}\left(f+h\right) \\ &= \Phi_{y_1,y_2}\left(f\right)\;\Phi_{x,x}\left(f\right)-\Phi_{y_1,x}\left(f\right)\;\Phi_{y_2,x}\left(f\right) \\ &\quad +D\Phi_{y_1,y_2}\left(f;h\right)\;\Phi_{x,x}\left(f\right)+\Phi_{y_1,y_2}\left(f\right)\;D\Phi_{x,x}\left(f;h\right) \\ &\quad -D\Phi_{y_1,x}\left(f;h\right)\;\Phi_{y_2,x}\left(f\right)-\Phi_{y_1,x}\left(f\right)\;D\Phi_{y_2,x}\left(f;h\right) \\ &\quad +R_N\end{aligned}$$

where

$$R_N = \Phi_{y_1,y_2}(f;h)\ \ [R\Phi_{x,x}(f+h)]$$

$$+D\Phi_{y_1,y_2}(f;h)\ \ [D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)]$$

$$+R\Phi_{y_1,y_2}(f;h)\ \ [\Phi_{x,x}(f+h) + D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)]$$

$$-\Phi_{y_1,x}(f;h)\ \ [R\Phi_{y_2,x}(f+h)]$$

$$-D\Phi_{y_1,x}(f;h)\ \ [D\Phi_{y_2,x}(f+h) + R\Phi_{y_2,x}(f+h)]$$

$$-R\Phi_{y_1,x}(f;h)\ \ [\Phi_{y_2,x}(f+h) + D\Phi_{y_2,x}(f+h) + R\Phi_{y_2,x}(f+h)]$$

and

$$\begin{aligned}
D' &= \Phi_{y_1,y_1}(f+h)\ \ \Phi_{x,x}(f+h) - \Phi^2_{y_1,x}(f+h) \\
&= \Phi_{y_1,y_1}(f)\ \ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f) \\
&\quad +D\Phi_{y_1,y_1}(f;h)\ \ \Phi_{x,x} + \Phi_{y_1,y_1}(f)\ \ D\Phi_{x,x}(f;h) \\
&\quad -2\ D\Phi_{y_1,x}(f;h)\ \ \Phi_{y_1,x}(f) \\
&\quad +R_D
\end{aligned}$$

where

$$\begin{aligned}
R_D &= \Phi_{y_1,y_1}(f;h)\ \ [R\Phi_{x,x}(f+h)] \\
&\quad +D\Phi_{y_1,y_1}(f;h)\ \ [D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)] \\
&\quad +R\Phi_{y_1,y1}(f;h)\ \ [\Phi_{x,x}(f+h) + D\Phi_{x,x}(f+h) + R\Phi_{x,x}(f+h)]
\end{aligned}$$

$$-\Phi_{y_1,x}(f;h)\ \ [R\Phi_{y_1,x}(f+h)]$$

$$-D\Phi_{y_1,x}(f;h)\ \ [D\Phi_{y_1,x}(f+h) + R\Phi_{y_1,x}(f+h)]$$

$$-R\Phi_{y_1,x}(f;h)\ \ [\Phi_{y_1,x}(f+h) + D\Phi_{y_1,x}(f+h) + R\Phi_{y_1,x}(f+h)]$$

Then,

$$
\begin{aligned}
& N'D - ND' \\
= \; & \left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right]\ \left[D\Phi_{y_1,y_2}(f;h)\ \Phi_{x,x}(f) + \Phi_{y_1,y_2}(f)\ D\Phi_{x,x}(f;h)\right] \\
& - \left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right]\ \left[D\Phi_{y_1,x}(f;h)\ \Phi_{y_2,x}(f) + \Phi_{y_1,x}(f)\ D\Phi_{y_2,x}(f;h)\right] \\
& - \left[\Phi_{y_1,y_2}(f)\ \Phi_{x,x}(f) - \Phi_{y_1,x}(f)\ \Phi_{y_2,x}(f)\right]\ \left[D\Phi_{y_1,y_1}(f;h)\ \Phi_{x,x} + \Phi_{y_1,y_1}(f)\ D\Phi_{x,x}(f;h)\right] \\
& +2\ \left[\Phi_{y_1,y_2}(f)\ \Phi_{x,x}(f) - \Phi_{y_1,x}(f)\ \Phi_{y_2,x}(f)\right]\ D\Phi_{y_1,x}(f;h)\ \Phi_{y_1,x}(f) \\
& + \left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right]\ R_N \\
& - \left[\Phi_{y_1,y_2}(f)\ \Phi_{x,x}(f) - \Phi_{y_1,x}(f)\ \Phi_{y_2,x}(f)\right]\ R_D
\end{aligned}
$$

and

$$
\begin{aligned}
& D' - D \\
= \; & D\Phi_{y_1,y_1}(f;h)\ \Phi_{x,x} + \Phi_{y_1,y_1}(f)\ D\Phi_{x,x}(f;h) - 2\ D\Phi_{y_1,x}(f;h)\ \Phi_{y_1,x}(f) + R_D
\end{aligned}
$$

