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# A NOTE ON ARROW–ENTHOVEN'S SUFFICIENCY THEOREM ON QUASI-CONCAVE PROGRAMMING: CORRIGENDUM TO: QUASI-CONCAVE PROGRAMMING

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In this note, we reconsider Arrow–Enthoven's (1961) Sufficiency Theorem 1 on quasi-concave programming [Econometrica, 29(4), p. 783]. One example illustrates the fact that condition (c) of their Theorem 1 should include the twice continuous differentiability of the quasi-concave objective function so that the conditions of the theorem are sufficient to identify global optima.

KEYWORDS: Continuous differentiability, quasi-concavity, optimization.

## 1. INTRODUCTION

The seminal contribution of Arrow and Enthoven (1961) on quasi-concave programming has enlarged the set of functions to which optimization theory can be applied. Arrow and Intriligator (1982), Intriligator (2000), and Samuelson (1983) put forward the importance of ordinal functions in economic theory. Koenker and Mizera (2010) used quasi-concavity density estimation in probability and statistics. In this note, we reconsider Theorem 1 of Arrow and Enthoven (1961) (henceforth AET), which provides a set of independent sufficient conditions so that the Kuhn and Tucker theorem holds in quasi-concave programming maximization problems. The AET was extended notably by Ferland (1972) to optimization problems with twice continuously differentiable objective functions. Some characterizations of quasi-concavity properties were provided in Diewert, Avriel, and Zang (1981).

It is common knowledge that the Kuhn and Tucker first-order conditions are not sufficient to determine, unlike the first-order conditions in concave programming, global optima of quasi-convex optimization problems, that is, problems with a quasi-concave objective function and a convex constraint set. In AET, the objective function and the constraint functions are assumed to be differentiable and quasi-concave, so that the Kuhn and Tucker conditions are sufficient to determine global optima provided some additional regularity conditions are met at the optimum point. More specifically, we focus on condition (c) in AET. Condition (c) states that, in the neighborhood of the optimum point, if the objective function is twice differentiable, and its gradient is different from zero, then the stationary point that meets the Kuhn and Tucker conditions solves the inequality constrained maximization problem. Through an example, we show that this condition also requires the twice continuous differentiability of the quasi-concave objective function.

The remainder of the paper is organized as follows. In Section 2, we restate and discuss the AET. In Section 3, we provide an example in which condition (c) fails to identify the maximum. In Section 4, we conclude.

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## 2. ARROW AND ENTHOVEN'S THEOREM 1

Consider the following notational convention. Vectors are in bold and are expressed in columns. The transpose of $\mathbf{x}$ is denoted by $\mathbf{x}^T$. Let $\mathbf{x} \in \mathbb{R}^n_+$. Then, $\mathbf{x} \geq \mathbf{0}$ means $x_i \geqslant 0$, $i = 1, \ldots, n$; $\mathbf{x} > \mathbf{0}$ means there is some $i$ such that $x_i > 0$, with $\mathbf{x} \neq \mathbf{0}$, and $\mathbf{x} \gg \mathbf{0}$ means $x_i > 0$ for all $i$, $i = 1, \ldots, n$. Let $B(\mathbf{x}, \delta)$ be the ball of center $\mathbf{x}$ and radius $\delta > 0$. Let $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$, $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$, be a vector function, with $n$, $p < \infty$, where $X$ is an open set. The Jacobian matrix $\mathcal{J}_{\mathbf{F}(\mathbf{x})} = [\frac{\partial(F_1,\ldots,F_p)}{\partial(x_1,\ldots,x_n)}]$ is a function from $X$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, the set of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^p$. When $p = 1$, $\mathcal{J}_{F(\mathbf{x})} = [\nabla F(\mathbf{x})]^T$, where $\nabla F(\mathbf{x})$ is the gradient of $F$. Let $\mathbf{h} \in \mathbb{R}^n$. The function $d\mathbf{F}_{\mathbf{x}^*}(\cdot) : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$, with $d\mathbf{F}_{\mathbf{x}^*}(\mathbf{h}) = \mathcal{J}_{\mathbf{F}(\mathbf{x}^*)}\mathbf{h}$, is the differential of $\mathbf{F}$ at $\mathbf{x}^*$. The notation $\mathbf{F} \in \mathcal{C}^1(X, \mathbb{R}^p)$ is used to say that $\mathbf{F}$ is continuously differentiable on $X$, which we now define.

