Econometrica Supplementary Material

SUPPLEMENT TO "GENERALIZED REDUCED-FORM AUCTIONS: A NETWORK FLOW APPROACH"

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APPENDIX B: FURTHER OMITTED PROOFS

B.1. *Structure of the Set of Reduced-Form Auctions*

We provide the proof of Theorem 4 in Remark 2, which shows that the two functions Ψ and Φ that set an upper bound and lower bound for the set of reduced-form auctions, respectively, form a paramodular pair.

PROOF OF THEOREM 4: We first observe that the operation $I(\theta, \cdot)$ as a function of T preserves the union, intersection, and complement of sets: that is, for any $\theta \in \Theta$ and $T, T' \subset D$, $I(\theta, T \cap T') = I(\theta, T) \cap I(\theta, T')$, $I(\theta, T \cup T') = I(\theta, T) \cup I(\theta, T')$, and $I(\theta, T \setminus T') = I(\theta, T) \setminus I(\theta, T')$. To see that the complement is preserved, for instance, note that $i \in I(\theta, T \setminus T')$ if and only if $\theta_i \in T \setminus T'$, that is, $\theta_i \in T$ and $\theta_i \notin T'$, which is equivalent to having $i \in I(\theta, T)$ and $i \notin I(\theta, T')$, that is, $i \in I(\theta, T) \setminus I(\theta, T')$. The other equalities can be checked similarly.

Given this, paramodularity of Ψ and Φ holds due to the fact that the paramodularity of C and L is not affected by the expectation operator. For instance, the compliance holds since for any $T, T' \subset D$,

$$\begin{aligned}
&\Psi(T') - \Phi(T) \\
&\quad = \sum_{\theta \in \Theta} \bigl[C\bigl(I(\theta, T')\bigr) - L\bigl(I(\theta, T)\bigr)\bigr] p(\theta) \\
&\quad \geq \sum_{\theta \in \Theta} \bigl[C\bigl(I(\theta, T') \setminus I(\theta, T)\bigr) - L\bigl(I(\theta, T) \setminus I(\theta, T')\bigr)\bigr] p(\theta) \\
&\quad = \sum_{\theta \in \Theta} \bigl[C\bigl(I(\theta, T' \setminus T)\bigr) - L\bigl(I(\theta, T \setminus T')\bigr)\bigr] p(\theta) \\
&\quad = \Psi(T' \setminus T) - \Phi(T \setminus T').
\end{aligned}$$

The first and last equalities follow from the fact that

$$\Psi(T) = \sum_{\theta \in Y(T)} C\bigl(I(\theta, T)\bigr) p(\theta) = \sum_{\theta \in \Theta} C\bigl(I(\theta, T)\bigr) p(\theta)$$

and

$$\Phi(T) = \sum_{\theta \in Y(T)} L\bigl(I(\theta, T)\bigr) p(\theta) = \sum_{\theta \in \Theta} L\bigl(I(\theta, T)\bigr) p(\theta)$$

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since, for any $\theta \in \Theta \setminus Y(T)$, $I(\theta, T) = \emptyset$, so $C(I(\theta, T)) = L(I(\theta, T)) = 0$. The next to last equality follows from the observation in the previous paragraph while the inequality follows from the compliance of C and L. An analogous argument can be used to show the sub- and supermodularity of Ψ and Φ, respectively. *Q.E.D.*

B.2. *General Type Distributions*

For the proof of Theorem 5, we denote the set of ex post allocation rules that respect (C, L) by $\mathcal{Q}_0(C, L)$ and denote the set of implementable interim allocation rules for given (C, L) by $\mathcal{Q}(C, L)$.

PROOF OF THEOREM 5: Let $\Lambda : \mathcal{Q}_0(C, L) \to \mathcal{Q}(C, L)$ be the function that maps an ex post allocation rule to its reduced form. Note that since $q \in \mathcal{Q}_0(C, L)$ is bounded and μ is a probability measure, $\mathcal{Q}_0(C, L)$ and $\mathcal{Q}(C, L)$ are subsets of the Hilbert space $L_2(\Theta, \mu, \mathbb{R}^{|I|})$. Along the lines of Lemma 5.4 in Border (1991), one can show that $\mathcal{Q}_0(C, L)$ and $\mathcal{Q}(C, L)$ are weakly compact and the linear mapping Λ is weakly continuous.

If $Q : \Theta \to [0, C(I)]^{|I|}$ satisfies (B^C), it is bounded and hence there exists a sequence of simple functions $(Q^n : \Theta \to [0, C(I)]^{|I|})_{n\in\mathbb{N}}$ with $Q_i^n(\theta) = Q_i^n(\theta_i)$, such that for $n \to \infty$, Q^n converges uniformly to Q and $Q^1 \le Q^2 \le Q^3 \le \cdots \le Q$. Since convergence is uniform, there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$ for $n \to \infty$, and such that for all $T = (T_i)_{i\in I}$, $T_i \in \mathcal{A}_i$,

$$(\mathrm{C}^n) \qquad \int_{Y(T)} L_{\varepsilon_n}\big(I(\theta, T)\big)\, d\mu(\theta) \le \sum_{i\in I} \int_{T_i} Q_i^n(\theta_i)\, d\mu_i(\theta_i) \le \int_{Y(T)} C\big(I(\theta, T)\big)\, d\mu(\theta),$$

where $L_{\varepsilon_n}(I(\theta, T)) = \max\{L(I(\theta, T)) - \varepsilon_n, 0\}$.

As Q^n is a simple function, we can write Q_i^n as

$$Q_i^n(\theta) = \sum_{k=1}^{K_i^n} \alpha_{ik}^n \chi_{A_{ik}^n}(\theta),$$

where $\alpha_{ik}^n \in [0, C(I)]$, $\{A_{ik}^n\}_k$ is a partition of Θ_i such that each $A_{ik}^n \in \mathcal{A}_i$, and χ_A is the indicator function of A.