Denote $D\Xi(f;h)$ and $R\Xi(f;h)$ by

$$
\begin{aligned}
& D\Xi(f;h) \\
= \; & \frac{\left[D\Phi_{y_1,y_2}(f;h)\ \Phi_{x,x}(f) + \Phi_{y_1,y_2}(f)\ D\Phi_{x,x}(f;h)\right]}{\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)} \\
& - \frac{\left[D\Phi_{y_1,x}(f;h)\ \Phi_{y_2,x}(f) + \Phi_{y_1,x}(f)\ D\Phi_{y_2,x}(f;h)\right]}{\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)} \\
& - \frac{\left[D\Phi_{y_1,y_1}(f;h)\ \Phi_{x,x} + \Phi_{y_1,y_1}(f)\ D\Phi_{x,x}(f;h)\right]\ \left[\Phi_{y_1,y_2}(f)\ \Phi_{x,x}(f) - \Phi_{y_1,x}(f)\ \Phi_{y_2,x}(f)\right]}{\left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right]^2} \\
& + \frac{\left[2\ D\Phi_{y_1,x}(f;h)\ \Phi_{y_1,x}(f)\right]\left[\Phi_{y_1,y_2}(f)\ \Phi_{x,x}(f) - \Phi_{y_1,x}(f)\ \Phi_{y_2,x}(f)\right]}{\left[\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f) - \Phi^2_{y_1,x}(f)\right]^2}
\end{aligned}
$$

and

$$
\begin{aligned}
& R\Xi(f;h) \\
= \; & \frac{D\ R_N - N\ R_D}{D^2} - \frac{\left[D' - D\right]\ \left[N'D - ND'\right]}{D^2\ D'}
\end{aligned}
$$

Then,

$$\Xi(f+h)-\Xi(f)=D\Xi(f;h)+R\Xi(f;h)$$

It remains to show that for some $c>0$,

$$|D\Xi(f;h)|\leq c\|h\| \text{ and } |R\Xi(f;h)|\leq c\|h\|^2$$

By Lemmas A.2 and A.3, for all h in F with $\|h\|\leq\widetilde{\delta}$,

$$\begin{aligned}&\Phi_{y_1,y_1}(f+h)\ \Phi_{x,x}(f+h)-\Phi^2_{y_1,x}(f+h)\\ =\ &[\Phi_{y_1,y_1}(f)+D\Phi_{y_1,y_1}(f;h)+R\Phi_{y_1,y_1}(f;h)][\Phi_{x,x}(f)+D\Phi_{x,x}(f;h)+R\Phi_{x,x}(f;h)]\\ &-[\Phi_{y_1,x}(f)+D\Phi_{y_1,x}(f;h)+R\Phi_{y_1,x}(f;h)]^2\end{aligned}$$

and, by the definition of b, for all $w,z\in\{y_1,y_2,x\}$, $|D\Phi_{w,z}(f;h)|\leq b\|h\|$ and $|R\Phi_{w,z}(f;h)|\leq b\|h\|^2$. Moreover, by the assumptions on $f_{Y,X}$ and μ, for all such w, z, $|\Phi_{w,z}(f)|\leq\left(8\|f\|^2/\delta^2\right)\leq b$. Hence,

$$\begin{aligned}&\Phi_{y_1,y_1}(f+h)\ \Phi_{x,x}(f+h)-\Phi^2_{y_1,x}(f+h)\\ =\ &\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f)-\Phi^2_{y_1,x}(f)+T\end{aligned}$$

for T such that

$$|T|\leq\ 16\ b^2\ \|h\|\leq 16\ b^2\ \ \widetilde{\delta}\leq\ \delta^*/2$$

Then, since

$$\Phi_{y_1,y_1}(f)\ \Phi_{x,x}(f)-\Phi^2_{y_1,x}(f)>\delta^*$$

it follows that

$$\Phi_{y_1,y_1}(f+h)\ \Phi_{x,x}(f+h)-\Phi^2_{y_1,x}(f+h)>\delta^*/2$$

and then, by the definitions of D and D',

$$D^2\ D'>(\delta^*)^3/2$$

Let $c=\max\left\{16\ b^2/(\delta^*)^2,\ 3776\ b^6\ /(\delta^*)^3\right\}$. Then, for all h such that $\|h\|\leq\widetilde{\delta}$

$$|D\Xi(f;h)|\leq\frac{16\ b^2}{(\delta^*)^2}\leq\ c\ \|h\|$$

and since for all h such that $\|h\|\leq\widetilde{\delta}$, $|D'|\leq 18b^2$, $|D|\leq 2b^2$, $|N|\leq 2b^2$, $|R_N|\leq 12b^2\|h\|^2$, $|R_D|\leq 12b^2\|h\|^2$, $|D'-D|\leq 16b^2\|h\|$, and $|N'D-ND'|\leq 64b^4\|h\|$, it follows that

$$\begin{aligned}|R\Xi(f;h)|\ &\leq\ \frac{18\ b^2(24b^4\|h\|^2+24b^4\|h\|^2)+(16b^2\|h\|)(64\ b^4\|h\|)}{(\delta^*)^3/2}\\ &\leq\ c\|h\|^2\end{aligned}$$

This completes the proof of Lemma A.4.