DEFINITION—Continuous Differentiability: The function $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$, $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$, is *continuously differentiable* on the open set $X$ if it is differentiable on $X$, and the $(p, n)$ matrix $\mathcal{J}_{\mathbf{F}(\mathbf{x})} = [\frac{\partial(F_1,\ldots,F_p)}{\partial(x_1,\ldots,x_n)}]$ is a continuous function from $X$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$. That is, for all $\mathbf{x}^* \in X$, and every $\varepsilon > 0$, there is a $\delta > 0$ for which if $\mathbf{x}' \in X$ and $\mathbf{x}' \in B(\mathbf{x}^*, \delta)$, then $\|d\mathbf{F}_{\mathbf{x}^*}(\mathbf{h}) - d\mathbf{F}_{\mathbf{x}'}(\mathbf{h})\| < \varepsilon$, where $\mathbf{h} \in \mathbb{R}^n$.

Therefore, $\mathbf{F} \in \mathcal{C}^s(X, \mathbb{R}^p)$ if each component function of $\mathbf{F}$ has first through $s$th continuous partial derivatives on $X$. Thus, $\mathbf{F}$ is twice continuously differentiable on $X$ if and only if $\mathbf{F} \in \mathcal{C}^2(X, \mathbb{R}^p)$: the first- and second-order partial derivatives of $\mathbf{F}$ exist and are continuous for all $\mathbf{x} \in X$. A continuous function is of class $\mathcal{C}^0$, and a smooth function is of class $\mathcal{C}^\infty$. A twice differentiable function is of class $\mathcal{C}^1$.

Consider now the following inequality constrained maximization problem:

$$(\mathcal{P}) : \quad \max_{\{\mathbf{x}\}} F(\mathbf{x})$$
$$\text{subject to } \mathbf{x} \in C = X \cap \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{G}(\mathbf{x}) \geq \mathbf{0}\right\}, \tag{1}$$

where $F : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$, and $\mathbf{G} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{G}(\mathbf{x})$, with $G_j : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto G_j(\mathbf{x})$, $j = 1, \ldots, m$. Let $F$ and $\mathbf{G}$ be differentiable on $X$.

Assume that there exists a solution to $(\mathcal{P})$ (Weierstrass's theorem holds), that is, there exists $\mathbf{x}^* \in X : F(\mathbf{x}^*) \geqslant F(\mathbf{x})$, for all $\mathbf{x} \in C$.

Define $L : X \times \mathbb{R}^m_+ \to \mathbb{R}$, with $(\mathbf{x}; \boldsymbol{\lambda}) \mapsto L(\mathbf{x}; \boldsymbol{\lambda}) := F(\mathbf{x}) + \sum_{j=1}^m \lambda_j G_j(\mathbf{x})$, where the Lagrangian $L$ is differentiable on $X$. The Kuhn and Tucker theorem provides necessary conditions for $(\mathcal{P})$ to have a local maximum at $\mathbf{x}^* \geq \mathbf{0}$, if the constraint qualification holds, that is, if the set of effective constraints has the maximum possible rank, in which case there exists a vector $(\boldsymbol{\lambda}^*)^T = (\lambda_1^*, \ldots, \lambda_m^*) \in \mathbb{R}^m_+$ such that

$$\forall i \quad x_i^* \geqslant 0, \qquad \left.\frac{\partial L(\mathbf{x}; \boldsymbol{\lambda})}{\partial x_i}\right|_{(\mathbf{x};\boldsymbol{\lambda})=(\mathbf{x}^*;\boldsymbol{\lambda}^*)} \leqslant 0, \quad \text{with } x_i^* \left.\frac{\partial L(\mathbf{x}; \boldsymbol{\lambda})}{\partial x_i}\right|_{(\mathbf{x};\boldsymbol{\lambda})=(\mathbf{x}^*;\boldsymbol{\lambda}^*)} = 0; \tag{KT1}$$

$$\forall j \quad \lambda_j^* \geqslant 0, \qquad G_j\big(\mathbf{x}^*\big) \geqslant 0, \quad \text{with } \lambda_j^* G_j\big(\mathbf{x}^*\big) = 0, \tag{KT2}$$

where $\frac{\partial L(\mathbf{x};\boldsymbol{\lambda})}{\partial x_i}|_{(\mathbf{x};\boldsymbol{\lambda})=(\mathbf{x}^*;\boldsymbol{\lambda}^*)} = \frac{\partial F(\mathbf{x})}{\partial x_i}|_{\mathbf{x}=\mathbf{x}^*} + \sum_{j=1}^m \lambda_j^* \frac{\partial G_j(\mathbf{x})}{\partial x_i}|_{\mathbf{x}=\mathbf{x}^*}$, $i = 1, \ldots, n$.