Next, for given n and each $i \in I$, we define a discretized type space $\tilde{\Theta}_i^n := \{A_{ik}^n\}_{k=1,\dots,K_i^n}$. The distribution over type profiles is given by

$$\tilde{p}\big(A_{1k_1}^n, \dots, A_{|I|k_{|I|}}^n\big) := \mu\big(A_{1k_1}^n \times \cdots \times A_{|I|k_{|I|}}^n\big).$$

Let $\tilde{Q}^n$ be the interim allocation rule for the discrete type space Θ^n defined by

$$\tilde{Q}_i^n(A_{ik}^n) := \alpha_{ik}^n.$$

We have chosen Q^n such that $\tilde{Q}^n$ is implementable for the relaxed constraints $(C, L-\varepsilon_n)$. Hence, for each n, there exists an allocation rule $\tilde{q}^n$ for the discrete type space that respects $(C, L-\varepsilon_n)$ and has reduced form $\tilde{Q}^n$. Hence we can define an allocation rule q^n for the continuous type space that respects $(C, L-\varepsilon_n)$ and has reduced form Q^n: If $\theta \in A_{1k_1}^n \times \cdots \times A_{|I|k_{|I|}}^n$, we define

$$q_i^n(\theta) := \tilde{q}_i^n(A_{1k_1}^n, \ldots, A_{|I|k_{|I|}}^n).$$

So we have shown that $Q^n \in \mathcal{Q}(C, L-\varepsilon_n)$.

Next, we take the limit $n \to \infty$ to show that $Q \in \mathcal{Q}(C, L)$. Since $q^n \in \mathcal{Q}_0(C, 0)$ for all n and $\mathcal{Q}_0(C, L)$ is weakly compact, there is a weakly convergent subsequence with limit $q \in Q_0(C, L)$. Moreover, since q^n respects $(C, L-\varepsilon_n)$ and $\varepsilon_n \to 0$, then q respects (C, L), that is, $q \in \mathcal{Q}_0(C, L)$. By continuity of Λ, there exists Q' such that $Q(\theta) = Q'(\theta)$ for almost every θ. Since $\mathcal{Q}(C, L)$ is a compact set, $Q' \in \mathcal{Q}(C, L)$. As in the proof of Proposition 3.1 in Border (1991), one can show that also $Q \in \mathcal{Q}(C, L)$. *Q.E.D.*

B.3. *Border Characterization in the Partitional Constraint Structure*

PROOF OF THEOREM 8: We first derive the effective constraints for arbitrary sets $G \subset I$. For any $G \subset I$, define

$$\mathcal{H}_G^L := \{G' \in \tilde{\mathcal{H}} | G' \subset G\} \quad \text{and} \quad \mathcal{H}_G^C := \{G' \in \tilde{\mathcal{H}} | G' \cap G \neq \emptyset\}.$$

First, we show that $C(G) = \phi(\mathcal{H}_G^C) = \min\{\sum_{G' \in \mathcal{H}_G^C} C_{G'}, C_I - \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} L_{G'}\}$. To begin, observe that $C(G) \leq \phi(\mathcal{H}_G^C)$. This follows from the fact that for any $q \in \mathcal{P}$,

$$\sum_{i \in G} q_i \leq \sum_{G' \in \mathcal{H}_G^C} \sum_{i \in G'} q_i \leq \sum_{G' \in \mathcal{H}_G^C} C_{G'}, \tag{B.1}$$

$$\sum_{i \in G} q_i \leq C_I - \sum_{i \in I \setminus G} q_i \leq C_I - \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} \sum_{i \in G'} q_i \leq C_I - \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} L_{G'}, \tag{B.2}$$

where the first inequality in (B.1) and the second inequality in (B.2) hold since $G \subset \bigcup_{G' \in \mathcal{H}_G^C} G'$ and $q_i \geq 0$ $\forall i$. We construct an allocation $q \in \mathcal{P}$ to show that

$\phi(\mathcal{H}_G^C)$ can be attained as a maximum of (3), so $C(G) = \phi(\mathcal{H}_G^C)$. To this end, note that

$$\sum_{G' \in \mathcal{H}_G^C} L_{G'} \le \phi(\mathcal{H}_G^C) \le \sum_{G' \in \mathcal{H}_G^C} C_{G'}, \tag{B.3}$$

$$\phi(\mathcal{H}_G^C) + \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} L_{G'} \le C_I \le \phi(\mathcal{H}_G^C) + \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} C_{G'}, \tag{B.4}$$

which follows from the definition of ϕ and the assumption that $C_{G'} \ge L_{G'}\ \forall G' \in \tilde{\mathcal{H}}$ and $\sum_{G' \in \tilde{\mathcal{H}}} L_{G'} \le L_I \le C_I \le \sum_{G' \in \tilde{H}} C_{G'}$. These two inequalities imply that there are $\lambda_1, \lambda_2 \in [0, 1]$ such that

$$\phi(\mathcal{H}_G^C) = \sum_{G' \in \mathcal{H}_G^C} \big[\lambda_1 L_{G'} + (1 - \lambda_1) C_{G'}\big], \tag{B.5}$$

$$C_I = \phi(\mathcal{H}_G^C) + \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} \big[\lambda_2 L_{G'} + (1 - \lambda_2) C_{G'}\big]. \tag{B.6}$$