Lemma A.5: *Let* $\overline{M}^y$, $\overline{M}^x$, $\overline{M}$, $\overline{M}^t$, $\mathtt{F}$, $\|\cdot\|$, $\alpha_{y_j}(\cdot)$, $\beta_{x_s}(\cdot)$, μ, $\Phi_{y_j,x_s}(\cdot)$, $\Phi_{y_j,y_s}(\cdot)$ *and* $\Phi_{x_j,x_s}(\cdot)$ *be as in the statements of Lemmas A.2 and A.3. Suppose that either Assumptions 2.7-2.10 or Assumptions 2.7, 3.4, 3.5, 2.9, and 2.10 are satisfied. Let* G *denote the dimension of the vector of observable endogenous variables* Y *and let* $\overline{M} = \{y\} \times \overline{M}^t$. *Then,*

$$\begin{aligned}
&\sqrt{N\sigma^{G+2}} D\Phi_{y_j,x_s}(f, \widehat{f} - f) \\
&= \sqrt{N\sigma^{G+2}} \int_{\overline{M}} \left(\widehat{f}_{y_j}(y,x) - f_{y_j}(y,x)\right) \left[\frac{\mu\ \left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\mu\ dx\right)}{f}\right] dx + o_p(1)
\end{aligned}$$

$$\begin{aligned}
\sqrt{N\sigma^{G+2}} \left[\Phi_{x_j,x_s}(\widehat{f}) - \Phi_{x_j,x_s}(f)\right] &= \sqrt{N\sigma^{G+2}} D\Phi_{x_j,x_s}(f, \widehat{f} - f) + \sqrt{N\sigma^{G+2}} R\Phi_{x_j,x_s}(f, \widehat{f} - f) \\
&= o_p(1)
\end{aligned}$$

$$\begin{aligned}
&\sqrt{N\sigma^{G+2}} D\Phi_{y_j,y_s}\left(f; \widehat{f} - f\right) \\
&= \sqrt{N\sigma^{G+2}} \int_{\overline{M}} \left(\frac{\widehat{f}_{y_j}(y,x) - f_{y_j}(y,x)}{f(y,x)}\right) \left[\frac{\mu\ \left(\alpha_{y_s}(f) - \int_{\overline{M}} \alpha_{y_s}(f)\mu\ dx\right)}{f}\right] dx \\
&\quad + \sqrt{N\sigma^{G+2}} \int_{\overline{M}} \left(\frac{\widehat{f}_{y_s}(y,x) - f_{y_s}(y,x)}{f(y,x)}\right) \left[\frac{\mu\ \left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f)\mu\ dx\right)}{f}\right] dx \\
&\quad + o_p(1) \ \text{ and}
\end{aligned}$$

$$\text{and} \quad \sqrt{N\sigma^{G+2}} \left\|\widehat{f} - f\right\|^2 \to 0 \ \text{ in probability.}$$

Proof: First note that

$$\begin{aligned}
(T.2) \quad &\int_{\overline{M}} D\beta_{x_s}(f;h) \left(\alpha_{y_j}(f) - \int_{\overline{M}} \alpha_{y_j}(f)\ \mu dx\right) \mu\ dx \\
&= \int_{\overline{M}} \frac{[h_{x_s} f - f_{x_s} h]}{f^2} \frac{f_{y_j}}{f} \mu\ dx - \int_{\overline{M}} \frac{\left[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}\right]}{\widetilde{f}^2} \frac{f_{y_j}}{f} \mu\ dx \\
&\quad - \left(\int_{\overline{M}} \frac{[h_{x_s} f - f_{x_s} h]}{f^2} \mu\ dx\right) \left(\int_{\overline{M}} \left(\frac{f_{y_j}}{f}\right) \mu dx\right) \\
&\quad + \left(\int_{\overline{M}} \frac{\left[\widetilde{h}_{x_s} \widetilde{f} - \widetilde{f}_{x_s} \widetilde{h}\right]}{\widetilde{f}^2} \mu dx\right) \left(\int_{\overline{M}} \left(\frac{f_{y_j}}{f}\right) \mu dx\right)
\end{aligned}$$

Since by Assumption 2.8 the values and derivatives of the continuously differentiable function μ when any coordinate of x is on the boundary of $\overline{M}^t$ are equal to zero and f and $\widetilde{f}$ are larger than δ everywhere on $\overline{M}$ and $\overline{M}^t$, it follows that the continuously differentiable functions $[h\ \mu]/f$, $\left[h\ \mu\ f_{y_j}\right]/f^2$, $\left[\widetilde{h}\ \mu\right]/\widetilde{f}$ and $\left[\widetilde{h}\ \mu\ f_{y_j}\right]/\left[\widetilde{f}f\right]$ vanish when any coordinate of x is on the boundary of $\overline{M}^t$. Hence, integration by parts of the terms in $(T.2)$ containing h_{x_s} or $\widetilde{h}_{x_s}$ gives

$$
\begin{aligned}
&\int_{\overline{M}} D\beta_{x_s}(f;h)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\ \mu\ dx \\
=\ & -\int_{\overline{M}} h\ \left[\frac{\partial}{\partial x_s}\left(\frac{\mu\ f_{y_j}}{f^2}\right)+\frac{f_{x_s}f_{y_j}\mu}{f^3}\right]\ dx+\int_{\overline{M}}\widetilde{h}\ \left[\frac{\partial}{\partial x_s}\left(\frac{\mu\ f_{y_j}}{\widetilde{f}\ f}\right)+\frac{\widetilde{f}_{x_s}f_{y_j}\mu}{\widetilde{f}^2 f}\right]\ dx \\
& +\left(\int_{\overline{M}} h\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{f}\right)+\frac{f_{x_s}\mu}{f^2}\right]\ dx\right)\ \left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\ \mu dx\right) \\
& -\left(\int_{\overline{M}}\widetilde{h}\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{\widetilde{f}}\right)+\frac{\widetilde{f}_{x_s}\mu}{\widetilde{f}^2}\right]dx\right)\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\ \mu dx\right)
\end{aligned}
$$