As emphasized by Arrow and Enthoven (1961), if the objective and constraints vector functions are concave, then (KT1) and (KT2) constitute a set of sufficient conditions for a constrained maximum (see Kuhn (1968)). In order to extend the set of functions on which the conditions of such a theorem might have applied, Arrow and Enthoven assumed that the objective function $F$ as well as the $m$-dimensional vector function $\mathbf{G}$ were quasi-concave (for a complete characterization of quasi-concave functions, see, for instance, Diewert, Avriel, and Zang (1981)). Before we restate the AET, we define the notion of a relevant variable. A variable $x_i^*$ is relevant if there is some point in the constraint set, say $\bar{x}_i^*$, at which $\bar{x}_i^* > 0$.

THEOREM—Arrow and Enthoven: *Let* $F : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto F(\mathbf{x})$, *be a quasi-concave differentiable function, and let* $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{G}(\mathbf{x})$, *be a quasi-concave differentiable vector function, where* $G_j : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto G_j(\mathbf{x})$, $j = 1, \dots, m$ , *both defined for* $\mathbf{x} \geq \mathbf{0}$. *Let* $\mathbf{x}^*$ *and* $\boldsymbol{\lambda}^*$ *satisfy* (KT1)–(KT2), *and let one of the following conditions be satisfied*:

(a) $\frac{\partial F(\mathbf{x})}{\partial x_k}|_{\mathbf{x}=\mathbf{x}^*} < 0$, *for at least one variable* $x_k^*$;

(b) $\frac{\partial F(\mathbf{x})}{\partial x_l}|_{\mathbf{x}=\mathbf{x}^*} > 0$, *for some relevant variable* $x_l^*$;

(c) $\mathcal{J}_{\mathbf{F}(\mathbf{x}^*)} = (\frac{\partial F(\mathbf{x})}{\partial x_1}|_{\mathbf{x}=\mathbf{x}^*}, \dots, \frac{\partial F(\mathbf{x})}{\partial x_n}|_{\mathbf{x}=\mathbf{x}^*}) \neq \mathbf{0}$ *and* $F(\mathbf{x})$ *is twice differentiable in the neighborhood of* $\mathbf{x}^*$;

(d) $F(\mathbf{x})$ *is concave*.

*Then* $\mathbf{x}^*$ *maximizes* $F(\mathbf{x})$ *subject to the constraints* $\mathbf{G}(\mathbf{x}) \geq 0$, $\mathbf{x} \geq 0$.

The AET stipulates that *only one* of these four conditions (there may be others) needs to be satisfied for $\mathbf{x}^*$ to maximize $F(\mathbf{x})$ subject to the constraints if (KT1) and (KT2) hold at $\mathbf{x}^*$. It is worth noticing that, from (a) and (b), it follows that the condition $\mathcal{J}_{\mathbf{F}(\mathbf{x}^*)} \neq \mathbf{0}$ is sufficient if all variables $x_i$ are relevant. In this note, we focus on condition (c) to put forward the importance of the twice continuous differentiability of the quasi-concave objective function.

Sufficient condition (c) says that $\mathbf{x}^*$ is not a stationary point of the quasi-concave objective function $F$, and that $F$ is twice differentiable (so $F \in \mathcal{C}^1(X, \mathbb{R})$). It is worth noticing that a concave function defined on an open and convex set is continuously differentiable everywhere on this set, except possibly at a set of points of Lebesgue measure zero. But a quasi-concave function may be discontinuous, and then it may not be differentiable, in the interior of its domain. We put forward that condition (c) should also include the *twice continuous* differentiability of the quasi-concave objective function. Therefore, the following example illustrates that condition (c) alone is not sufficient for AET to hold.