Now define q as follows: for each $G' \in \mathcal{H}_G^C$, $q_i = \frac{\lambda_1 L_{G'} + (1-\lambda_1) C_{G'}}{|G \cap G'|}$ if $i \in G' \cap G$, while $q_i = 0$ if $i \in G' \setminus G$; for each $G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C$ and all $i \in G'$, let $q_i = \frac{\lambda_2 L_{G'} + (1-\lambda_2) C_{G'}}{|G'|}$. Given this,

$$\begin{aligned}
\sum_{i \in G} q_i &= \sum_{G' \in \mathcal{H}_G^C} \sum_{i \in G \cap G'} q_i = \sum_{G' \in \mathcal{H}_G^C} \sum_{i \in G \cap G'} \left(\frac{\lambda_1 L_{G'} + (1 - \lambda_1) C_{G'}}{|G \cap G'|} \right) \\
&= \sum_{G' \in \mathcal{H}_G^C} \big[\lambda_1 L_{G'} + (1 - \lambda_1) C_{G'}\big], \\
\sum_{i \in I \setminus G} q_i &= \sum_{G' \in \mathcal{H}_G^C} \sum_{i \in G' \setminus G} q_i + \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} \sum_{i \in G'} q_i = \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} \sum_{i \in G'} q_i \\
&= \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C} \big[\lambda_2 L_{G'} + (1 - \lambda_2) C_{G'}\big].
\end{aligned}$$

Given (B.5) and (B.6), these equalities mean $\sum_{i \in G} q_i = \phi(\mathcal{H}_G^C)$ and $\sum_{i \in I} q_i = C_I$. Thus, it only remains to verify that $q \in \mathcal{P}$. The fact that $\sum_{i \in I} q_i = C_I \ge L_I$ means that the capacity constraint for $G = I$ is satisfied. For each $G' \in \mathcal{H}_G^C$, we have $\sum_{i \in G'} q_i = \lambda_1 L_{G'} + (1 - \lambda_1) C_{G'} \in [L_{G'}, C_{G'}]$, so the capacity constraint is satisfied. Analogously, the capacity constraint is satisfied for each $G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^C$.

Since establishing $L(G) = \psi(\mathcal{H}_G^L)$ is analogous, we only provide a sketch of the proof. First, it is easy to see that $L(G) \ge \psi(\mathcal{H}_G^L)$, following a similar derivation as in (B.1) and (B.2). Also, (B.3)–(B.6) hold with ϕ, $\mathcal{H}_G^C$, and C_I being replaced by ψ, $\mathcal{H}_G^L$, and L_I, respectively, and with some $\lambda_1, \lambda_2 \in [0, 1]$. Construct an allocation $q \in \mathcal{P}$ that achieves $\psi(\mathcal{H}_G^L)$, as follows: for each $G' \in \mathcal{H}_G^L$ and all $i \in G'$, $q_i = \frac{\lambda_1 L_{G'} + (1-\lambda_1)C_{G'}}{|G'|}$; for each $G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^L$, $q_i = \frac{\lambda_2 L_{G'} + (1-\lambda_2)C_{G'}}{|G' \setminus G|}$ if $i \in G' \setminus G$, while $q_i = 0$ if $i \in G' \cap G$. Given this, it is straightforward to see that $\sum_{i \in G} q_i = \sum_{G' \in \mathcal{H}_G^L} [\lambda_1 L_{G'} + (1-\lambda_1)C_{G'}]$ and $\sum_{i \in I \setminus G} q_i = \sum_{G' \in \tilde{\mathcal{H}} \setminus \mathcal{H}_G^L} [\lambda_2 L_{G'} + (1-\lambda_2)C_{G'}]$. The rest of the proof is parallel to that in the previous paragraph.

To summarize, we have shown that for any $G \subset I$, the effective constraints are given by $L(G) = \psi(\mathcal{H}_G^L)$ and $C(G) = \phi(\mathcal{H}_G^C)$. Lemma 1 implies that the effective constraints (C, L) are paramodular. Now we are ready to prove the theorem.

(i) Fix any $\theta = (\theta_i)_{i \in I}$ and define $T = \bigsqcup_{i \in I} T_i$, where $T_i = [\theta_i, \overline{\theta}_i]$. For any profile, we have $C(I(\tilde{\theta}, T)) = \phi(\mathcal{H}^C_{I(\tilde{\theta}, T)})$. Inserting this into (the general type-space version of) (BU) in Theorem 6 and noting that $C(I(\tilde{\theta}, T)) = 0$ if $\tilde{\theta} \notin Y(T)$, we get

$$\begin{aligned}
\sum_{i \in I} \int_{\theta_i}^{\overline{\theta}_i} Q_i(s_i)\, dF_i(s_i) &\leq \int_{\Theta_1} \cdots \int_{\Theta_{|I|}} C\big(I(\tilde{\theta}, T)\big)\, dF_1(\tilde{\theta}_1) \cdots dF_{|I|}(\tilde{\theta}_{|I|}) \\
&= \sum_{\mathcal{H}' \subset \mathcal{H}} \phi\big(\mathcal{H}'\big) \Pr\big\{\mathcal{H}^C_{I(\tilde{\theta}, T)} = \mathcal{H}'\big\} \\
&= \sum_{\mathcal{H}' \subset \mathcal{H}} \phi\big(\mathcal{H}'\big) \cdot \prod_{G \in \mathcal{H}'} \big(1 - \mathcal{F}_G(\theta)\big) \cdot \prod_{G \in \tilde{\mathcal{H}} \setminus \mathcal{H}'} \mathcal{F}_G(\theta).
\end{aligned}$$

Meanwhile, consider $T = \bigsqcup_i T_i$, where $T_i = [\underline{\theta}_i, \theta_i]$. We have $L(I(\tilde{\theta}, T)) = \psi(\mathcal{H}^L_{I(\tilde{\theta}, T)})$. Inserting this into (the general type-space version of) (BL) in Theorem 6, we have

$$\begin{aligned}
\sum_{i \in I} \int_{\underline{\theta}_i}^{\theta_i} Q_i(s_i)\, dF_i(s_i) &\geq \int_{\Theta_1} \cdots \int_{\Theta_{|I|}} L\big(I(\tilde{\theta}, T)\big)\, dF_1(\theta_1) \cdots dF_{|I|}(\theta_{|I|}) \\
&= \sum_{\mathcal{H}' \subset \mathcal{H}} \psi\big(\mathcal{H}'\big) \Pr\big\{\mathcal{H}^L_{I(\tilde{\theta}, T)} = \mathcal{H}'\big\} \\
&= \sum_{\mathcal{H}' \subset \mathcal{H}} \psi\big(\mathcal{H}'\big) \cdot \prod_{G \in \mathcal{H}'} \mathcal{F}_G(\theta) \cdot \prod_{G \in \tilde{\mathcal{H}} \setminus \mathcal{H}'} \big(1 - \mathcal{F}_G(\theta)\big).
\end{aligned}$$

(ii) Last, the proof of (ii) follows from application of Corollary 3 to (i). *Q.E.D.*

APPENDIX C: THE ROLE OF THE COMPLIANCE PROPERTY

The compliance condition ensures that the submodular upper bounds and supermodular lower bounds constitute effective bounds in the following sense.