Let $h=\widehat{f}-f$,

$$
\begin{aligned}
\omega_a\left(x\right) &= -\ \frac{f_{y_j}\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\ \mu dx\right)\ \mu}{f^2} \\
&\quad -\left[\frac{\partial}{\partial x_s}\left(\frac{\mu\ f_{y_j}}{f^2}\right)+\frac{f_{x_s}f_{y_j}\mu}{f^3}\right]\ +\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\ \mu dx\right)\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{f}\right)+\frac{f_{x_s}\mu}{f^2}\right]
\end{aligned}
$$

$$
\omega_b(x)=\left[\frac{\partial}{\partial x_s}\left(\frac{\mu\ f_{y_j}}{\widetilde{f}\ f}\right)+\frac{\widetilde{f}_{x_s}f_{y_j}\mu}{\widetilde{f}^2 f}-\left(\int_{\overline{M}}\left(\frac{f_{y_j}}{f}\right)\ \mu dx\right)\left[\frac{\partial}{\partial x_s}\left(\frac{\mu}{\widetilde{f}}\right)+\frac{\widetilde{f}_{x_s}\mu}{\widetilde{f}^2}\right]\right]
$$

Then, by Lemmas A.2 and A.3 and by $(T.2)$,

$$
\begin{aligned}
& D\Phi_{y_j,x_s}\left(f;\widehat{f}-f\right) \\
=\ & \int_{\overline{M}} D\alpha_{y_j}(f;\widehat{f}-f)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\ \mu dx\right)\ \mu\ dx \\
& +\int_{\overline{M}} D\beta_{x_s}(f;\widehat{f}-f)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\ \mu\ dx
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{\overline{M}} \left[\frac{\left[\left(\widehat{f}_{y_j} - f_{y_j} \right) f \right]}{f^2} \right] \left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\ \mu dx \right)\ \ \mu\ dx \\
& - \int_{\overline{M}} \left[\frac{f_{y_j} \left(\widehat{f} - f \right)}{f^2} \right] \left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\ \mu dx \right)\ \ \mu\ dx \\
& - \int_{\overline{M}} \left(\widehat{f} - f \right)\ \left[\frac{\partial}{\partial x_s} \left(\frac{\mu\ f_{y_j}}{f^2} \right) + \frac{f_{x_s} f_{y_j} \mu}{f^3} \right]\ dx \\
& + \int_{\overline{M}} \left(\widehat{\widetilde{f}} - \widetilde{f} \right)\ \left[\frac{\partial}{\partial x_s} \left(\frac{\mu\ f_{y_j}}{\widetilde{f}\ f} \right) + \frac{\widetilde{f}_{x_s} f_{y_j} \mu}{\widetilde{f}^2 f} \right]\ dx \\
& + \left(\int_{\overline{M}} \left(\widehat{f} - f \right) \left[\frac{\partial}{\partial x_s} \left(\frac{\mu}{f} \right) + \frac{f_{x_s} \mu}{f^2} \right]\ dx \right) \left(\int_{\overline{M}} \left(\frac{f_{y_j}}{f} \right)\ \mu dx \right) \\
& - \left(\int_{\overline{M}} \left(\widehat{\widetilde{f}} - \widetilde{f} \right) \left[\frac{\partial}{\partial x_s} \left(\frac{\mu}{\widetilde{f}} \right) + \frac{\widetilde{f}_{x_s} \mu}{\widetilde{f}^2} \right] dx \right) \left(\int_{\overline{M}} \left(\frac{f_{y_j}}{f} \right)\ \mu dx \right)
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{\overline{M}} \left(\widehat{f}_{y_j} - f_{y_j} \right) \frac{\left(\beta_{x_s}(f) - \int_{\overline{M}} \beta_{x_s}(f)\ \mu dx \right)}{f}\ \ \mu\ dx \\
& + \int_{\overline{M}} \left(\widehat{f} - f \right)\ \omega_a(x)\ dx + \int_{\overline{M}} \left(\widehat{\widetilde{f}} - \widetilde{f} \right)\ \omega_b(x)\ dx
\end{aligned}
$$

By Assumptions 2.7, 2.9, and 2.10, it follows by Lemma 5.3 in Newey (1994), by letting in that lemma, $k_1 = G$, $k_2 = \dim(x)$, $l = 0$, $t = x$, $h_0 = f$, $m(\widehat{h}) = \left(\int_{\overline{M}}\ \widehat{f}(y,x)\ \omega_a(x)\ dx \right)$, and $m(h_0) = \left(\int_{\overline{M}}\ f(y,x)\ \omega_a(x)\ dx \right)$,that for a finite constant matrix V_a

$$
\sqrt{N\sigma^G} \left(\int_{\overline{M}} \left(\widehat{f}(y,x) - f(y,x) \right) \omega_a(x)\, dx \right) \to N(0, V_a).
$$