## 3. CONDITION (C): AN EXAMPLE

Consider the problem $(\mathcal{P})$ with $X = \mathbb{R}_+^n$ and $F$ and $\mathbf{G}$ given by

$$F(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} = \bar{\mathbf{x}}; \\ \left(\sum_{i=1}^n x_i - \frac{1}{r}\right)^4 \sin\left(\frac{1}{\sum_{i=1}^n x_i - \frac{1}{r}}\right) + \sum_{i=1}^n x_i - \frac{1}{r}, & \text{if } \mathbf{x} \neq \bar{\mathbf{x}}, \end{cases} \tag{2}$$

where $\bar{\mathbf{x}} = (\frac{1}{nr}, \dots, \frac{1}{nr})^T$, with $1 \leqslant r < \infty$, and

$$G_1(\mathbf{x}) = \frac{\pi + r}{\pi r} - \sum_{i=1}^{n} x_i \quad \text{and} \quad G_2(\mathbf{x}) = \begin{cases} \left(\dfrac{2}{r} - \displaystyle\sum_{i=1}^{n} x_i\right)^2 - \dfrac{1}{r^2}, & \text{if } \mathbf{0} \leq \mathbf{x} \leq \bar{\mathbf{x}}, \\ 0, & \text{if } \mathbf{x} > \bar{\mathbf{x}}. \end{cases} \tag{3}$$

As the objective function $F$ is continuous on the compact constraint set $C = \{\mathbf{x} \in \mathbb{R}^n_+ : G_1(\mathbf{x}) \geqslant 0 \text{ and } G_2(\mathbf{x}) \geqslant 0\}$, then, by the Weierstrass theorem, there exists a global maximum to $(\mathcal{P})$.

The remainder of this section is devoted to the characterization of the set of optima. First, we check that (2) and (3) satisfy the assumptions as well as condition (c) of AET. Second, we set up the Lagrangian and determine its stationary points by using (KT1) and (KT2). Third, we show condition (c) of AET is not sufficient.

The functions $G_1(\mathbf{x})$ and $G_2(\mathbf{x})$ are differentiable and quasi-concave on $\mathbb{R}^n_+$, and the function $F(\mathbf{x})$ is twice differentiable on $\mathbb{R}^n_+$. As $F(\mathbf{x})$ is differentiable on $\mathbb{R}^n_+$, the Jacobian matrix $\mathcal{J}_{F(\mathbf{x})}$ of $F(\mathbf{x})$ exists, and it may be written

$$[\mathcal{J}_{F(\mathbf{x})}]^T = \begin{bmatrix} \dfrac{\partial F(\mathbf{x})}{\partial x_1} \\ \vdots \\ \dfrac{\partial F(\mathbf{x})}{\partial x_j} \\ \vdots \\ \dfrac{\partial F(\mathbf{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 4a^3 \sin\left(\dfrac{1}{a}\right) - a^2 \cos\left(\dfrac{1}{a}\right) + 1 \\ \vdots \\ 4a^3 \sin\left(\dfrac{1}{a}\right) - a^2 \cos\left(\dfrac{1}{a}\right) + 1 \\ \vdots \\ 4a^3 \sin\left(\dfrac{1}{a}\right) - a^2 \cos\left(\dfrac{1}{a}\right) + 1 \end{bmatrix}, \tag{4}$$

where $a \equiv \sum_{i=1}^n x_i - \frac{1}{r}$. As $\mathcal{J}_{F(\mathbf{x})} \gg \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n_+$, the function $F$ is strictly increasing with respect to each $x_i$, then, it is quasi-concave in $\mathbf{x}$ on $\mathbb{R}^n_+$. In addition, as $\mathcal{J}_{F(\bar{\mathbf{x}})} = [1, \dots, 1] \neq \mathbf{0}$, the function $F$ satisfies condition (c) of AET. Finally, as $F$ is twice differentiable, for all $\mathbf{x} \in \mathbb{R}^n_+$, the bordered Hessian matrix $\bar{\mathcal{H}}_{F(\mathbf{x})} = \left[\begin{smallmatrix} 0 & \mathcal{J}_{\mathbf{F(x)}} \\ [\mathcal{J}_{\mathbf{F(x)}}]^T & \mathcal{H}_{\mathbf{F(x)}} \end{smallmatrix}\right]$ of $F$ exists, and the $ij$th element of the Hessian matrix $\mathcal{H}_{F(\mathbf{x})} = [\frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j}]_{1 \leqslant i,j \leqslant n}$ of $F$ may be written

$$\forall i, j \quad \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j} = (12a - 1)\sin\left(\frac{1}{a}\right) - 6a\cos\left(\frac{1}{a}\right). \tag{5}$$