LEMMA 2—Frank and Tardos (1988, p. 502, Proposition 2.3): *If* (C, L) *is paramodular, then* $C(G) = \max\{\sum_{i\in G} q_i | q = (q_i)_{i\in I}$ *respects* $(C, L)\}$ *and* $L(G) = \min\{\sum_{i\in G} q_i | q = (q_i)_{i\in I}$ *respects* $(C, L)\}$ *for each* $G \subset I$.

Furthermore, there is a sense in which compliance constitutes a weakest sufficient condition or a maximal domain for submodular upper bounds and supermodular lower bounds to be effective. Note first that a violation of compliance can only occur for sets $G, G' \subset I$ such that $G \cap G' \neq \emptyset$, because otherwise $C(G' \setminus G) - L(G \setminus G') = C(G') - L(G)$. Suppose that the four constraints $C(G')$, $C(G' \setminus G)$, $L(G)$, and $L(G \setminus G')$ are given for sets $G, G' \subset I$ with $G \cap G' \neq \emptyset$, and compliance is violated for these sets. The following lemma shows that if it is possible to extend the constraints to all subsets such that C is submodular, L is supermodular, and the set of feasible allocations is nonempty, then there exists such an extension for which *at least one constraint is not effective.*

LEMMA 3: *Let* $G, G' \in I$ *with* $G \cap G' \neq \emptyset$ *and let* $C(G'), C(G' \setminus G), L(G), L(G \setminus G') \in \mathbb{R}_+$ *such that* $C(G') - L(G) < C(G' \setminus G) - L(G \setminus G')$. *If there exists an extension* $(C(\tilde{G}), L(\tilde{G}))_{\tilde{G}\subset I}$ *of these constraints to* 2^I *such that* C *is submodular,* L *is supermodular, and* $\mathcal{P} := \{x \in \mathbb{R}_+^{|I|} | L(\tilde{G}) \leq \sum_{i\in\tilde{G}} x_i \leq C(\tilde{G}), \forall \tilde{G} \subset I\} \neq \emptyset$, *then there also exists an extension with these properties for which* $C(G' \setminus G) > \max\{\sum_{i\in G'\setminus G} x_i | x \in \mathcal{P}\}$ *or* $L(G \setminus G') < \min\{\sum_{i\in G\setminus G'} x_i | x \in \mathcal{P}\}$.

PROOF: Note first that (a) if $C(G') < C(G' \setminus G)$, then $C(G' \setminus G)$ is not effective; (b) if $G \subset G'$, the violation of compliance implies $C(G' \setminus G) > C(G') - L(G)$ so that $C(G' \setminus G)$ is not effective; and (c) if $G' \subset G$, then $L(G \setminus G')$ is ineffective because $L(G \setminus G') < L(G) - C(G')$. Hence the statement of the lemma follows in all three cases.

Second, supermodularity of L implies that L is monotonic. Therefore, we can assume that $L(G) \geq L(G \setminus G')$, because otherwise no supermodular extension exists.

After these preliminary considerations, we only have to consider the case that $G \not\subset G'$, $G' \not\subset G$, $C(G') \geq C(G' \setminus G)$, and $L(G) \geq L(G \setminus G')$. For this case, we define $C(G \cap G') = L(G \cap G') = C(G') - C(G' \setminus G)$. Then the violation of compliance implies that $C(G \cap G') = C(G') - C(G' \setminus G) < L(G) - L(G \setminus G')$ and hence $L(G \setminus G') < L(G) - C(G \cap G')$, which means that $L(G \setminus G')$ is not effective.

The proof will be complete once we define (C, L) for the remaining sets. We simplify notation by denoting $G_1 = G' \setminus G$, $G_2 = G \setminus G'$, and $G_3 = G \cap G'$. We fix a large number K that is greater than the sum of all upper and lower bounds imposed on these sets, and define for any $H \subset I$,

$$C(H) := \begin{cases} \sum_{k \in \{1,3\}: G_k \cap H \neq \emptyset} C(G_k), & \text{if } H \subset G', \\ K, & \text{otherwise,} \end{cases}$$

and

$$L(H) := \begin{cases} L(G_k), & \text{if } \emptyset \neq G_k \subset H \text{ for some } k \in \{2,3\} \text{ and } G \not\subseteq H. \\ L(G), & \text{if } G \subset H, \\ 0, & \text{if } G_k \not\subseteq H \text{ for all } k \in \{2,3\}. \end{cases}$$

It is easy to check that the upper and lower bounds defined here are consistent with those given above. It is also easy to check that $C(H) \geq L(H)$ for any $H \subset I$, while both C and L are monotonic, that is, $C(H) \leq C(H')$ for any $H \subset H' \subset I$, and similarly for L. To see that $\mathcal{P}$ is nonempty, choose an element $i_k \in G_k$ for each $k = 1, 2, 3$ and define $x \in \mathbb{R}_+^{|I|}$ by assigning $x_{i_1} = C(G') - C(G \cap G') = C(G' \setminus G)$, $x_{i_2} = K = C(G \setminus G') \geq L(G \setminus G')$, $x_{i_3} = L(G \cap G') = C(G \cap G')$, and $x_i = 0$ for each $i \in I \setminus \{i_1, i_2, i_3\}$. It is then straightforward to verify that x satisfies (C, L), so $x \in \mathcal{P}$.