Similarly, letting in that same lemma, $k_1 = 0$, $k_2 = \dim(x)$, $l = 0$, $t = x$, $h_0 = \widetilde{f}$, $m(\widehat{h}) = \left(\int_{\overline{M}}\ \widehat{\widetilde{f}}(x)\ \omega_b(x)\ dx \right)$ and $m(h_0) = \left(\int_{\overline{M}}\ \widetilde{f}(x)\ \omega_b(x)\ dx \right)$, it follows that for a finite constant matrix V_b

$$
\sqrt{N} \left(\int_{\overline{M}} \left(\widehat{\widetilde{f}}(x) - \widetilde{f}(x) \right) \omega_b(x)\ dx \right) \to N(0, V_b).
$$

Hence,

$$
\begin{aligned}
& \sqrt{N\sigma^{G+2}} D\Phi_{y_j,x_s}(f, \widehat{f} - f) \\
= & \sqrt{N\sigma^{G+2}} \int_{\overline{M}} \left(\widehat{f}_{y_j}(y,x) - f_{y_j}(y,x) \right) \left[\frac{\mu \left(\beta_{x_s}(f) - \int_{\overline{M}}\ \beta_{x_s}(f)\ \mu\ dx \right)}{f} \right]\ dx + o_p(1)
\end{aligned}
$$

Using analogous arguments for $D\Phi_{x_j,x_s}(f,\widehat{f}-f)$ and for $D\Phi_{y_j,y_s}(f,\widehat{f}-f)$, it follows that

$$\begin{aligned}
&\sqrt{N\sigma^{G+2}}D\Phi_{x_j,x_s}(f,\widehat{f}-f) \\
=\ &\sqrt{N\sigma^{G+2}}\int_{\overline{M}} D\beta_{x_j}(f;\widehat{f}-f)\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\ \mu dx\right)\ \ \mu\ dx \\
&+\sqrt{N\sigma^{G+2}}\int_{\overline{M}} D\beta_{x_s}\left(f;\widehat{f}-f\right)\ \left(\beta_{x_j}(f)-\int_{\overline{M}}\beta_{x_j}(f)\ \mu dx\right)\ \ \mu\ dx \\
=\ &o_p(1)
\end{aligned}$$

and that

$$\begin{aligned}
&\sqrt{N\sigma^{G+2}}D\Phi_{y_j,y_s}\left(f;\widehat{f}-f\right) \\
=\ &\sqrt{N\sigma^{G+2}}\int_{\overline{M}} D\alpha_{y_j}(f;\widehat{f}-f)\left(\alpha_{y_s}(f)-\int_{\overline{M}}\alpha_{y_s}(f)\ \mu dx\right)\ \ \mu\ dx \\
&+\sqrt{N\sigma^{G+2}}\int_{\overline{M}} D\alpha_{y_s}(f;\widehat{f}-f)\ \left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\ \ \mu \\
=\ &\sqrt{N\sigma^{G+2}}\int_{\overline{M}}\left(\widehat{f}_{y_j}-f_{y_j}\right)\left(\frac{\left(\alpha_{y_s}(f)-\int_{\overline{M}}\alpha_{y_s}(f)\ \mu dx\right)\ \mu}{f}\right)dx \\
&+\sqrt{N\sigma^{G+2}}\int_{\overline{M}}\left(\widehat{f}_{y_s}-f_{y_s}\right)\left(\frac{\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\ \mu}{f}\right)dx \\
&+o_p\left(1\right)
\end{aligned}$$

To show that

$$\sqrt{N\sigma^{G+2}}\left\|\widehat{f}-f\right\|^2\to 0\ \ \textit{in probability.}$$

we note that from Appendix B.3 in Newey (1994),

$$\left\|\widehat{f}-f\right\|=O_p\left(\sqrt{\log(N)/\left(N\sigma^{2G+2}\right)}+\sigma^s\right)$$

Since by Assumption 2.10, $\sqrt{N\sigma^{G+2}}\left(\sqrt{\log\left(N\right)/\left(N\sigma^{2G+2}\right)}+\sigma^s\right)^2$ converges to zero, it follows that

$$\sqrt{N\sigma^{G+2}}\left\|\widehat{f}-f\right\|^2\to 0\ \ \textit{in probability.}$$

Lemma A.6: *Under the same definitions, notations, and assumptions in Lemma A.5, for all j,s and all $w,z\in\{y_j,y_s,x_j,x_s\}$, for finite $\overline{c}>0$, and for $\left\|\widehat{f}-f\right\|$ sufficiently small,*

$$\sup_{(y,x)\in\overline{M}}\ \left|\left(\Delta\partial_w\log\widehat{f}_{Y|X=x}\left(y\right)\right)\left(\Delta\partial_z\log\widehat{f}_{Y|X=x}\left(y\right)\right)\frac{\mu^2}{\widehat{f}(y,x)}\right.$$
$$\left.-\left(\Delta\partial_w\log f_{Y|X=x}\left(y\right)\right)\left(\Delta\partial_z\log f_{Y|X=x}\left(y\right)\right)\frac{\mu^2}{f(y,x)}\right|\leq\overline{c}\ \left\|\widehat{f}-f\right\|$$