Let us now characterize the solution to $(\mathcal{P})$. The Lagrangian associated with this inequality constrained maximization problem may be written

$$L(\mathbf{x}; \lambda_1, \lambda_2) := \left(\sum_{i=1}^n x_i - \frac{1}{r}\right)^4 \sin\left(\frac{1}{\displaystyle\sum_{i=1}^n x_i - \frac{1}{r}}\right) + \sum_{i=1}^n x_i - \frac{1}{r}$$
$$+ \lambda_1\left(\frac{\pi + r}{\pi r} - \sum_{i=1}^n x_i\right) + \lambda_2\left(\left(\frac{2}{r} - \sum_{i=1}^n x_i\right)^2 - \frac{1}{r^2}\right), \tag{6}$$

where $\lambda_1, \lambda_2 \geqslant 0$, and with $\mathbf{x} \geq \mathbf{0}$.

The stationary points of $L$ are given by (KT1) and (KT2), which may be written

$$\forall i \quad x_i \geqslant 0, \quad \frac{\partial F}{\partial x_i} - \lambda_1 - 2\lambda_2\left(\frac{1}{r} - a\right) \leqslant 0, \quad \text{with } x_i\left(\frac{\partial F}{\partial x_i} - \lambda_1 - 2\lambda_2\left(\frac{1}{r} - a\right)\right) = 0; \quad (7)$$

and

$$\lambda_1 \geqslant 0, \frac{1}{\pi} - a \geqslant 0, \quad \text{with } \lambda_1\left(\frac{1}{\pi} - a\right) = 0; \quad (8)$$

$$\lambda_2 \geqslant 0, \left(\frac{1}{r} - a\right)^2 - \frac{1}{r^2} \geqslant 0, \quad \text{with } \lambda_2\left(\left(\frac{1}{r} - a\right)^2 - \frac{1}{r^2}\right) = 0, \quad (9)$$

where $\frac{\partial F}{\partial x_i}$ is given by (4), and $G_1$, $G_2$, $\frac{\partial G_1}{\partial x_i}$, and $\frac{\partial G_2}{\partial x_i}$ are expressed in terms of $a$.

If $\mathbf{x} = \bar{\mathbf{x}}$ (in which case $a = 0$), then, by the complementary slackness conditions given by (8)–(9), as $G_1(\bar{\mathbf{x}}) > 0$ and $G_2(\bar{\mathbf{x}}) = 0$, we deduce $\lambda_1 = 0$, and from (7), as, for each $i$, $\frac{\partial F(\mathbf{x})}{\partial x_i}|_{\mathbf{x}=\bar{\mathbf{x}}} = 1$, we deduce $\lambda_2 \geqslant \frac{r}{2}$. Then, $\mathbf{x} = \bar{\mathbf{x}}$ satisfies (7)–(9).[1] Therefore, there is a pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2_+$, which satisfies (8)–(9). But $\bar{\mathbf{x}}$ does not maximize $F(\mathbf{x})$ subject to $\mathbf{x} \in C$, while the point $\mathbf{x} = \mathbf{x}^*$ (in which case $a = \frac{1}{\pi}$), where $\mathbf{x}^* = (\frac{\pi+r}{n\pi r}, \dots, \frac{\pi+r}{n\pi r}, \dots, \frac{\pi+r}{n\pi r})^T$, with $\lambda_1^* \geqslant 1$ and $\lambda_2^* = 0$, does![2]

Therefore, the stationary point $\mathbf{x} = \bar{\mathbf{x}}$ satisfies condition (c) of AET, but does not solve problem $(\mathcal{P})$. The reason stems from the fact that the objective function is not *twice continuously* differentiable at $\mathbf{x} = \bar{\mathbf{x}}$, that is, $F \notin \mathcal{C}^2(\mathbb{R}^n_+, \mathbb{R})$, even if $F \in \mathcal{C}^1(\mathbb{R}^n_+, \mathbb{R})$. Indeed, as $F$ is differentiable on $\mathbb{R}^n_+$, that is, $dF_{\mathbf{x}}(\mathbf{h}) \in \mathcal{L}(\mathbb{R}^n_+, \mathbb{R})$, $\mathbf{h} \in \mathbb{R}^n$, it is differentiable at $\bar{\mathbf{x}}$, and we have

$$dF_{\bar{\mathbf{x}}}(\mathbf{h}) = \mathcal{J}_{F(\bar{\mathbf{x}})} d\mathbf{h} + \|\mathbf{h}\| \eta_{\bar{\mathbf{x}}}(\mathbf{h}), \quad \text{with} \lim_{\|\mathbf{h}\| \to \mathbf{0}, \|\mathbf{h}\| \neq \mathbf{0}} \eta_{\bar{\mathbf{x}}}(\mathbf{h}) = 0. \quad (10)$$