We next show that C is submodular: for any two sets H and $H' \supset H$, and any $i \in I \setminus H'$, $C(H' \cup \{i\}) - C(H') \leq C(H \cup \{i\}) - C(H)$. This is immediate if $H' \not\subseteq G'$ or $i \notin G'$ since in the former case, $C(H' \cup \{i\}) = C(H') = K$ and $C(H \cup \{i\}) \geq C(H)$, while in the latter case, $C(H' \cup \{i\}) = C(H \cup \{i\}) = K$ and $C(H') \geq C(H)$. Thus we assume from now on that $H \subset H' \subset G'$ and $i \in G'$. Then $i \in G_k$ for some $k = 1, 3$. If $H' \cap G_k = \emptyset$, then $C(H' \cup \{i\}) - C(H') = C(G_k) = C(H \cup \{i\}) - C(H)$. If $H' \cap G_k \neq \emptyset$, then $C(H' \cup \{i\}) - C(H') = 0 \leq C(H \cup \{i\}) - C(H)$.

Last, we show that L is supermodular: for any two sets H and $H' \supset H$, and any $i \in I \setminus H'$, $L(H' \cup \{i\}) - L(H') \geq L(H \cup \{i\}) - L(H)$. Observe first that for any such $H \subset I$ and $i \in I$, we have $L(H \cup \{i\}) - L(H) = 0$ unless $G_k \not\subseteq H$ and $G_k \subset (H \cup \{i\})$ for some $k = 2, 3$, in which case we have either (i) $i \in G_k$, $G_k \setminus \{i\} \subset H \cap G$, and $H \cap G \neq G \setminus \{i\}$ or (ii) $i \in G_k$ and $H \cap G = G \setminus \{i\}$. This implies that to show the supermodularity, it suffices to consider the two cases (i) and (ii). If (i) holds and $H' \cap G \neq G \setminus \{i\}$, then $L(H \cup \{i\}) - L(H) = L(G_k) = L(H' \cup \{i\}) - L(H')$, as desired. If (i) holds and $H' \cap G = G \setminus \{i\}$, then we have $G_k \not\subseteq H'$, $G_{k'} \subset H'$ for $k' \in \{2,3\} \setminus \{k\}$, and $G = G_k \cup G_{k'} \subset H' \cup \{i\}$, which implies $L(H \cup \{i\}) - L(H) = L(G_k) < L(G) - L(G_{k'}) = L(H' \cup \{i\}) - L(H')$. Here the strict inequality follows from the fact that $L(G) > L(G \setminus G') + L(G \cap G') = L(G_k) + L(G_{k'})$. Finally, in case (ii) holds, we have $L(H \cup \{i\}) - L(H) = L(G) - L(G_{k'}) = L(H' \cup \{i\}) - L(H')$, as desired. *Q.E.D.*

APPENDIX D: THE CONNECTION WITH BUDISH, CHE, KOJIMA, AND MILGROM (2013)

The characterization of feasible interim allocation rules we study has a connection with the characterization of implementable expected allocations studied by Budish et al. (2013) (hereafter BCKM). BCKM studied the constraint structure—the set of agent–object pairs whose assignment probability must obey some arbitrary integer-valued ceiling and floor constraints—that permits any expected assignment that satisfies these constraints to be implemented by a lottery of deterministic assignments, each of which satisfies the same constraints. As mentioned in that paper, that requirement boils down to requiring that the set of feasible fractional assignments, which forms a bounded polytope, have integer-valued extreme points. While both characterizations deal with implementability of some marginals via some joint distribution, there are several differences: (i) The integrality of the feasible set is the main issue in BCKM's characterization, but it is not an issue in the current characterization, (ii) our main challenge arises from the fact that there are different types of each agent, whereas no such problem arises in BCKM, and (iii) BCKM adopted the notion of "universal implementation," which requires implementation to hold for all arbitrary quotas for the identified constraint structures. In contrast to this, we allow for arbitrary constraint structures, but require the effective constraints to be paramodular. For the specific case of a hierarchical constraint structure, our Lemma 1 shows that paramodularity of the effective constraints is universal, that is, it holds for arbitrary constraints on the hierarchical family. This is similar to BCKM, except their corresponding condition is that the constraint sets form a pair of hierarchies.

Despite the differences, these two results have a common mathematical foundation, which is provided by Edmonds' polymatroid intersection theorem. This connection will also explain why the universal implementation in BCKM can be attained by bi-hierarchical constraint sets, whereas it can be attained only by hierarchical constraint sets in the current context. For simplicity, we focus on the case in which the constraints are only in the upper bounds. This assumption can be dropped in most of the discussion, except for Appendix E.

To begin, let us define a polymatroid. Let Ω be a finite set, called the *ground set*, and consider a weight function $x : \Omega \to \mathbb{R}_+$. Let $\mathcal{X}$ denote all such functions. A bounded convex set

$$\mathcal{P} = \left\{ x \in \mathcal{X} \,\middle|\, \sum_{\omega \in U} x(\omega) \leq f(U), \forall U \in 2^{\Omega} \right\}$$

is said to be a *polymatroid* if $f : 2^{\Omega} \to \mathbb{R}_+$ is submodular.

Edmonds' polymatroid intersection theorem[31] has the following results:

[31] See, for instance, Theorem 46.1 and Corollary 46.1a of Schrijver (2000).

THEOREM 9: *Let $\mathcal{P}$ and $\mathcal{P}'$ be two polymatroids defined by f and f'.*

(i) *Primal Integrality* (*PI*). *All extreme points of $\mathcal{P} \cap \mathcal{P}'$ are integer-valued whenever f and f' are integer-valued.*

(ii) *Total Dual Integrality* (*TDI*). *For any integer-valued n-vector c, the dual of maximizing $c^T x$ over $x \in \mathcal{P} \cap \mathcal{P}'$, where f and f' are rationals, has an integer optimal solution.*

We now show how the characterizations given by these two papers relate to the two distinct parts of this theorem: BCKM relates to part (i) and our characterization relates to part (ii) of Theorem 9.