Proof: By Lemma A.1, it follows that for all $\widehat{f}$ such that $\left\|\widehat{f}-f\right\| \leq \widetilde{\delta}_0$ and for all $(y,x)\in\overline{M}$,

$$\begin{aligned}\left|\frac{\widehat{f}_{y_j}(y,x)}{\widehat{f}(y,x)}-\frac{f_{y_j}(y,x)}{f(y,x)}\right| &= \left|\alpha_{y_j}\left(\widehat{f}\right)-\alpha_{y_j}(f)\right| \leq \left|D\alpha_{y_j}\left(f;\widehat{f}-f\right)\right|+\left|R\alpha_{y_j}\left(f;\widehat{f}-f\right)\right| \\ &\leq a\left\|\widehat{f}-f\right\|+a\left\|\widehat{f}-f\right\|^2\end{aligned}$$

Hence,

$$\sup_{(y,x)\in\overline{M}}\left|\alpha_{y_j}\left(\widehat{f}\right)-\alpha_{y_j}(f)\right| \leq a\left\|\widehat{f}-f\right\|+a\left\|\widehat{f}-f\right\|^2$$

Since μ is bounded, $\int_{\overline{M}}\mu(y,x)dx=1$, and $\overline{M}$ is compact, this implies that

$$\begin{aligned}&\left|\int_{\overline{M}}\alpha_{y_j}\left(\widehat{f}\right)\mu\, dx-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right| \\ \leq\ &\int_{\overline{M}}\left|\alpha_{y_j}\left(\widehat{f}\right)-\alpha_{y_j}(f)\right|\ \mu dx \\ \leq\ & a\left\|\widehat{f}-f\right\|+a\left\|\widehat{f}-f\right\|^2\end{aligned}$$

and therefore, for all $(y,x)\in\overline{M}$

$$\begin{aligned}&\left|\left(\alpha_{y_j}\left(\widehat{f}\right)-\int_{\overline{M}}\alpha_{y_j}\left(\widehat{f}\right)\ \mu dx\right)-\left(\alpha_{y_j}(f)-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right)\right| \\ \leq\ &\left|\alpha_{y_j}\left(\widehat{f}\right)-\alpha_{y_j}(f)\right|+\left|\int_{\overline{M}}\alpha_{y_j}\left(\widehat{f}\right)\ \mu dx-\int_{\overline{M}}\alpha_{y_j}(f)\ \mu dx\right| \\ \leq\ & 2\ a\left\|\widehat{f}-f\right\|+2\ a\left\|\widehat{f}-f\right\|^2\end{aligned}$$

By an analogous argument for β_{x_s}, it can shown employing Lemma A.1 that for all $(y,x)\in\overline{M}$

$$\begin{aligned}&\left|\left(\beta_{x_s}\left(\widehat{f}\right)-\int_{\overline{M}}\beta_{x_s}\left(\widehat{f}\right)\ \mu dx\right)-\left(\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}\ \mu dx\right)\right| \\ \leq\ & 2\ a\left\|\widehat{f}-f\right\|+2\ a\left\|\widehat{f}-f\right\|^2\end{aligned}$$

Since for all $(y,x)\in\overline{M}$, $f(y,x)>\delta$ and $f(x)>\delta$, it also follows that, since by the definition of $\widetilde{\delta}_0$ in the proof of Lemma A.1, $\widetilde{\delta}_0=\min\{\delta/2,1\}$, for all $\widehat{f}$ such that $\left\|\widehat{f}-f\right\|\leq\widetilde{\delta}_0$,

$$\left|\frac{1}{\widehat{f}(y,x)}-\frac{1}{f(y,x)}\right| \leq \frac{\left|\widehat{f}(y,x)-f(y,x)\right|}{\left|\widehat{f}(y,x)\right|\,|f(y,x)|} \leq \frac{2\left\|\widehat{f}-f\right\|}{\delta^2}$$

Denote $\widehat{\alpha}=\left(\alpha_{y_j}\left(\widehat{f}\right)-\int_{\overline{M}}\alpha_{y_j}\left(\widehat{f}\right)\ \mu dx\right),\ \widehat{\beta}=\left[\beta_{x_s}(\widehat{f})-\int_{\overline{M}}\beta_{x_s}(\widehat{f})\ \mu dx\right],$
$\alpha=\left(\alpha_{y_j}\left(f\right)-\int_{\overline{M}}\alpha_{y_j}\left(f\right)\ \mu dx\right)$, and $\beta=\left[\beta_{x_s}(f)-\int_{\overline{M}}\beta_{x_s}(f)\ \mu dx\right]$. Note that

$$\frac{\widehat{\alpha}\widehat{\beta}}{\widehat{f}}-\frac{\alpha\beta}{f}=\frac{\left(\widehat{\alpha}-\alpha\right)\widehat{\beta}f+\left(\widehat{\beta}-\beta\right)\alpha f-\alpha\beta\left(\widehat{f}-f\right)}{\widehat{f}\ f}$$