Thus, for each $i \in \{1, \dots, n\}$, let $\mathbf{e}_i \in \mathbb{R}^n_+$ be the vector in the $i$th direction. Let $\mathbf{h} = t\mathbf{e}_i$, $t \in \mathbb{R}$. The directional derivative of $F$ in the direction $\mathbf{e}_i$ at $\bar{\mathbf{x}}$ is given by

$$\begin{aligned} \left.\frac{\partial F(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} &= \lim_{t \to 0}\left(\frac{F(\bar{\mathbf{x}} + t\mathbf{e}_i) - F(\bar{\mathbf{x}})}{t}\right) \\ &= \lim_{t \to 0}\left(\frac{t^4 \sin\left(\frac{1}{t}\right) + t}{t}\right) = 1, \end{aligned} \quad (11)$$

as $F(\bar{\mathbf{x}}) = 0$, and $|t^3 \sin(\frac{1}{t})| \leqslant t^3$ as $|\sin(\frac{1}{t})| \leqslant 1$. Likewise, we have $\frac{\partial F(\mathbf{x})}{\partial x_j}|_{\mathbf{x}=\bar{\mathbf{x}}} = 1$, for all $j \neq i$. In addition, by using (4), with $x_j = \frac{1}{nr}$, for all $j \neq i$, we deduce

$$\forall i \quad \lim_{x_i \to \frac{1}{nr}} \frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{a \to 0}\left[4a^3 \sin\left(\frac{1}{a}\right) - a^2 \cos\left(\frac{1}{a}\right) + 1\right] = 1, \quad (12)$$

[1] It is worth noticing that the constraint qualification holds at $\mathbf{x} = \bar{\mathbf{x}}$ as $G_2(\mathbf{x})$ is the only effective constraint, that is, $G_2(\bar{\mathbf{x}}) = 0$, and $\operatorname{rank}[\mathcal{J}_{G_2(\bar{\mathbf{x}})}] = 1$, with $\mathcal{J}_{G_2(\bar{\mathbf{x}})} = (-\frac{2}{r}, \dots, -\frac{2}{r})$, while the constraint $G_1(\mathbf{x}) \geqslant 0$ is slack at $\mathbf{x} = \bar{\mathbf{x}}$. Thus, (KT1)–(KT2) are here also necessary.

[2] Indeed, we have $F(\frac{\pi+r}{n\pi r}, \dots, \frac{\pi+r}{n\pi r}, \dots, \frac{\pi+r}{n\pi r}) = \frac{1}{\pi}$, as $\sin(\pi) = 0$ (in radians). Then, from (7), we have $\lambda_1^* \geqslant \frac{r}{2}$, and from (9), we have $\lambda_2^* = 0$, as $G_2 = 0$ when $\mathbf{x} > \bar{\mathbf{x}}$.

as $|\sin(\frac{1}{a})| \leqslant 1$ and $|\cos(\frac{1}{a})| \leqslant 1$, so (12) is well defined. Then, $dF_{\bar{\mathbf{x}}}(\mathbf{h})$ given by (10) satisfies the definition, so $F \in \mathcal{C}^1(\mathbb{R}^n_+, \mathbb{R})$ in the neighborhood of $\mathbf{x} = \bar{\mathbf{x}}$.