D.1. *BCKM*

It is easy to see how Theorem 9(i) implies the universal implementation characterization result of BCKM. In their model, the set $\Omega = N \times O$ is simply a set of agent–object pairs, with N representing the set of agents and O representing the set of objects, and for each $(i, o) \in \Omega$, the weight function $x(i, o)$ describes a (fractional) assignment of the object to agent i. BCKM then considered an arbitrary family $\mathcal{F} \subset 2^{\Omega}$ of subsets of Ω and required the fractional assignment to be in the set

$$\mathcal{Q} := \left\{ x \in \mathcal{X} \Big| \sum_{\omega \in U} x(\omega) \leq f(U), \forall U \in \mathcal{F} \right\}.$$

Their universal implementation result then boils down to the statement that *every extreme point of $\mathcal{Q}$ is integer-valued for any integer-valued f if $\mathcal{F}$ comprises a pair of disjoint hierarchies, that is, $\mathcal{F} = \mathcal{H} \cup \mathcal{H}'$, where $\mathcal{H}$ and $\mathcal{H}'$ are hierarchies.* To see how Theorem 9(i) implies this statement, observe first that, given the hypothesis,

$$\mathcal{Q} = \mathcal{P} \cap \mathcal{P}',$$

where

$$\mathcal{P} := \left\{ x \in \mathcal{X} \Big| \sum_{\omega \in U} x(\omega) \leq f(U), \forall U \in \mathcal{H} \right\}$$

and

$$\mathcal{P}' := \left\{ x \in \mathcal{X} \Big| \sum_{\omega \in U} x(\omega) \leq f(U), \forall U \in \mathcal{H}' \right\}.$$

To see now that the desired universal implementation characterization holds, it suffices to recall Lemma 1, which asserts that $\mathcal{P}$ and $\mathcal{P}'$ (each set generated by quotas defined on hierarchical sets) are polymatroids. Hence, BCKM's main result follows from Theorem 9(i).

This perspective provides a new mathematical insight on BCKM. More interestingly, it suggests a way to extend BCKM. Suppose the assignment must satisfy upper bounds $f: 2^{\Omega} \to \mathbb{Z}_+$ and lower bounds $g: 2^{\Omega} \to \mathbb{Z}_+$. We say that (f, g) is *bi-paramodular* if there exist (f_1, g_1) and (f_2, g_2) such that $(f_i, g_i)_{i=1,2}$ is paramodular, and $f = \min\{f_1, f_2\}$ and $g = \max\{g_1, g_2\}$. Then we get the following result.

THEOREM 10: *Any fractional assignment x is implementable with respect to (f, g) if (f, g) is bi-paramodular.*

D.2. *The Current Paper*

The connection of Theorem 9 with the current paper is much more difficult to see; so far, we have been able to establish it only for the upper bound case. The upshot is that at least in the case of upper bound only, we can see why Theorem 9(ii) implies that the type of characterization as in Theorem 3 should obtain.

To begin, let $\tilde{q}_i(\theta) = q_i(\theta)p(\theta)$ and $\tilde{q} = (\tilde{q}_i(\theta))_{i\in I,\theta\in\Theta}$. For any interim allocation rule Q, consider the linear programming problem

$$\text{(P1)} \qquad \max_{\tilde{q}\geq 0} \sum_{i\in I,\theta\in\Theta} \tilde{q}_i(\theta)$$

subject to

$$\text{(D.1)} \qquad \sum_{i\in G} \tilde{q}_i(\theta) \leq C(G)p(\theta), \quad \forall G\subset I, \forall\theta\in\Theta, \qquad \big[x(G,\theta)\big]$$

and

$$\text{(D.2)} \qquad \sum_{\theta_{-i}\in\Theta_{-i}} \tilde{q}_i(\theta_i,\theta_{-i}) \leq Q_i(\theta_i)p_i(\theta_i), \quad \forall\theta_i\in\Theta_i, \forall i\in I, \qquad \big[z(i,\theta_i)\big],$$

where each variable in the square brackets is the dual variable for the corresponding constraint. The constraints (D.1) correspond to the capacity constraints we have in our model for subsets of agents. The constraints (D.2) correspond to the requirement that Q is a reduced form (or implementable).

Note that given the last constraint, the optimal value of this problem cannot exceed the aggregate interim allocation probability, that is, $\sum_{i\in I}\sum_{\theta_i\in\Theta_i} p_i(\theta_i)\times Q_i(\theta_i)$. Note also that the interim allocation rule $(Q_i(\theta_i))_{\theta_i\in\Theta_i,i\in I}$ is a reduced form if and only if the optimal value equals $\sum_{i\in I}\sum_{\theta_i\in\Theta_i} p_i(\theta_i)Q_i(\theta_i)$.

To see how this program is related to our characterization, observe that the coefficients in the primal objective function are all 1's. Hence, if the feasible set associated with constraints (D.1) and (D.2) are TDI, then the dual of (P1) has an optimal integer solution, as implied by Theorem 9(ii). It turns out that this implication gives rise to a Border type characterization, which will be established in the next section, Appendix E.

Hence, the important question with regard to our characterization boils down to whether the feasible set associated with constraints (D.1) and (D.2) is TDI. The answer to this question is given by observing that each constraint gives rise to a polymatroid.

LEMMA 4: *Each of the constraints* (D.1) *and* (D.2) *gives rise to a polymatroid with* $\Omega = I \times \Theta$ *as a ground set*.

PROOF: Given the ground set $\Omega = I \times \Theta$, for each $\omega = (i, \theta) \in \Omega$ and $U \subset \Omega$, let $x(\omega) = \tilde{q}_i(\theta)$ and $x(U) = \sum_{\omega \in U} x(\omega)$.