Then,

$$\begin{aligned}\left|\frac{\widehat{\alpha}\widehat{\beta}}{\widehat{f}}-\frac{\alpha\beta}{f}\right| &= \left|\frac{\left(\widehat{\alpha}-\alpha\right)\widehat{\beta}f+\left(\widehat{\beta}-\beta\right)\alpha f-\alpha\beta\left(\widehat{f}-f\right)}{\widehat{f}\ f}\right| \\ &\leq \frac{\left|\left(\widehat{\alpha}-\alpha\right)\right|\ \left|\widehat{\beta}\right|f+\left|\widehat{\beta}-\beta\right|\ \left|\alpha\ f\right|+\left|\widehat{f}-f\right|\ \left|\alpha\beta\right|}{\left|\widehat{f}\right|\left|f\right|} \\ &\leq \frac{2\ \left[\left\|\left(\widehat{\alpha}-\alpha\right)\right\|\ \left|\widehat{\beta}\right|\ \left|f\right|+\left\|\widehat{\beta}-\beta\right\|\ \left|\alpha\ f\right|+\left\|\widehat{f}-f\right\|\ \left|\alpha\beta\right|\right]}{\delta^2}\end{aligned}$$

Since $\left|\widehat{\beta}\right|=\left|\beta+D\beta+R\beta\right|\leq\left|\beta\right|+\left|D\beta\right|+\left|R\beta\right|$ and all the three last term are either uniformly bounded, by Assumption 2.7, or bounded by a constant times $\left\|\widehat{f}-f\right\|$, by Lemma A.1, it follows that for a finite $\overline{c}>0$, for all $(y,x)\in\overline{M}$,
$\left|\left(\Delta\partial_w\log\widehat{f}_{Y|X=x}\left(y\right)\right)\left(\Delta\partial_z\log\widehat{f}_{Y|X=x}\left(y\right)\right)\frac{\mu^2}{\widehat{f}(y,x)}-\left(\Delta\partial_w\log f_{Y|X=x}\left(y\right)\right)\left(\Delta\partial_z\log f_{Y|X=x}\left(y\right)\right)\frac{\mu^2}{f(y,x)}\right|$
$\leq\overline{c}\ \left\|\widehat{f}-f\right\|,$
where $w=y_j$ and $z=x_s$. A similar argument holds for $w\in\{y_s.x_j,x_s\}$ and $z\in\{y_j,y_s,x_j\}$.

Lemma B.1: *Suppose that* (3.2) *and Assumptions 3.1-3.3 are satisfied. Then,*

$$\begin{aligned}&\left(r_x'-\widetilde{r}_x'\ \left(\widetilde{r}_y'\right)^{-1}\ r_y'\right)\ q_\varepsilon \\ &=\left[1-\widetilde{r}_{y_1}^1\left(\frac{\widetilde{r}_x^2}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r_{y_1}^1\ r_x^2}\right)\right]\left(\mathbf{g}_x-\gamma_x\right)-\left[\widetilde{r}_{y_1}^1\left(\frac{\widetilde{r}_x^2}{|\widetilde{r}_y|}\right)\left(\frac{r_{y_2}^1}{r_{y_1}^1}\right)-\widetilde{r}_{y_2}^1\left(\frac{\widetilde{r}_x^2}{|\widetilde{r}_y|}\right)\right]\left(\mathbf{g}_{y_1}-\gamma_{y_1}\right)\end{aligned}$$

and

$$\begin{aligned}&-\left(\gamma_x-\widetilde{\gamma}_x\right)+\widetilde{r}_x'\ \left(\widetilde{r}_y'\right)^{-1}\left(\gamma_y-\widetilde{\gamma}_y\right) \\ &=-\left(\gamma_x-\widetilde{\gamma}_x\right)-\widetilde{r}_{y_2}^1\left(\frac{\widetilde{r}_x^2}{|\widetilde{r}_y|}\right)\left(\gamma_{y_1}-\widetilde{\gamma}_{y_1}\right)+\widetilde{r}_{y_1}^1\left(\frac{\widetilde{r}_x^2}{|\widetilde{r}_y|}\right)\left(\gamma_{y_2}-\widetilde{\gamma}_{y_2}\right)\end{aligned}$$

where the notation corresponds to that used in the proof of Theorem 3.1.

Proof: Taking logs and differentiating both sides of $(T.3.1)$, in the proof of Theorem 3.1, with respect to x, leaving the arguments implicit, we get

$$(B.1.1)\qquad \mathbf{g}_x=r_x^2\ q_{\varepsilon_2}+\gamma_x$$

and taking logs and differentiation both sides of $(T.3.1)$ with respect to y_1, we get

$$(B.1.2) \qquad \mathbf{g}_{y_1} = r^1_{y_1}\ q_{\varepsilon_1} + r^2_{y_1}\ q_{\varepsilon_2} + \gamma_{y_1}$$

Since the matrix

$$\begin{pmatrix} 0 & r^2_x \\ r^1_{y_1} & r^2_{y_1} \end{pmatrix}$$

is invertible, because by assumption $r^1_{y_1}$ and r^2_x are different from zero, the unique value of the vector $(q_{\varepsilon_1}, q_{\varepsilon_2})$ that solves $(B.1.1) - (B.1.2)$ is

$$(B.1.3) \qquad \begin{pmatrix} q_{\varepsilon_1} \\ q_{\varepsilon_2} \end{pmatrix} = \begin{pmatrix} \frac{-\ r^2_{y_1}}{r^1_{y_1} r^2_x} & \frac{1}{r^1_{y_1}} \\ \frac{1}{r^2_x} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{g}_x - \gamma_x \\ \mathbf{g}_{y_1} - \gamma_{y_1} \end{pmatrix}$$