Next, as $F$ is twice differentiable on $\mathbb{R}^n_+$, that is, $d^2F_{\mathbf{x}}(\mathbf{h}, \mathbf{k}) \in \mathcal{L}(\mathbb{R}^n_+, \mathcal{L}(\mathbb{R}^n_+, \mathbb{R}))$, $\mathbf{h}, \mathbf{k} \in \mathbb{R}^n$, it is twice differentiable at $\bar{\mathbf{x}}$. Then, there exists a Hessian matrix with $n^2$ terms, that is, $\mathcal{H}_{F(\bar{\mathbf{x}})} = [\frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j}|_{\mathbf{x}=\bar{\mathbf{x}}}]_{1 \leqslant i,j \leqslant n}$, such that

$$\forall (\mathbf{h}, \mathbf{k}) \in \mathbb{R}^n \times \mathbb{R}^n \quad d^2F_{\bar{\mathbf{x}}}(\mathbf{h}, \mathbf{k}) = \mathbf{h}^T \mathcal{H}_{F(\bar{\mathbf{x}})} \mathbf{k}. \tag{13}$$

In particular, by using (4), we have

$$\begin{aligned} \forall i, j \quad \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} &= \lim_{t \to 0} \frac{1}{t} \left( \frac{\partial F(\mathbf{x})}{\partial x_j}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}+t\mathbf{e}_i} - \frac{\partial F(\mathbf{x})}{\partial x_j}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} \right) \\ &= \lim_{t \to 0} \left( \frac{4t^3 \sin\left(\frac{1}{t}\right) - t^2 \cos\left(\frac{1}{t}\right)}{t} \right) = 0. \end{aligned} \tag{14}$$

But, by using (5), and proceeding in the same way as for (12), we deduce

$$\begin{aligned} \forall i, j \quad \lim_{a \to 0} \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j} &= \lim_{a \to 0} \left[ (12a - 1) \sin\left(\frac{1}{a}\right) - 6a \cos\left(\frac{1}{a}\right) \right] \\ &= -\lim_{a \to 0} \sin\left(\frac{1}{a}\right), \end{aligned} \tag{15}$$

which is undefined at $\mathbf{x} = \bar{\mathbf{x}}$. Then, $\lim_{\mathbf{x} \to \bar{\mathbf{x}}} \mathcal{H}_{F(\mathbf{x})}$ is undefined, so $d^2F_{\mathbf{x}}(\mathbf{h}, \mathbf{k})$ is not continuous at $\bar{\mathbf{x}}$. We conclude that $F \notin C^2(\mathbb{R}^n_+, \mathbb{R})$ in the neighborhood of $\mathbf{x} = \bar{\mathbf{x}}$. Therefore, while $F$ satisfies condition (c) of AET at $\bar{\mathbf{x}}$, $F$ does not satisfy the conditions for a local maximum, so it does not reach a global maximum at $\mathbf{x} = \bar{\mathbf{x}}$.

## 4. CONCLUDING REMARKS

The conclusion follows: if the objective function is not of class $\mathcal{C}^2$ everywhere on the constraint set, some points could satisfy the Kuhn and Tucker conditions without being optima. Then, condition (c) is replaced by condition (c′) in AET:

(c′) $\mathcal{J}_{\mathbf{F}(\mathbf{x}^*)} \neq \mathbf{0}$, *and* $F(\mathbf{x})$ *is twice continuously differentiable in the neighborhood of* $\mathbf{x}^*$.

It turns out that this stronger condition holds in the original proof of Arrow and Enthoven's result (see on pages 785 to 787). In what follows, we only point out the role played by the twice continuous differentiability of the quasi-concave objective function in their proof. To this end, consider on page 787, equation (2.23), that is, $u(v)\frac{\phi_u(0,v)}{v} \geqslant \phi_v(0, v)$. To generate their contradiction, they studied the limit of $\frac{\phi_u(0,v)}{v}$ as $v$ approaches zero, that is, $\lim_{v \to 0} \frac{1}{v}(\frac{\phi_u(0,v)}{v}) = \lim_{v \to 0} \frac{\phi_u(0,v) - \phi_u(0,0)}{v-0}$ as $\phi_u(0, 0) = 0$ in equation (2.13) on page 785. This limit should be $\phi_{uv}(0, 0)$, with $\phi_{uv}(0, 0) = \frac{\partial \phi_u(u,v)}{\partial v}|_{(u,v) \to (0,0)} = \frac{\partial^2 \phi_u(u,v)}{\partial v \partial u}|_{(u,v) \to (0,0)}$. But, even if we have $\phi_{uv}(0, 0) = \lim_{v \to 0} \frac{\phi_u(0,v) - \phi_u(0,0)}{v-0}$, the limit as $v$ approaches 0 of $\phi_u(u, v)$ need not be well defined for $v$ in a neighborhood of 0 unless the partial derivative of $\frac{\phi_u(0,v)}{v}$ is continuous at zero.

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