We first show that the set of $\tilde{q}$'s that satisfy (D.1) is a polymatroid. To do so, define a weight function $f_1 : 2^\Omega \to \mathbb{R}_+$ as follows: For each $U \subset \Omega$, let $\alpha(\theta, U) := \{i \in I | (i, \theta) \in U\}$ and

$$f_1(U) = \sum_{\theta \in \Theta} C\big(\alpha(\theta, U)\big) p(\theta).$$

Letting $\mathcal{P}_1 := \{x \in \mathbb{R}_+^{|\Omega|} : x(U) \le f_1(U)\}$, it is straightforward to check that $\mathcal{P}_1$ is equivalent to the set of allocations that satisfy (D.1), which is thus a polymatroid if f_1 is submodular. To show this, consider any subsets $U, U' \subset \Omega$ with $U \subset U'$ and any $\omega = (i, \theta) \notin U'$. Then we have $f_1(U \cup \{\omega\}) - f_1(U) = [C(\alpha(\theta, U) \cup \{i\}) - C(\alpha(\theta, U))]p(\theta) \ge [C(\alpha(\theta, U') \cup \{i\}) - C(\alpha(\theta, U'))]p(\theta) = f_1(U' \cup \{\omega\}) - f_1(U')$, where the inequality holds due to the fact that $\alpha(\theta, U) \subset \alpha(\theta, U')$ and C is submodular.

We next show that the set of $\tilde{q}$'s that satisfy (D.2) is a polymatroid. To do so, define another weight function $f_2 : 2^\Omega \to \mathbb{R}_+$ as follows: For each $U \subset \Omega$, let $(i, \theta_i, \Theta_{-i}) = \{(i, \theta_i, \theta_{-i}) : \theta_{-i} \in \Theta_{-i}\}$ (by some abuse of notation) and let

$$f_2(U) = \sum_{(i,\theta_i):(i,\theta_i,\Theta_{-i}) \cap U \neq \emptyset} p_i(\theta_i) Q_i(\theta_i).$$

Letting $\mathcal{P}_2 := \{x \in \mathbb{R}_+^{|\Omega|} : x(U) \le f_2(U)\}$, it is again straightforward to check that $\mathcal{P}_2$ is equivalent to the set of allocations that satisfy (D.2), which is thus a polymatroid if f_2 is submodular. To show this, consider any subsets $U, U' \subset \Omega$ with $U \subset U'$ and any $\omega = (i, \theta_i, \theta_{-i}) \notin U'$. If $(i, \theta_i, \Theta_{-i}) \cap U \neq \emptyset$, then we have $f_2(U \cup \{\omega\}) - f_2(U) = 0 = f_2(U' \cup \{\omega\}) - f_2(U')$. If $(i, \theta_i, \Theta_{-i}) \cap U = \emptyset$ and $(i, \theta_i, \Theta_{-i}) \cap U' \neq \emptyset$, then $f_2(U' \cup \{\omega\}) - f_2(U') = 0 \le p_i(\theta_i) Q_i(\theta_i) = f_2(U \cup \{\omega\}) - f_2(U)$. If $(i, \theta_i, \Theta_{-i}) \cap U' = \emptyset$, then $f_2(U \cup \{\omega\}) - f_2(U) = p_i(\theta_i) Q_i(\theta_i) = f_2(U' \cup \{\omega\}) - f_2(U')$. *Q.E.D.*

REMARK 4—Universal Implementation: When the sets of agents facing quota constraints form a hierarchy, we have a universal implementation in the sense that regardless of the specific values of the quotas, the Border type characterization, specifically Theorem 3, holds. The reason for this is that by Lemma 1, the quota constraints (D.1) form a polymatroid regardless of the specific values of the quotas. The reason that we cannot accommodate more (e.g., bihierarchy), as also proven by Remark 1, is because we have already used up another polymatroid in our reduced-form requirement (D.2). This is precisely the reason why bihierarchy is possible under BCKM but not in our case; they did not face additional constraints such as (D.2) that we have to deal with.

APPENDIX E: POLYMATROID METHOD FOR THE BORDER CHARACTERIZATION

In this subsection, we show that the polymatroid optimization problem stated in (P1) provides an alternative way to obtain the Border characterization. As mentioned earlier, this result is established by using the fact that the constraints of (P1) are TDI, so the dual problem has an integer solution. For this argument, we need to assume that p and Q are all rational numbers. We note that the argument below is not readily adaptable to the general case with both upper and lower bound constraints. This illustrates the advantage of using our network-flow approach to obtain the generalized characterization as in Theorem 3.

To begin, let us write the dual problem to (P1) as

$$\text{(Dual-1)} \quad \min_{x(\cdot),z(\cdot)} \sum_{G\subset I,\theta\in\Theta} p(\theta)C(G)x(G,\theta)+\sum_{i\in I}\sum_{\theta_i\in\Theta_i}\big[Q_i(\theta_i)p_i(\theta_i)z(i,\theta_i)\big]$$

subject to

$$\text{(E.1)} \qquad \sum_{G:i\in G} x(G,\theta)+z(i,\theta_i)\geq 1, \quad \forall i\in I, \forall\theta\in\Theta$$

and $x(G,\theta), z(i,\theta_i)\geq 0\ \forall G,\theta,i,\theta_i$. To show the sufficiency of the Border condition for implementability of Q,[32] suppose that Q is not a reduced form, which means that the optimal value of the primal, and thus the dual, problem is smaller than $\sum_{i\in I}\sum_{\theta_i\in\Theta_i} p_i(\theta_i)Q_i(\theta_i)$. We show that this leads to the violation of the upper bound condition in (B) for some $T\subset D$.[33]

[32]The proof of necessity is straightforward and thus is omitted.

[33]The duality argument we use below is similar to that in Cai, Daskalakis, and Weinberg (2011). Unlike Cai, Daskalakis, and Weinberg (2011), however, our argument exploits the TDI property to yield the Border characterization, which is much tighter than the characterization in Cai, Daskalakis, and Weinberg (2011).