Since $r'_x = (0, r^2_x)$, it follows by $(B.1.3)$ that

$$(B.1.4T.3.5) \qquad r'_x\ q_\varepsilon = r^2_x\ q_{\varepsilon_2} = \mathbf{g}_x - \gamma_x$$

By the definition of r'_y and $(B.1.3)$,

$$(B.1.5) \qquad r'_y\ q_\varepsilon = \begin{pmatrix} r^1_{y_1} & r^2_{y_1} \\ r^1_{y_2} & r^2_{y_2} \end{pmatrix} \begin{pmatrix} \frac{-\ r^2_{y_1}}{r^1_{y_1} r^2_x} & \frac{1}{r^1_{y_1}} \\ \frac{1}{r^2_x} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{g}_x - \gamma_x \\ \mathbf{g}_{y_1} - \gamma_{y_1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ \frac{-\ r^1_{y_2}\ r^2_{y_1}}{r^1_{y_1} r^2_x} + \frac{r^2_{y_2}}{r^2_x} & \frac{r^1_{y_2}}{r^1_{y_1}} \end{pmatrix} \begin{pmatrix} \mathbf{g}_x - \gamma_x \\ \mathbf{g}_{y_1} - \gamma_{y_1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{g}_{y_1} - \gamma_{y_1} \\ \left(\frac{r^1_{y_2}}{r^1_{y_1}}\right)(\mathbf{g}_{y_1} - \gamma_{y_1}) + \left(\frac{|r_y|}{r^1_{y_1}\ r^2_x}\right)(\mathbf{g}_x - \gamma_x) \end{pmatrix}$$

Note that

$$(B.1.6) \qquad \widetilde{r}'_x\ \left(\widetilde{r}'_y\right)^{-1} = \begin{pmatrix} 0 & , & \frac{\widetilde{r}^2_x}{|\widetilde{r}_y|} \end{pmatrix} \begin{pmatrix} \widetilde{r}^2_{y_2} & -\widetilde{r}^2_{y_1} \\ -\widetilde{r}^1_{y_2} & \widetilde{r}^1_{y_1} \end{pmatrix}$$

$$= \begin{pmatrix} -\widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) & , & \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) \end{pmatrix}$$

By $(B.1.5)$ and $(B.1.6)$, it then follows that

$$(B.1.7) \qquad \left[\widetilde{r}'_x\ \left(\widetilde{r}'_y\right)^{-1}\right]\left[r'_y\ q_\varepsilon\right]$$

$$= \left[\begin{pmatrix} -\widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) & , & \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) \end{pmatrix}\right]\left[\begin{pmatrix} \mathbf{g}_{y_1} - \gamma_{y_1} \\ \left(\frac{r^1_{y_2}}{r^1_{y_1}}\right)(\mathbf{g}_{y_1} - \gamma_{y_1}) + \left(\frac{|r_y|}{r^1_{y_1}\ r^2_x}\right)(\mathbf{g}_x - \gamma_x) \end{pmatrix}\right]$$

$$= \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{r^1_{y_2}}{r^1_{y_1}}\right) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\right](\mathbf{g}_{y_1} - \gamma_{y_1}) + \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r^1_{y_1}\ r^2_x}\right)\right](\mathbf{g}_x - \gamma_x)$$

By $(B.1.4)$ and $(B.1.7)$

$$\left(r'_x - \widetilde{r}'_x\ \left(\widetilde{r}'_y\right)^{-1}\ r'_y\right)\ q_\varepsilon$$

$$= [r'_x \ q_\varepsilon] - \left[\widetilde{r}'_x \ \left(\widetilde{r}'_y\right)^{-1}\right] \ [r'_y \ q_\varepsilon]$$

$$= (\mathbf{g}_x - \gamma_x) - \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{r^1_{y_2}}{r^1_{y_1}}\right) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\right](\mathbf{g}_{y_1 -} \gamma_{y_1}) - \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r^1_{y_1} \ r^2_x}\right)\right](\mathbf{g}_x - \gamma_x)$$

$$= \left[1 - \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{|r_y|}{r^1_{y_1} \ r^2_x}\right)\right](\mathbf{g}_x - \gamma_x) - \left[\widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\frac{r^1_{y_2}}{r^1_{y_1}}\right) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\right](\mathbf{g}_{y_1 -} \gamma_{y_1})$$

By $(B.1.6)$,

$$-(\gamma_x - \widetilde{\gamma}_x) + \widetilde{r}'_x \ \left(\widetilde{r}'_y\right)^{-1}\left(\gamma_y - \widetilde{\gamma}_y\right)$$

$$= -(\gamma_x - \widetilde{\gamma}_x) + \left(-\widetilde{r}^1_{y_2}\left(\tfrac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) \quad , \quad \widetilde{r}^1_{y_1}\left(\tfrac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right) \right) \begin{pmatrix} \gamma_{y_1} - \widetilde{\gamma}_{y_1} \\ \\ \gamma_{y_2} - \widetilde{\gamma}_{y_2} \end{pmatrix}$$

$$= -(\gamma_x - \widetilde{\gamma}_x) - \widetilde{r}^1_{y_2}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\gamma_{y_1} - \widetilde{\gamma}_{y_1}\right) + \widetilde{r}^1_{y_1}\left(\frac{\widetilde{r}^2_x}{|\widetilde{r}_y|}\right)\left(\gamma_{y_2} - \widetilde{\gamma}_{y_2}\right)$$