To this end, recall first that the constraints of (P1) are TDI, so its dual (Dual-1) has an integer solution, which then implies $z(i,\theta_i)=0$ or 1 for all (i,θ_i), since otherwise one could reduce $z(i,\theta_i)$, and thereby the value of the objective function, without violating (E.1).

Given any such optimal $z(\cdot)$, the dual problem (Dual-1) can be decomposed into the following subproblems: for each $\theta\in\Theta$,

$$\text{(Dual-2)}\quad \min_{x(\cdot,\theta),y(\cdot,\theta)} p(\theta)\sum_{G\subset I} C(G)x(G,\theta)$$

subject to

$$\text{(E.2)}\qquad \sum_{G:i\in G} p(\theta)x(G,\theta)\geq p(\theta)\bigl[1-z(i,\theta_i)\bigr],\quad \forall i\in I.$$

With $\gamma(i,\theta)$ denoting the dual variable for the constraint (E.2), the dual problem to (Dual-2) can be written as

$$\text{(P2)}\qquad \max_{\gamma(\cdot,\theta)}\sum_{i\in I} p(\theta)\bigl[1-z(i,\theta_i)\bigr]\gamma(i,\theta)$$

subject to

$$\text{(E.3)}\qquad \sum_{i\in G}\gamma(i,\theta)\leq C(G),\quad \forall G\subset I.$$

To solve (P2), let $T_i=\{\theta_i\in\Theta_i|z(i,\theta_i)=0\}$ for each $i\in I$, so $z(i,\theta_i)=1$ for any $\theta_i\in\Theta_i\setminus T_i$. Recall that with $T=\bigsqcup_{i\in I}T_i$, $I(\theta,T)=\{i\in I|\theta_i\in T_i\}$. Then the objective function of (P2) becomes

$$\sum_{i:z(i,\theta_i)=0} p(\theta)\gamma(i,\theta)=p(\theta)\sum_{i\in I(\theta,T)}\gamma(i,\theta),$$

which clearly attains its maximum when $\sum_{i\in I(\theta,T)}\gamma(i,\theta)=C(I(\theta,T))$, given the constraint (E.3). Plug this into the objective function of (Dual-1) to obtain

$$\sum_{\theta\in\Theta} p(\theta)C\bigl(I(\theta,T)\bigr)+\sum_{i\in I}\sum_{\theta_i\in\Theta_i} p_i(\theta_i)Q_i(\theta_i)z(i,\theta_i).$$

Noting that this expression must be smaller than $\sum_{i\in I}\sum_{\theta_i\in\Theta_i}p_i(\theta_i)Q_i(\theta_i)$ by assumption, we get

$$\begin{aligned}0&>\sum_{\theta\in\Theta} p(\theta)C\bigl(I(\theta,T)\bigr)+\sum_{i\in I}\sum_{\theta_i\in\Theta_i} p_i(\theta_i)Q_i(\theta_i)\bigl[z(i,\theta_i)-1\bigr]\\&=\sum_{\theta\in Y(T)} p(\theta)C\bigl(I(\theta,T)\bigr)-\sum_{i\in I}\sum_{\theta_i\in T_i} p_i(\theta_i)Q_i(\theta_i),\end{aligned}$$

which means that (B) is violated for T, as desired.

E.1. *A Characterization for General Constraints*

Without assuming supermodularity of the upper bounds, Cai, Daskalakis, and Weinberg (2011) derived a characterization that involves a continuum of constraints. To state their result, we define

$$\mathcal{A}(C) := \left\{ x \in [0,1]^{|I|} \middle| \sum_{i \in G} x_i \leq C(G), \forall G \subset I \right\}$$

as the set of allocations that is feasible for given upper bounds $C : 2^I \to [0, n]$. In the following theorem, C need not be submodular.

THEOREM 11—Cai, Daskalakis, and Weinberg (2011): *Let Q be an interim allocation rule. The allocation rule Q is the reduced form of an allocation rule that respects $(C, 0)$ if and only if for all weights $(W_i(\theta_i))_{i\in I, \theta_i \in \Theta_i} \in [0,1]^{\sum_i |\Theta_i|}$,*

$$\sum_{i\in I} \sum_{\theta_i \in \Theta_i} W_i(\theta_i)\big[p_i(\theta_i)Q_i(\theta_i)\big] \leq \sum_{\theta\in\Theta} \max_{x \in \mathcal{A}(C)} \left\{ \sum_{i\in I} W_i(\theta_i) x_i \right\}. \tag{E.4}$$

This characterization is obtained from the dual linear program (Dual-1) and the weights W are the dual variables z. Therefore, submodularity implies that (E.4) has to be checked only for integer-valued weights. But for $(W_i(\theta_i))_{i\in I, \theta_i\in\Theta_i} \in \{0,1\}^{\sum_i |\Theta_i|}$, (E.4) is equivalent to (B) with $T = \{\theta_i \in D | W_i(\theta_i) = 1\}$.

Conversely, if submodularity is violated, some of the constraints in (E.4) induced by noninteger weights are binding. To see this, consider the first example in Table I in Remark 1. If we maximize the objective function subject to (B), a maximizer is given by $Q_i^*(\underline{\theta}_i) = 13/8$ and $Q_i^*(\overline{\theta}_i) = 9/4$ for all $i \in I$. For this interim allocation rule, (E.4) is, for example, violated for weights $W_i(\underline{\theta}_i) = 1/2$ and $W_i(\overline{\theta}_i) = 1$ for all $i \in I$. Indeed, a straightforward calculation shows that for these weights and the interim allocation rule Q^*, the LHS of (E.4) is 147/32, whereas the RHS is 9/2, which is strictly smaller. This demonstrates that the additional constraints can, in general, not be neglected and the characterization obtained in the absence of submodularity is much less tractable than our characterization in Theorem 3.

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